

MODELLING THE MARKET IN A RISK-AVERSE WORLD

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ABSTRACT

In this paper, descriptive models of real returns on the market portfolio of the United Kingdom are developed and analysed. The ‘market portfolio’ is taken to comprise listed equity and government bonds, aggregated in proportion to their market capitalisation from time to time. The models have the attributes that, conditionally on information at the start of a year:

- the real return on the market portfolio during that year is normally distributed; and
- the market price of risk during that year is reasonably greater than zero.

The best of the models considered was found to be a regime-switching model. For the purposes of predictive modelling, however, it was decided to use ex-ante estimates of expected returns. This led to bias in the observed mean returns, which negates the rational expectations hypothesis. In the light of the literature on the subject, this is considered acceptable for these purposes.

KEYWORDS

Market portfolio, Risk aversion, United Kingdom, Bias, Rational expectations

1. INTRODUCTION

In this paper, descriptive models of returns on the market portfolio of the United Kingdom are developed and analysed. For the purposes of the paper, ‘returns’ are defined as real annual forces of return. The market portfolio is taken to comprise listed equity and government bonds (both conventional and index-linked), aggregated in proportion to their market capitalisation from time to time. The models have the attributes that, conditionally on information at the start of a year:

- the return on the market portfolio during that year is normally distributed; and
- the market price of risk during that year is reasonably greater than zero.

For the purposes of the latter attribute, the market price of risk is taken to be:

$$R_t = \frac{\mu_{M;t} - \delta_{I;t}}{\sigma_{M;t}} \text{ for } t = 1, \dots, N; \quad (1)$$

where:

$$\mu_{M;t} = E\{\delta_{M;t} | F_{t-1}\};$$

$$\sigma_{M;t}^2 = E\{(\delta_{M;t} - \mu_{M;t})^2 | F_{t-1}\};$$

$\delta_{I;t}$ is the real return on a one-year risk-free zero-coupon bond during the year $[t-1, t]$;

$\delta_{M;t}$ is the real return on the market portfolio during that year; and

F_t is the information at time t , including $\delta_{I;t}$.

The purpose of the development of the descriptive models is to inform the definition of predictive stochastic models for use with the equilibrium models developed in Thomson & Gott (unpublished a, b). (The distinctions between ‘descriptive’, ‘predictive’ and ‘normative’ models follow Thomson (2006).) In the development of the descriptive models, it is therefore borne in mind that the purpose of estimation is to derive ex-post estimates of ex-ante parameters. The rational expectations hypothesis is applied so far as it is possible to do so. However, where that hypothesis conflicts with this purpose, constraints on the estimates are accepted.

By the same token, the role of $\delta_{I;t}$ in the models is not primarily to explain the variability of $\delta_{M;t}$; other variables might do so better. It is primarily to satisfy the required attributes. Attention is drawn below to instances in which these issues arise. The predictive model envisaged is not intended to constitute ‘the real-world model’ in any unique or logical positivist sense of that concept. It is merely intended to be a reasonable model for the purposes of ex-ante decision-making. For this reason, no hypothesis testing is undertaken, and no out-of-sample tests are made.

The requirement that, conditionally on information at the start of a year, the return on the market portfolio during that year is normally distributed, does not prevent the use of a model in which the unconditional distribution of the return on the market portfolio is otherwise distributed. Indeed, it is largely the purpose of this paper to explore the use of other models.

It also follows that the market price of risk should not only be reasonably greater than zero in the descriptive model, but should also be so for any reasonable value of $\delta_{I;t}$ that may occur in a predictive model. The interpretation of ‘reasonably greater than zero’ is amplified in sections 3 and 4 below.

It is envisaged that the predictive model will be parameterised so that the user will not be able to outperform the market on a risk-adjusted basis. This means that the model can be used, for example, to determine market-consistent prices of market-related instruments, and to determine market-consistent liability-based mandates for investment management.

In section 2, relevant literature is reviewed. In section 3 the models are described. In section 4 the parameterisation of the models is presented and the results are discussed. In section 5 the use of the models for predictive purposes is discussed

2. LITERATURE REVIEW

2.1 RISK AVERSION

As pointed out by Merton (1980: 327):

“... a necessary condition for equilibrium is that the expected return on the market must be greater than the riskless rate.... A sufficient condition for this proposition to obtain is that all investors are strictly risk-averse expected utility maximizers.”

For this proposition to obtain, the market price of risk must be positive. While some models of market equilibrium do not rule out a negative market price of risk (e.g. Conrad & Kaul, 1988: 410; Derrig & Orr, 2004: 46), it must be accepted that the long-term financial institutions advised by actuaries (principally life offices and pension funds), effectively being custodians of trust moneys, are risk-averse. These clients are participants in the process of equilibrium-formation in the capital market. For actuarial purposes, therefore, the models used by actuaries for advising such clients must assume risk-aversion.

Since the publication of the Wilkie (1986) model, numerous stochastic models of returns on assets have been published. Most of these suffer from the drawback that, conditionally on information at the start of a period, they may produce negative market prices of risk during that period. Among the models that may exhibit this phenomenon (not all of which are published in detail), particularly in the case of equities, are:

- the Wilkie (1986, 1995b) model for the United Kingdom (also calibrated for other countries);
- the Carter (1991) model for Australia;
- the Dyson & Exley (1995) model for the U.K.;
- the Thomson (1996) model for South Africa;
- the Harris (1997) model for Australia;
- the CAP:Link scenario generation system (Mulvey & Thorlacius, 1998);
- the Boender, Van Aalst & Heemskerk (1998) model for the Netherlands;
- the Whitten & Thomas (1999) model for the U.K.;
- the TY model for the U.K. (Yakoubov, Teeger & Duval, 1999); and
- the Hilli et al (2007) model for Finland.

The Hibbert, Mowbray & Turnbull (unpublished) model for the U.K. avoids this problem.

2.2 THE MARKET PORTFOLIO

None of the models listed in the preceding section includes a model of the return on the market portfolio. While they do produce models of major constituents of the market portfolio, their aggregation into a model of the market portfolio would require a model of the

composition of that portfolio. The advantage in the explicit modelling of a market-portfolio proxy is that it permits the equilibrium modelling of the various asset categories.

As stated above, in this paper the market portfolio is taken to comprise listed equity and government bonds, aggregated in proportion to their market capitalisation from time to time. As Roll (1977) points out, a model of the market portfolio should include not only equities and bonds, but also all other capital assets, including non-traded assets such as human capital. While this would indeed be required for a true descriptive model, the requirements of a normative model for decision-making purposes are less exacting. Again it must be appreciated that the institutional clients of actuaries invest in a market of traded assets and participate in the process of equilibrium formation within that market. Eun (1994) analyses the capital asset-pricing model (CAPM) into the observable and latent portfolios comprising the market portfolio. He finds that, if the correlation between the two is positive, then, for the observable constituent of the market portfolio, the securities market line has a higher intercept than the risk-free rate. The excess is proportional to the risk premium on the latent constituent. If, however, equilibrium occurs between participants excluded from the latent portfolio, then, for those participants, the intercept must revert to the risk-free rate. This can be accounted for only if the homogeneity of expectations differs as between those participants and others.

The Thomson & Gott (unpublished b) model for the U.K. avoids the problem of negative market prices of risk and it includes a model of the market portfolio. However, the specification of the model of the market portfolio in that article was tentative. The exploration of alternative market models was left to further research, which is the subject of this paper.

2.3 BIAS AND RATIONAL EXPECTATIONS

The approach adopted in this paper admits the possibility of bias in conditional expected returns on the market portfolio.

Merton (1980, 125–6) observed that, while substantial effort had been expended on the estimation of the volatilities of returns, little work had been done on expected returns. He suggested that this was due to the relative difficulty of estimating expected returns. However, as Derrig & Orr (*op. cit.*: 46), Campbell (2000: 1522) and Grant & Quiggin (2006) point out, since Mehra & Prescott's (1985) exploration of the 'equity risk-premium puzzle', there have been numerous articles reviewing the expected returns on equity. Conrad & Kaul (*op. cit.*) postulate an autoregressive process for conditional expected return, but their model does not exclude negative market prices of risk. Fama & French (1989) find that expected returns follow a business-cycle pattern and contain a risk premium that is related to longer-term aspects of business conditions. Derrig & Orr (*op. cit.*) document numerous different approaches to the estimation of the equity risk premium, with widely differing results. Wilkie (1995a) contributes yet another. Thomson (2006b) suggests that reference to the equity risk premium 'puzzle' suggests a paradigmatic metanarrative that needs to be deconstructed.

An often unstated assumption underlying the calibration of stochastic models of returns on assets is that the rational expectations hypothesis (REH) (Muth, 1960) holds. While some authors (e.g. Thomson, 1996: 798–9) have cautioned prospective users that their descriptive models may not be appropriate for predictive purposes, the calibration of those models to ex-post observations suggests that, in the absence of information to the contrary, those observations are unbiased estimates of the corresponding ex-ante values.

Numerous studies (Cuthbertson, 1996: 116–201) show that, on certain assumptions, for certain markets at certain times, the REH may be rejected. While many of these relate to short-term effects or to individual shares relative to the market, some of them (e.g. Shiller, 1981; LeRoy & Porter, 1981) are of importance in the long-term modelling of the market.

Even in those cases, it has been shown (e.g. Marsh & Merton, 1986) that, with different assumptions, different conclusions may be drawn and Fama (1991: 1586) argues that they do not necessarily reject the REH. Nevertheless, the REH remains questionable. As Cuthbertson (*ibid.*: 97) points out, tests of the REH that rely on an assumed model such as the capital-asset pricing model (CAPM) involve joint assumptions; rejection does not necessarily imply rejection of the REH. Conversely, however, they would not necessarily imply rejection of the CAPM.

As Roll & Ross (1994) pointed out:

“... a decade of empirical studies [had] reported little evidence of a significant cross-sectional relation between average returns and betas.”

A possible explanation, they suggested, is that market-portfolio proxies are mean–variance inefficient. Another possible explanation is that the REH does not apply. They refer to the phenomenon as a ‘puzzle’; like the equity risk-premium puzzle, this begs the question whether the paradigm presupposed by the REH is true.

3. THE MARKET PORTFOLIO

In Thomson & Gott (*op. cit.*) a simple model of the return on the market portfolio was adopted, without consideration of more complex but possibly better models. In this section four models are considered: the basic model used in that article, a Markov regime-switching model, an exponential autoregressive (AR) model and an autoregressive conditional heteroskedasticity (ARCH) model. These models are defined below.

The data used are the returns on the U.K. market-portfolio proxy and on the one-year risk-free zero-coupon bond for the period from 1980 to 2006 as determined in that article. As noted above the market-portfolio proxy comprises U.K. listed equity and government bonds (both conventional and index-linked). These were the same data as used in Thomson & Gott (unpublished b).

As mentioned in section 2, the market-portfolio should include all assets in which the actuary’s client may invest. A notable absence is fixed property. Since many fixed properties are owned by listed companies, it would be necessary to avoid the double-counting involved. Corporate debt should also be included. Until such time as the necessary data are available, the portfolio used in this paper is an approximation to the best proxy available.

This data set is small, comprising only 27 values of each of the variables. It would have been possible to use quarterly data, but for the purpose of annual decision-making that would be of questionable value (Thomson, 1996). As Merton (*op. cit.*) points out, the precision of the estimate of expected returns depends on the total length of calendar time, rather than on the number of observations *per se*. On the other hand, in a rapidly changing world, it is questionable whether long data sets are relevant to the future.

In view of the small data set, particular care needs to be taken to avoid the treatment of spurious or fortuitous relationships as important. Also, in specifying models involving autoregressive effects, long lags should not be considered. If such a lag is of greater significance than a shorter lag, the effect would have to be regarded as fortuitous. In this paper only one-year lags are considered.

Since, as explained above, it is not intended that any predictive model based on the descriptive models developed in this paper is uniquely valid, it is considered better to retain reasonable uncertainty in the model.

3.1 THE BASIC MODEL

As explained in Thomson & Gott (unpublished b), $\mu_{M,t}$ cannot be modelled as a constant because this would result in negative risk premiums from time to time. Instead, as in that paper, we may model $\delta_{M,t}$ as:

$$\delta_{M,t} = g\delta_{I,t} + h + \sigma_M \varepsilon_t; \quad (2)$$

where:

$$\begin{aligned} \varepsilon_t &\sim N(0,1); \\ \text{cov}\{\varepsilon_t, \varepsilon_s\} &= 0 \text{ for } s \neq t; \\ g &\geq g^* \geq 1; \text{ and} \\ h &\geq h^* \geq 0; \end{aligned}$$

so that:

$$\mu_{M,t} = g\delta_{I,t} + h.$$

In order to avoid negative market prices of risk, we may take $g^* = 1$ and $h^* = 0$. These are referred to below as the ‘basic constraints’. However, the purpose of setting g^* and h^* greater than or equal to 0 is to not merely to ensure that the market price of risk is non-negative, but that it is reasonably greater than 0. For this purpose it is required either that $g^* = 1.2$ and $h^* = 0$ or that $g^* = 1$ and $h^* = 0.01$. (The concept ‘reasonably greater’ is necessarily arbitrary.) These are referred to below as the ‘required constraints’.

In Thomson & Gott (unpublished b) it was found that h was not significant at the 95% level. With $h = 0$ an estimate of $g = 1.8$ was obtained. The 95% confidence limits of g were 0.5 and 3.1, so that this estimate was not reliable. It was nevertheless used in that paper for the purposes of illustration. The volatility parameter σ_M was estimated at 0.12.

3.2 THE REGIME-SWITCHING MODEL

Another possible approach would be to use a Markov regime-switching model (Hamilton, 1989), with a similar structure in each regime, i.e.:

$$\delta_{M,t} = g_{S_t} \delta_{I,t} + h_{S_t} + \sigma_{S_t} \varepsilon_t; \quad (3)$$

where:

$$\begin{aligned} S_t &\in \{0,1\}; \\ \Pr\{S_t = 0 | S_{t-1} = 0\} &= p_{00}; \\ \Pr\{S_t = 1 | S_{t-1} = 0\} &= p_{01} = 1 - p_{00}; \\ \Pr\{S_t = 0 | S_{t-1} = 1\} &= p_{10}; \\ \Pr\{S_t = 1 | S_{t-1} = 1\} &= p_{11} = 1 - p_{10}; \end{aligned}$$

and $\varepsilon_t \sim N(0,1)$ is serially independent, so that, conditionally on information at time $t-1$:

$$\delta_{M,t} \sim N(\mu_{M,t}, \sigma_{M,t}^2);$$

where:

$$\begin{aligned} \mu_{M,t} &= g_{S_t} \delta_{I,t} + h_{S_t}; \\ \sigma_{M,t} &= \sigma_{S_t}; \\ g_s &\geq g_s^* \geq 0; \text{ and} \\ h_s &\geq h_s^* \geq 0. \end{aligned}$$

As for the basic model, it is required, for each s , either that $g_s^* = 1.2$ and $h_s^* = 0$ or that $g_s^* = 1$ and $h_s^* = 0.01$. In the basic model, the estimate of σ_M required no constraint. In the regime-switching model, however, one of the regimes may produce an estimate of σ_s that will yield unreasonably large market prices of risk. The need for constraints on σ_s is considered in the parameterisation of the model in section 4.2 below.

As mentioned in section 1 above, it is required that, conditionally on information at the start of a year, the return on the market portfolio during that year be normally distributed. In order to accommodate this requirement it is assumed that F_{t-1} includes S_t ; i.e. that the regime is known at the start of the year. It is for this reason that the parameters must satisfy the required constraints in each regime; otherwise the distribution of the return would have a mixture density.

3.3 THE EXPONENTIAL AR MODEL

$\mu_{M,t}$ can also not be modelled as a linear autoregressive moving-average (ARMA) time series because this would also result in negative risk premiums from time to time. However, $\delta_{M,t}$ may for example be modelled as:

$$\delta_{M,t} = g\delta_{I,t} + h \exp\left\{\alpha\left(\delta_{M,t-1} - g\delta_{I,t-1}\right)\right\} + \sigma_M \varepsilon_t; \quad (4)$$

where:

$$g \geq g^* \geq 1;$$

$$h \geq h^* \geq 0; \text{ and}$$

$$\varepsilon_t \sim N(0,1) \text{ is serially independent;}$$

so that, conditionally on information at time $t-1$:

$$\delta_{M,t} \sim N\left(\mu_{M,t}, \sigma_M^2\right);$$

where:

$$\mu_{M,t} = g\delta_{I,t} + h \exp\left\{\alpha\left(\delta_{M,t-1} - g\delta_{I,t-1}\right)\right\}.$$

Again it is required either that $g^* = 1.2$ and $h^* = 0$ or that $g^* = 1$ and $h^* = 0.01$.

This model is referred to in this paper as the ‘exponential AR model’.

3.4 THE ARCH MODEL

A fourth possibility is to include ARCH effects (Engle, 1982). $\delta_{M,t}$ may, for example, be modelled as:

$$\delta_{M,t} = g\delta_{I,t} + h + z_t;$$

where:

$$z_t = \sigma_t \varepsilon_t;$$

$$\sigma_t^2 = a + bz_{t-1}^2; \text{ and}$$

$$\varepsilon_t \sim N(0,1) \text{ is serially independent.}$$

3.5 GENERAL REMARKS

It may be noted that the basic model is a particular case of each of the other models. The use of the latter models must therefore be justified in terms of their additional descriptive value.

For each model, the likelihood function of the model and the maximum-likelihood estimates of the parameters were determined, as well as the confidence limits of those estimates.

For the purposes of comparison of the descriptive value of the respective models, but subject to the required attributes, the Akaike information criterion (AIC) was used, viz.:

$$A = 2k - 2l;$$

where:

k is the number of parameters;

$l = \ln(L)$; and

L is the likelihood of the observed values. (Akaike, 1974).

For each model considered (or, in the case of the regime-switching model, for each regime), the mean market price of risk was calculated, viz.:

$$R = \frac{\hat{\mu}_M - \bar{\delta}_I}{\hat{\sigma}_M};$$

where:

$\hat{\mu}_M = g\bar{\delta}_I + h$; and

$$\bar{\delta}_I = \frac{1}{N} \sum_{i=1}^N \delta_{I,t}.$$

Also, the implied bias in the mean was calculated, viz.:

$$B = \bar{\delta}_M - \hat{\mu}_M;$$

where:

$$\bar{\delta}_M = \frac{1}{N} \sum_{i=1}^N \delta_{M,t}.$$

Finally, a Q-Q plot (Wilk & Gnanadesikan, 1968) was produced.

4. THE PARAMETERISATION OF THE MODELS

In this section the parameterisation of each of the models is presented in turn, and the results are discussed. The results are then compared between the models considered.

4.1 THE BASIC MODEL

For the basic model the likelihood function and the estimates and confidence limits of the parameters were obtained in closed form following (e.g.) Hocking (1996: 136–45). Table 1 shows the results of the parameterisation.

The unconstrained version of the basic model gave an estimate of $h < 0$. That version is therefore not considered further, and the table shows only the results for $g = 1$ and $h = 0$. As shown in Table 1, both these cases satisfy the required constraints.

It may be noted that, in relation to their respective values, the confidence intervals of the parameters are wide, particularly in the cases of g and h . This leaves considerable scope for discretion in the determination of parameters for the predictive use of the model.

Figure 1 plots the observed values, estimates and confidence limits of $\delta_{M,t}$ against $\delta_{I,t}$ for the basic model. There is evidence that, for $h = 0$, the gradient is affected by the apparent outlier at $\delta_{I,t} = .0875$. There is no reason to ignore this point, but its effect underscores the

issue raised in the previous paragraph. Figures 2 and 3 show the same information in time-series form.

Table 1. Parameterisation of the basic model

Parameter	Details	$g = 1$	$h = 0$
g	estimate		1.83
	confidence limits		1.00, 3.59
h	estimate	0.023	
	confidence limits	0.00, 0.067	
σ_M	estimate	0.117	0.115
	confidence limits	0.083, 0.146	0.082, 0.144
k		2	2
l		20.20	20.54
A		-36.41	-37.08
R		0.195	0.226
B		0.000	-0.003

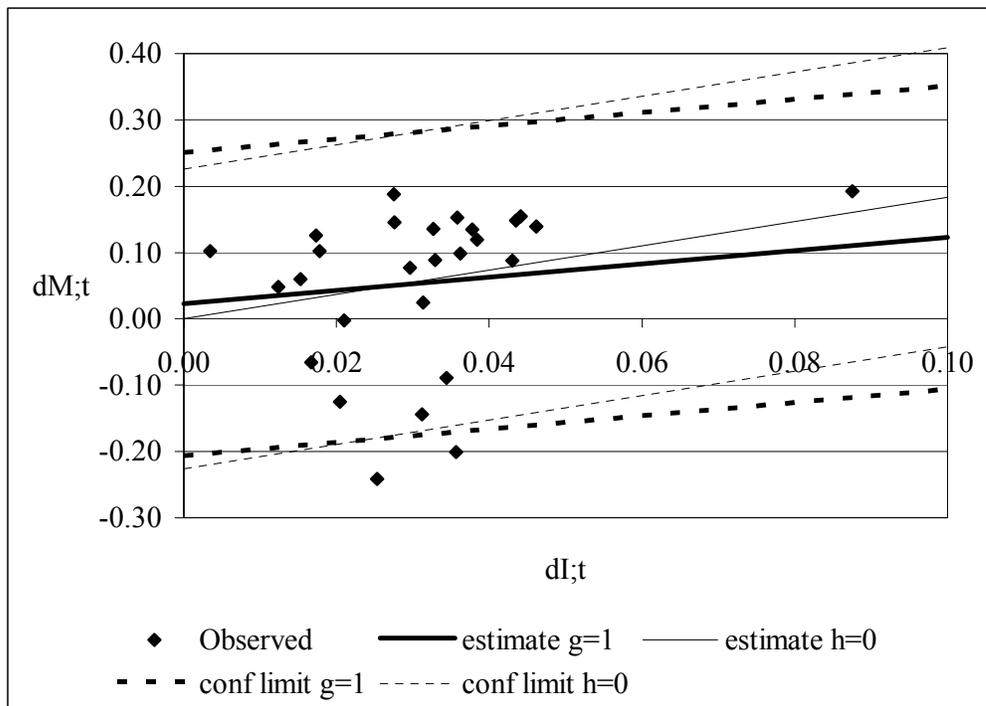


Figure 1. Basic model: $\delta_{M;t}$ against $\delta_{I;t}$

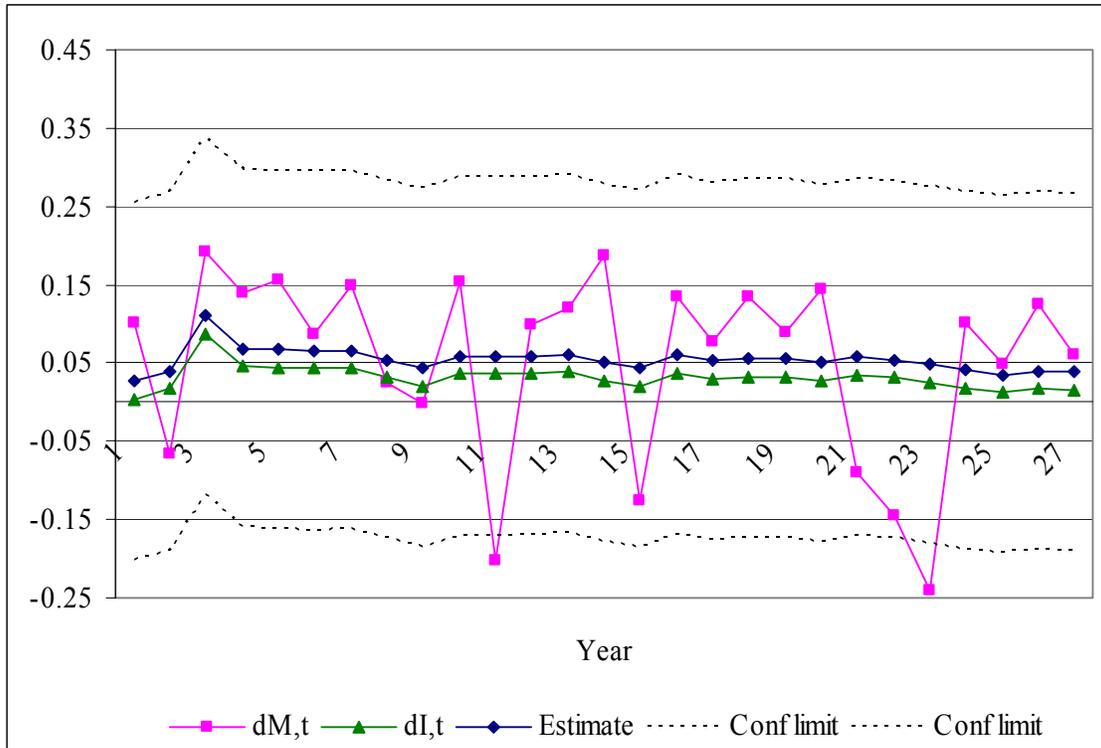


Figure 2. Basic model: time series for $g = 1$

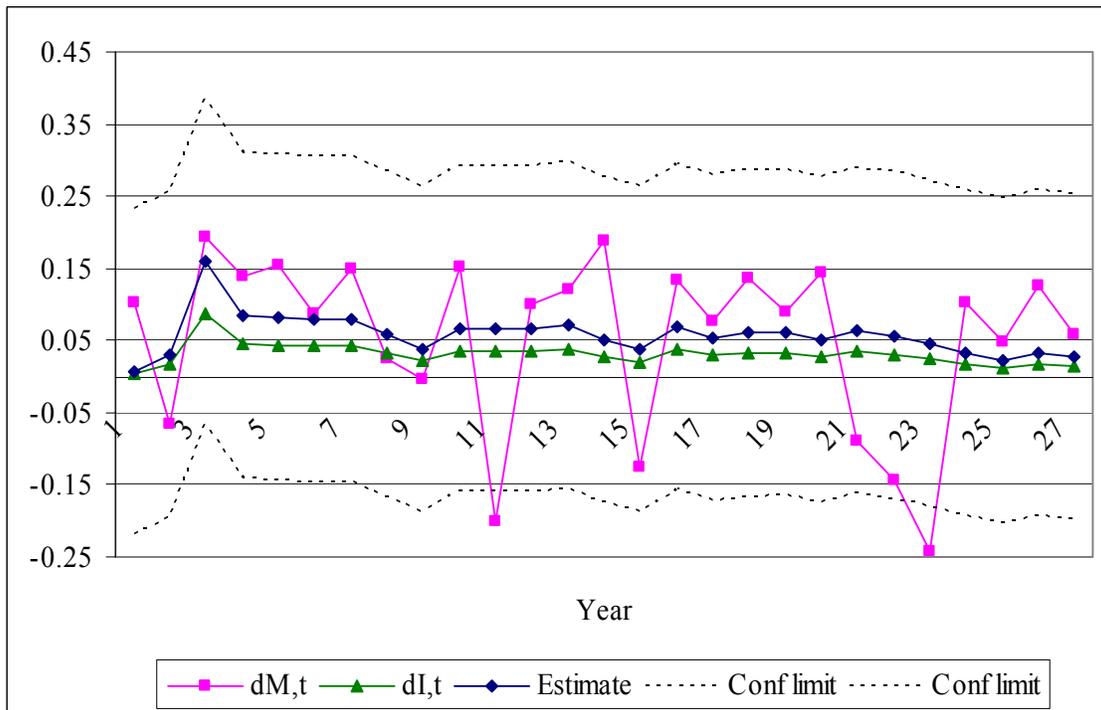


Figure 3. Basic model: time series for $h = 0$

It is clear that $\delta_{I,t}$ explains very little of the variability in $\delta_{M,t}$. However, as explained above, this is not the point; the purpose of including $\delta_{I,t}$ is not to explain the variability in

$\delta_{M,t}$, but to ensure that the market price of risk is positive. The constraint $h = 0$ produces a lower AIC and this version of the basic model is therefore considered further below. As shown in Table 1, this version also has the advantage of implying no bias.

For the purposes of the Q–Q plot, the set $\{z_t = \delta_{M,t} - g\delta_{I,t} + h \mid t = 1, \dots, N\}$ was ordered to give $\{z^{(r)} \mid r = 1, \dots, N\}$ such that:

$$z_s^{(r)} \geq z_s^{(r-1)} \text{ for } r = 2, \dots, N.$$

The Q–Q plot was then defined by the points:

$$\left(\Phi^{-1}\left(y_s^{(r)}\right), z_s^{(r)}\right);$$

where:

$$y_s^{(r)} = \frac{r - \frac{1}{2}}{N}. \quad (5)$$

In equation (5) the numerator and denominator represent the number of observations less than $z^{(r)}$ and $z^{(N)}$ respectively, the second term in the numerator being an adjustment for symmetry. This method of adjustment was used so as to correspond to that used for the regime-switching model below.

The Q–Q plot of the constrained versions of the basic model is shown in Figure 4. It is clear that, for both versions, the assumption of the normality of the residuals is not tenable. In fact, apart from the extreme quantiles, it appears that the lower quantiles and the upper quantiles follow fairly straight but different lines. This suggests that a regime-switching model may provide a better description of the data.

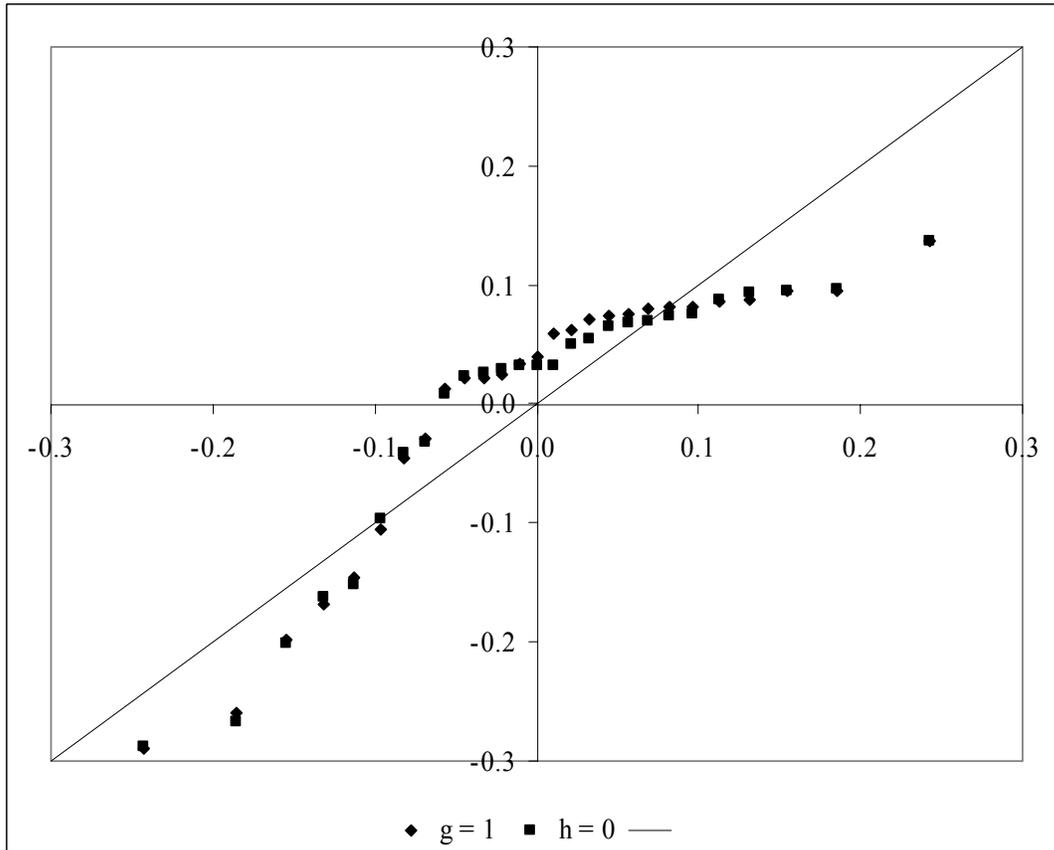


Figure 4. Basic model: Q–Q plot

4.2 THE REGIME-SWITCHING MODEL

For the regime-switching model it was not possible to obtain the likelihood function or the estimates and confidence limits of the parameters in closed form. The likelihood function follows Hamilton (1989). The maximum likelihood was found by means of the Nelder–Mead algorithm (Nelder & Mead, 1965).

In the initial application of the algorithm, the following basic constraints were applied:

- $0 \leq p_{s0} \leq 1$;
- $g_s \geq 1$;
- $h_s \geq 0$; and
- $\sigma_s \geq 0.001$.

The constraints on p_{s0} are trivial. Those on g_s and h_s , while weaker than those required in section 3.2, are common to them. The constraint on σ_s is used merely to avoid the discontinuities that occur for $\sigma_s = 0$.

Over some regions of the domain of the likelihood function of the regime-switching model, local maxima occurred. As a result it was necessary to implement a two-stage maximisation process. First a stopping criterion was used that was based on the change in the parameter vector over the last nine iterations of the Nelder–Mead algorithm. The maximisation process was then commenced ab initio, with the new parameter vector as the initial value and the original trial adjustments Δ_j as the initial trial adjustments. This was repeated until a stopping condition based on the change in the value of the likelihood was attained. Even then, it was found that, depending on the initial value of p_{00} , a local maximum might be obtained in a different region of the parameter domain than that of the global maximum. For reasons that become apparent below, it is necessary to consider both of these maxima.

In order to illustrate this problem, Table 2 shows the results of the parameterisation of the regime-switching model, for both maxima, with the basic constraints shown above. Here R and B are calculated not only for the sample as a whole, but also for each regime. They are defined as:

$$R_s = \frac{\hat{\mu}_s - \bar{\delta}_{I,s}}{\hat{\sigma}_s}; \text{ and}$$

$$B_s = \bar{\delta}_{M,s} - \hat{\mu}_s;$$

where:

$$\hat{\mu}_s = g_s \bar{\delta}_{I,s} + h_s; \text{ and}$$

$$\bar{\delta}_{I,s} = \frac{\sum_{t=1}^N P\{S_t = s | \delta_{M;t}, \dots, \delta_{M;1}\} \delta_{I;t}}{\sum_{t=1}^N P\{S_t = s | \delta_{M;t}, \dots, \delta_{M;1}\}}; \text{ and}$$

$$\bar{\delta}_{M,s} = \frac{\sum_{t=1}^N P\{S_t = s | \delta_{M;t}, \dots, \delta_{M;1}\} \delta_{M;t}}{\sum_{t=1}^N P\{S_t = s | \delta_{M;t}, \dots, \delta_{M;1}\}}.$$

Table 2. Parameterisation of the regime-switching model with basic constraints: local and global maxima

Parameter	Local maximum	Global maximum
p_{00}	0.61	0.00
p_{10}	0.50	0.65
g_0	1.37	1.07
h_0	0.073	0.100
σ_0	0.027	0.011
g_1	1.00	1.00
h_1	0.000	0.000
σ_1	0.144	0.125
l	29.63	34.31
R_0	3.191	10.344
R_1	0.000	0.000
R	0.791	0.709
B_0	0.001	0.000
B_1	-0.059	-0.027
B	-0.025	-0.017

The normal rationale for regime switching is that, in one regime, which occurs for relatively short periods, the expected values of returns are lower, and their standard deviations are higher, than in the more normal regime (e.g. Hardy, 2003). In other words, the market price of risk is lower in the abnormal regime. Taking regime 0 as the normal regime, this means that:

$$- \pi_0 > \pi_1; \text{ i.e. } p_{10} > p_{01} \text{ and } p_{00} > p_{11}; \quad (6)$$

$$- g_0 > g_1; \quad (7)$$

$$- h_0 > h_1; \text{ and} \quad (8)$$

$$- \sigma_0 < \sigma_1. \quad (9)$$

In fact, as Table 2 shows, while inequalities (7) to (9) hold at both maxima, inequality (6) does not hold at the global maximum. (Of course, if none of the inequalities held, the relationships indicated could be achieved merely by reversing the regimes.) This means that, at the global maximum, regime 0 is abnormal instead of normal; in fact, the estimates of p_{00} and p_{10} mean that, while regime 0 will frequently apply, it will never do so for a second consecutive year.

It may also be observed from Table 2 that, particularly at the global maximum, the volatility of returns is very low. In regime 0, the market price of risk is excessive, particularly for the global maximum. Both parameterisations imply substantial bias in regime 1.

It is clear that the critical parameter is p_{00} . Figure 5 shows the profile maximum likelihood as a function of that parameter, with the basic constraints shown above for the other parameters. That figure shows that the likelihood function follows two major segments.

It was found that the problems dealt with by the two-stage maximisation process related to local maxima on the plateau in the vicinity of $p_{00} \in (0.5, 0.7)$.

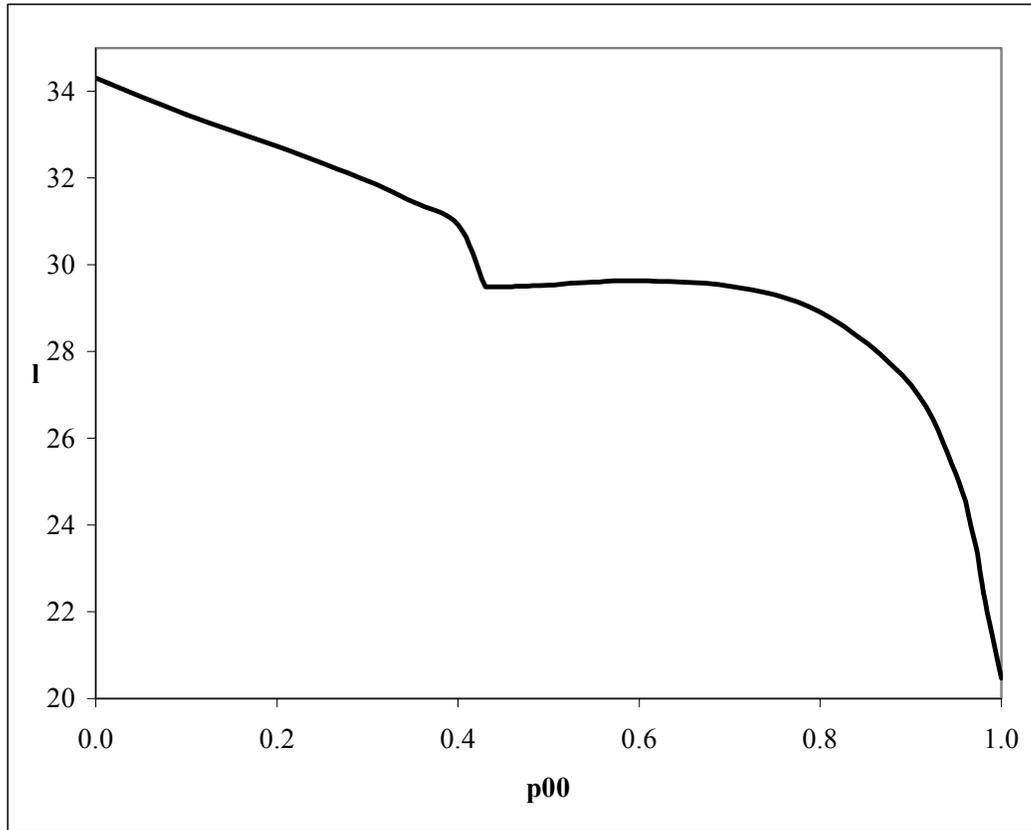


Figure 5. Regime-switching model: profile maximum likelihood as a function of p_{00} .

Figure 6 plots the observed values, estimates and confidence limits of $\delta_{M;t}$ against $\delta_{I;t}$ for the global maximum. The observed values have been identified as being in regime 0 if the posterior probability of regime 1 (i.e. $P\{S_t = 1 | \delta_{M;t}, \dots, \delta_{M;1}\}$) is less than 10%, regime 1 if it exceeds 90% and indeterminate otherwise. The estimates and confidence limits are those of the respective regimes. From that figure it is evident that:

- the estimate for regime 0 is effectively a straight line from the point representing the lowest value of $\delta_{I;t}$ to the point representing the highest;
- that line happens to pass very close to the points identified as being in regime 0, all of which fall within the very tight confidence limits produced by the low standard deviation in that regime;
- regime 1 comprises those points which fall outside of the confidence limits of regime 0;
- because of the large standard deviation in regime 1, the latter points (with one exception) are accommodated within the confidence limits of that regime; and
- the slope and intercept of the estimate for regime 1 are not influenced by the locations of the points identified as being in that regime.

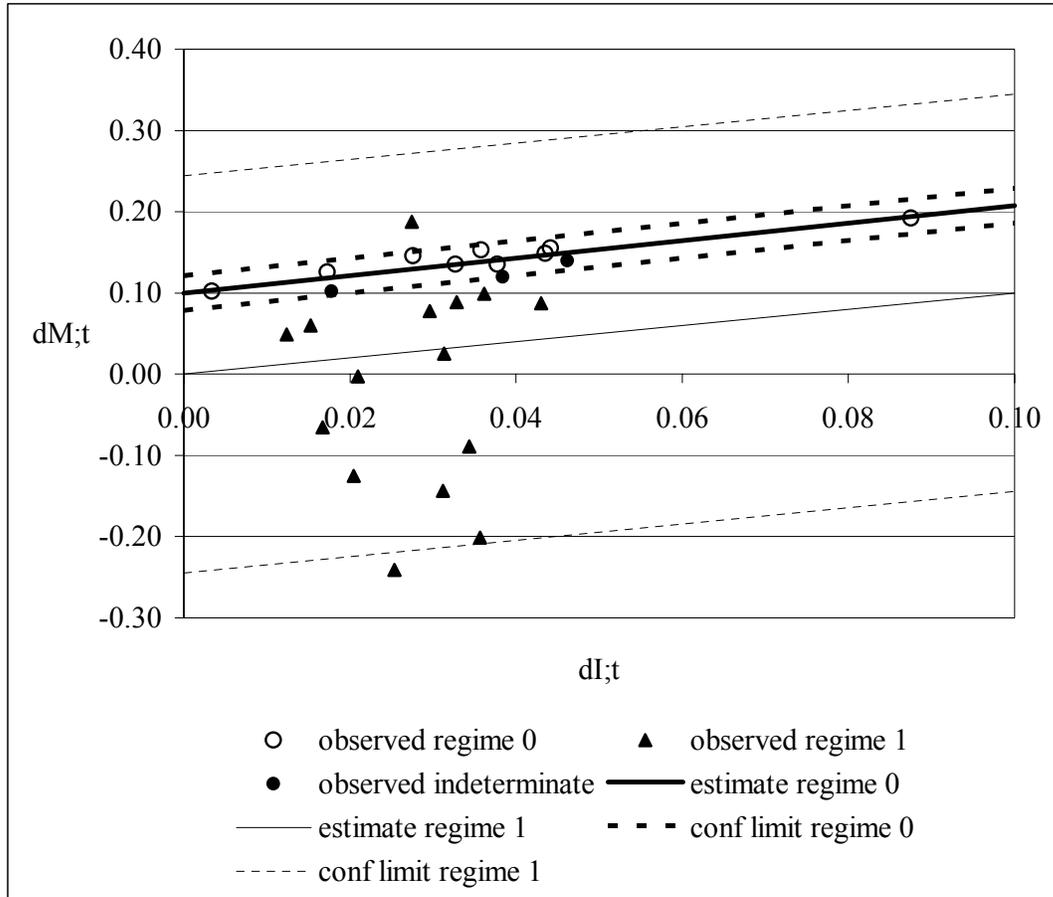


Figure 6. Regime-switching model, global maximum: $\delta_{M;t}$ against $\delta_{I;t}$

Figure 7 plots the observed values, estimates and confidence limits of $\delta_{M;t}$ against $\delta_{I;t}$ for the local maximum. The observed values have been identified as being in regime 0 if the posterior probability of regime 1 is less than 20%, regime 1 if it exceeds 80% and indeterminate otherwise. The estimates and confidence limits are those of the respective regimes. From that figure it is evident that:

- in comparison with the global maximum, this case identifies large-scale relationships, although the estimate for regime 0 still passes through the point representing the highest value of $\delta_{I;t}$;
- as before, regime 1 largely comprises those points which fall outside of the confidence limits of regime 0;
- here the standard deviation in regime 1 is even larger, and the latter points are all accommodated within the confidence limits of that regime; and
- as before, the slope and intercept of the estimate for regime 1 are not influenced by the locations of the points identified as being in that regime.

In the light of the above comparison, it is evident that, despite the greater likelihood of the global maximum, the local maximum is more credible.

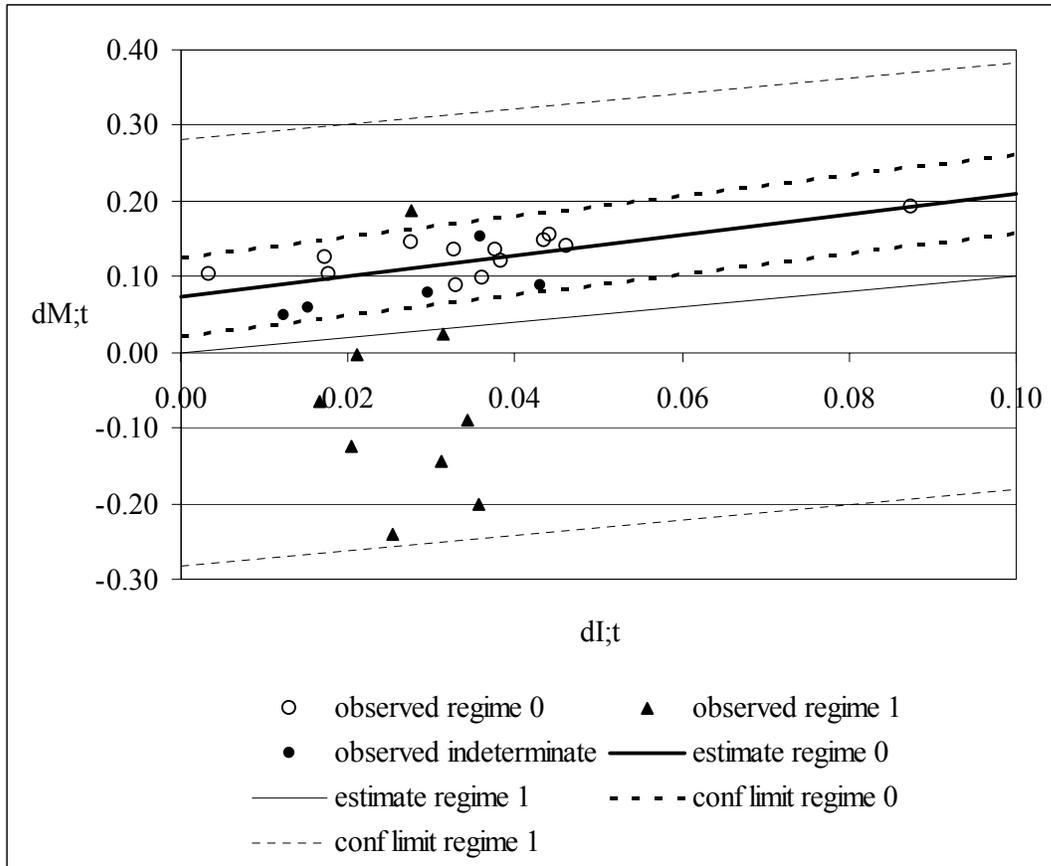


Figure 7. Regime-switching model, local maximum: $\delta_{M,t}$ against $\delta_{I,t}$

With the required constraints, the likelihood at the local maximum was greater for $g_1 = 1.2$ than for $h_1 = 0.01$ (29.45 as against 29.28). Table 3 shows the results of the parameterisation of the regime-switching model at the former, including 95% confidence limits.

The confidence limits of the estimates were determined by simulation. For this purpose a pseudo-random sample

$$\{\delta_{I,t,u}, \delta_{M,t,u} \mid t=1, \dots, N; u=1, \dots, 1000\}$$

was used to calculate a corresponding set of estimates $\{\theta_u \mid u=1, \dots, 1000\}$. For each parameter, the 95% upper and lower confidence limits were taken to be the means of the 25th- and 26th-largest, and of the 25th- and 26th-smallest estimates in the sample, respectively.

The confidence limits of the probabilities are extremely wide. While the confidence limits of g_0 in this model are slightly narrower than those of g in the basic model, those of h_0 are wider than those of h in that model. The confidence limits of σ_0 are narrower than those of σ in the basic model, while those of σ_1 are wider.

From Table 3 the question arises whether, with due regard to the confidence limits shown, the number of parameters estimated can be reduced. As regards the transition probabilities, it would be inappropriate, for the reasons discussed above, to set $p_{00} = 0$. On the other hand, it would not be inappropriate to set $p_{10} = 1$ (i.e. $p_{11} = 0$). As regards the regime-specific coefficients, we could set $g_0 = g_1 = 1.2$. The likelihoods and the values of the Akaike Information criterion with these additional constraints are reported in Table 4.

Table 3. Parameterisation of the regime-switching model: local maxima; $g_1 = 1.2$

Parameter	Details	
p_{00}	estimate	0.61
	confidence limits	0.00, 0.91
p_{10}	estimate	0.49
	confidence limits	0.12, 1.00
g_0	estimate	1.38
	confidence limits	1.00, 3.10
h_0	estimate	0.073
	confidence limits	0.015, 0.098
σ_0	estimate	0.027
	confidence limits	0.006, 0.046
g_1		1.20
h_1		0.00
σ_1	estimate	0.147
	confidence limits	0.077, 0.212
k		6
l		29.45
A		-46.89
R_0		3.190
R_1		0.038
R		0.795
B_0		0.000
B_1		-0.064
B		-0.028

Table 4. Likelihoods and values of the Akaike Information criterion with additional parameter constraints

Additional parameter constraints	l	k	A
none (as per Table 3)	29.45	6	-46.90
$p_{10} = 1$	21.92	5	-33.84
$g_0 = g_1 = 1.2$	29.37	5	-48.72

From Table 4 it is evident that the constraint $g_0 = g_1 = 1.2$ is the best. Table 5 shows the full results with those additional constraints.

Figure 8 shows, for this parameterisation:

$$P\{S_t = 1 \mid \delta_{M,t}, \dots, \delta_{M,t}\};$$

and Figure 9 shows the observed values, estimates and confidence limits of $\delta_{M,t}$ against $\delta_{I,t}$.

As might be expected, there is very little difference between Figure 9 and Figure 7.

Figure 10 shows the same information as Figure 9, in time-series format. In that figure, the confidence limits are based on posterior probabilities.

Table 5. Parameterisation of the regime-switching model: local maxima; $g_s = 1.2$

Parameter	Details	
p_{00}	estimate	0.59
	confidence limits	0.00, 0.91
p_{10}	estimate	0.51
	confidence limits	0.12, 1.00
g_0		1.20
h_0	estimate	0.080
	confidence limits	0.058, 0.099
σ_0	estimate	0.027
	confidence limits	0.007, 0.047
g_1		1.20
h_1		0.00
σ_1	estimate	0.145
	confidence limits	0.074, 0.217
k		5
l		29.36
A		-48.73
R_0		3.213
R_1		0.039
R		0.513
B_0		0.000
B_1		-0.063
B		-0.028

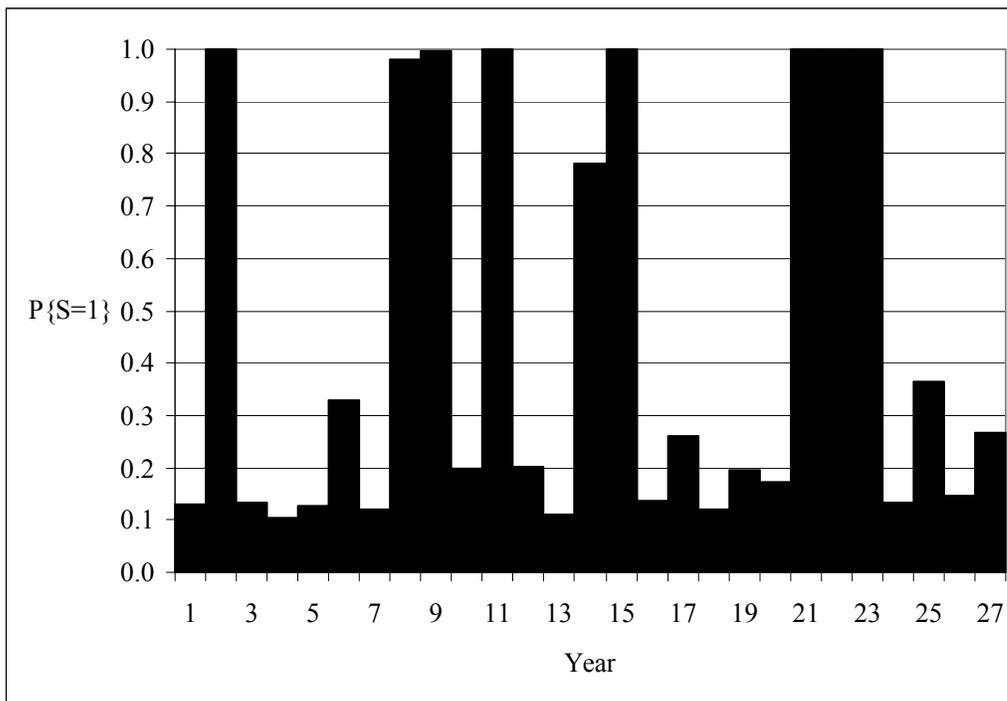


Figure 8. Regime-switching model, local maximum, $g_s = 1.2$: $P\{S_t = 1 | \delta_{M,t}, \dots, \delta_{M,1}\}$

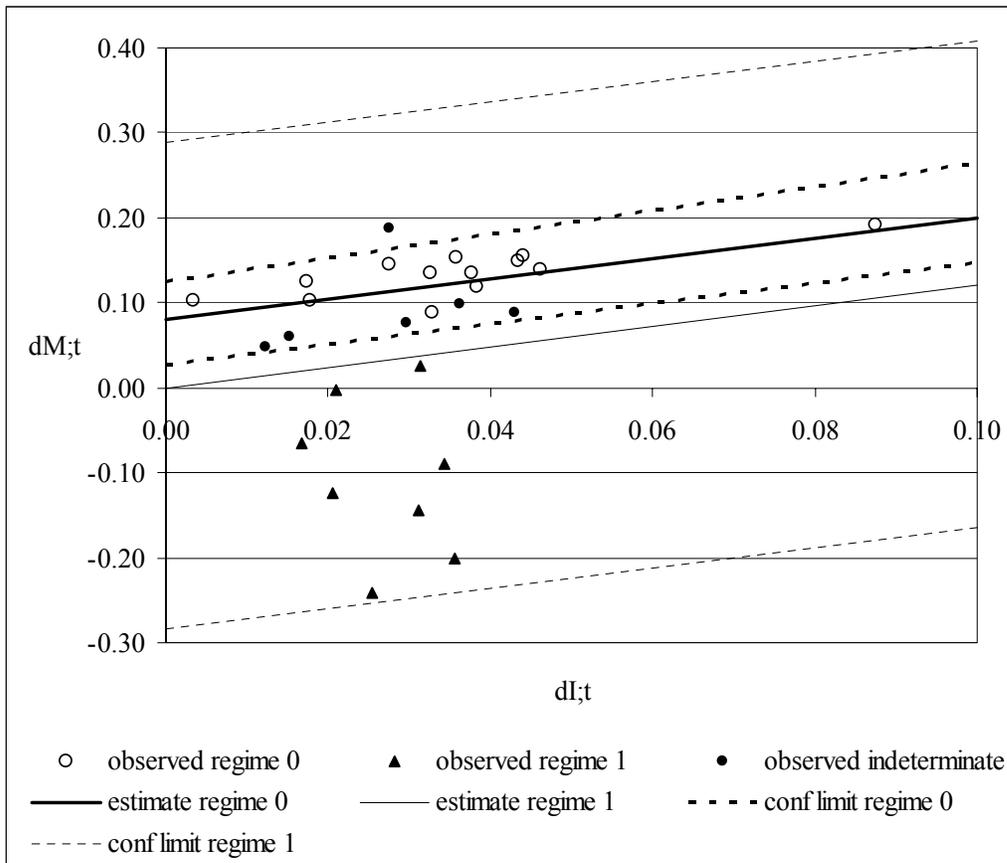


Figure 9. Regime-switching model, local maximum, $g_s = 1.2$: $\delta_{M,t}$ against $\delta_{I,t}$

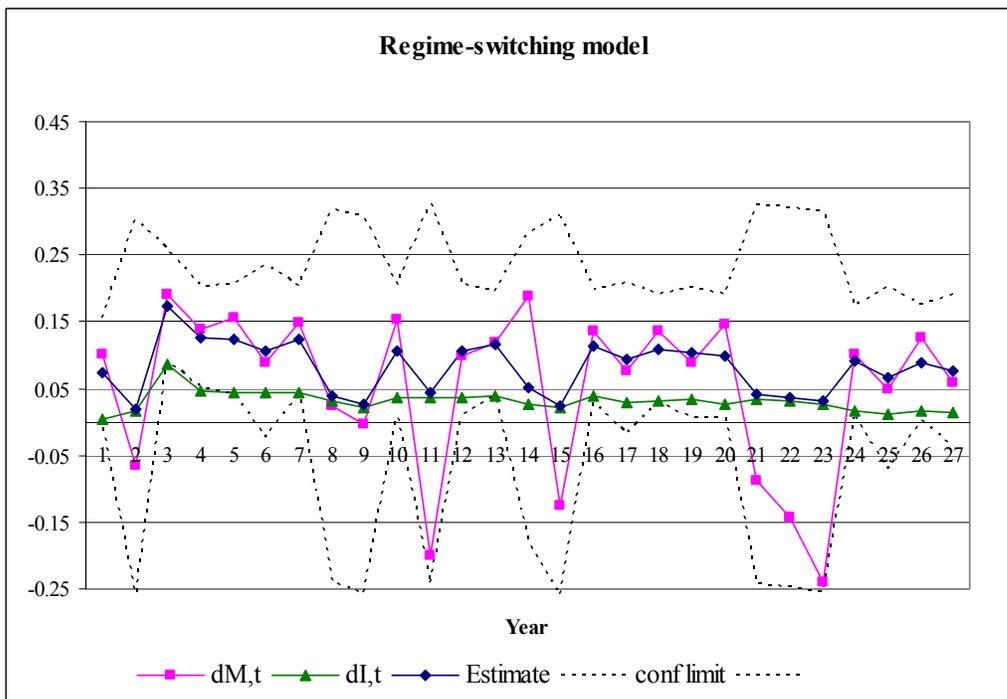


Figure 10. Regime-switching model, time series, local maximum: $g_s = 1.2$

A Q–Q plot was produced for each regime. These were defined as follows. For each observation the residual in each regime s was calculated as:

$$z_{ts} = \delta_{M,t} - g_s \delta_{1,t} - h_s.$$

For each regime the set $\left\{ \left(\delta_{M,t}, z_{ts}, P\{S_t = s \mid \delta_{M,t}\} \right) \mid t = 1, \dots, N \right\}$ was ordered with reference to

z_{ts} to give $\left\{ \left(\delta_s^{(r)}, z_s^{(r)}, P_s^{(r)} \right) \mid r = 1, \dots, N \right\}$ such that:

$$z_s^{(r)} \geq z_s^{(r-1)} \text{ for } r = 2, \dots, N.$$

The Q–Q plots were then defined by the points:

$$\left(\Phi^{-1}\left(y_s^{(r)}\right), z_s^{(r)} \right);$$

where:

$$y_s^{(r)} = \frac{\sum_{u=1}^r P_s^{(u)} - \frac{1}{2} P_s^{(r)}}{\sum_{u=1}^N P_s^{(u)}}; \quad (10)$$

and $\Phi(\bullet)$ is the normal distribution function with mean 0 and standard deviation σ_s . In equation (10) the numerator and denominator represent the expected number of observations less than $z^{(r)}$ and $z^{(N)}$ respectively, in regime s , the second term in the numerator being an adjustment for symmetry. This is analogous to the definition used for the basic model in equation (5).

Q–Q plots of the constrained version of the regime-switching model are shown in Figures 11 and 12 for regimes 0 and 1 respectively. In Figure 11, to avoid scaling issues, extreme values with $P\{S^{(r)} = 0\} < 0.01$ have been omitted as they are irrelevant to that regime. That plot is clearly an improvement on that of the basic model in Figure 4. While the plot for regime 1 is also closer to a straight line than that for the basic model, the bias is notable in that regime.

4.3 THE EXPONENTIAL AR MODEL

For the exponential AR model the likelihood function was obtained in closed form, but it was not possible to obtain the estimates of the parameters in closed form. As for the regime-switching model, the estimates were obtained by means of the Nelder–Mead algorithm.

Subject to the required constraints, the maximum-likelihood estimate of h is 0. This means that the model reduces to the basic model. It is therefore not considered further.

4.4 THE ARCH MODEL

The maximum-likelihood estimates of g and h are the same as those for the basic model. The likelihood function was obtained in closed form and the Nelder–Mead algorithm was used to determine the maximum-likelihood estimates of the parameters. The 95% confidence limits of the estimates were again determined by simulation.

Table 7 shows the results of the parameterisation of the ARCH model. Because the parameterisation is based on 26 observations instead of 27, the value of l has been multiplied by 27/26. This adjustment is not completely accurate, as the contribution of year 1 does not necessarily correspond to the average log-likelihood.

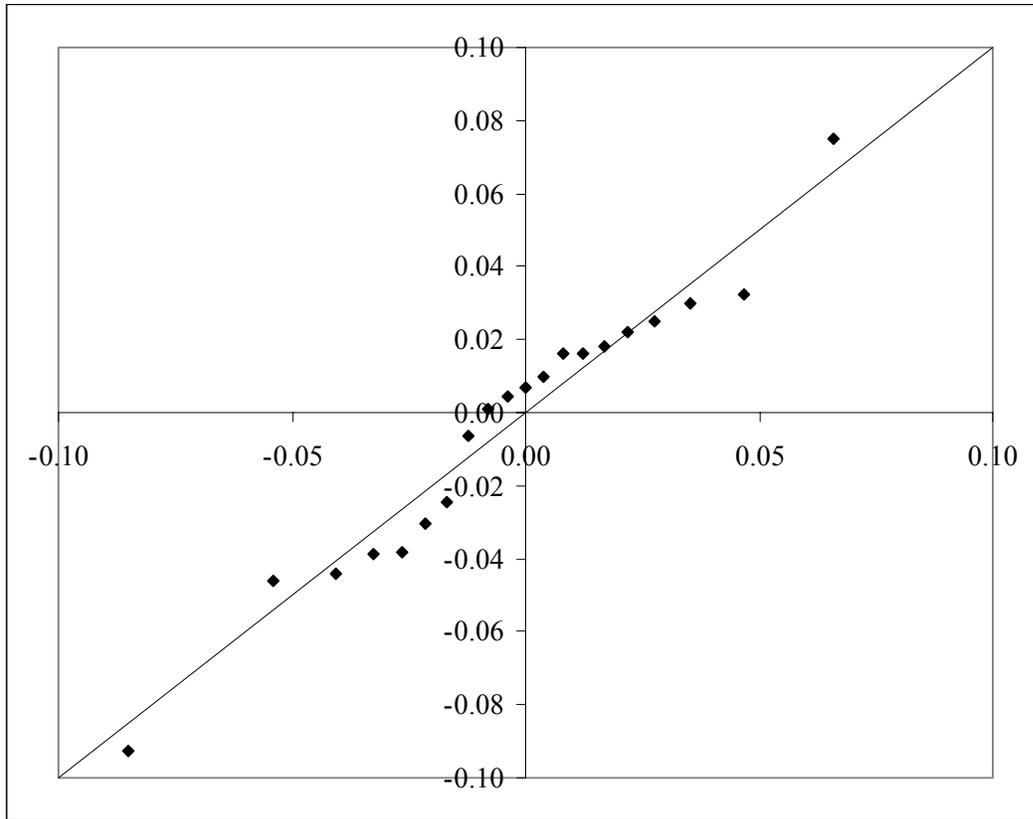


Figure 11. Regime-switching model, Q-Q plot regime 0, local maximum: $p_{10} = p_{01}$, $g_s = 1.2$

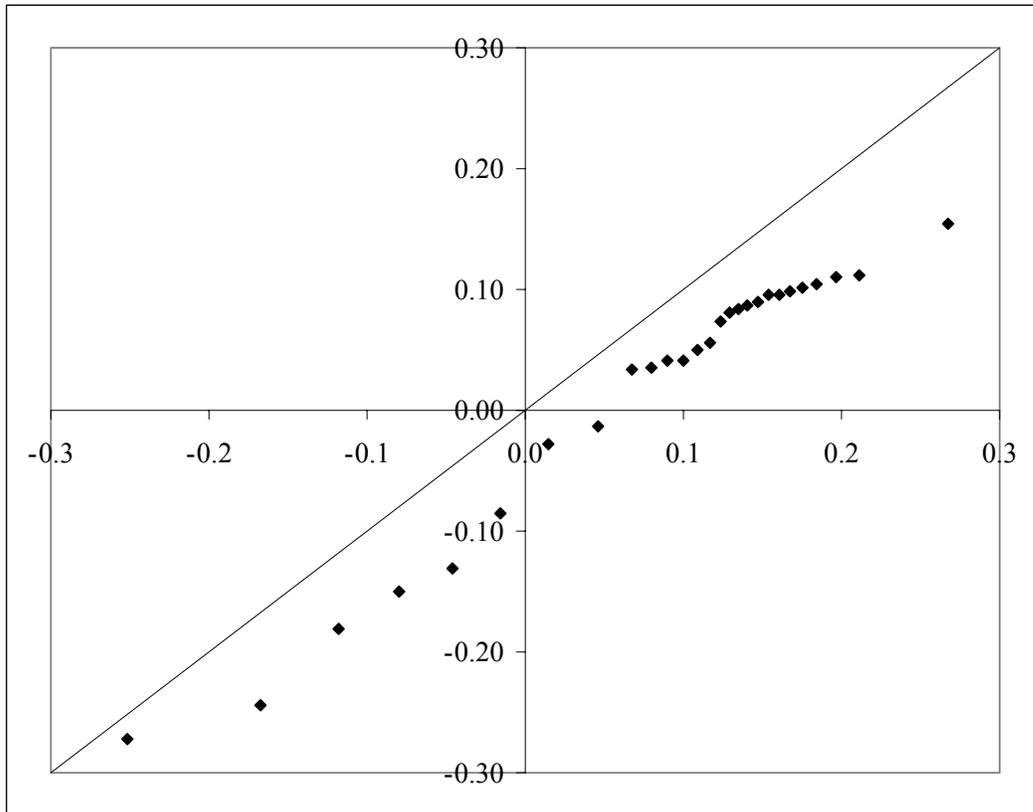


Figure 12. Regime-switching model, Q-Q plot regime 1, local maximum: $g_s = 1.2$

Table 7. Parameterisation of the ARCH model

Parameter	Details	
g	estimate	1.83
	confidence limits	1.00, 3.59
h		0.00
a	estimate	0.0053
	confidence limits	0.0018, 0.0093
b	estimate	0.759
	confidence limits	0.359, 1.158
k		3
l		22.34
A		-38.68
R		0.176
B		-0.003

It may be noted from Table 7 that the bias is the same as under the basic model. This is because, as noted above, the values of g and h are the same.

A Q-Q plot of the ARCH model was determined in the same manner as that of the basic model. The plot is shown in Figure 13. This reflects an improvement over the basic model.

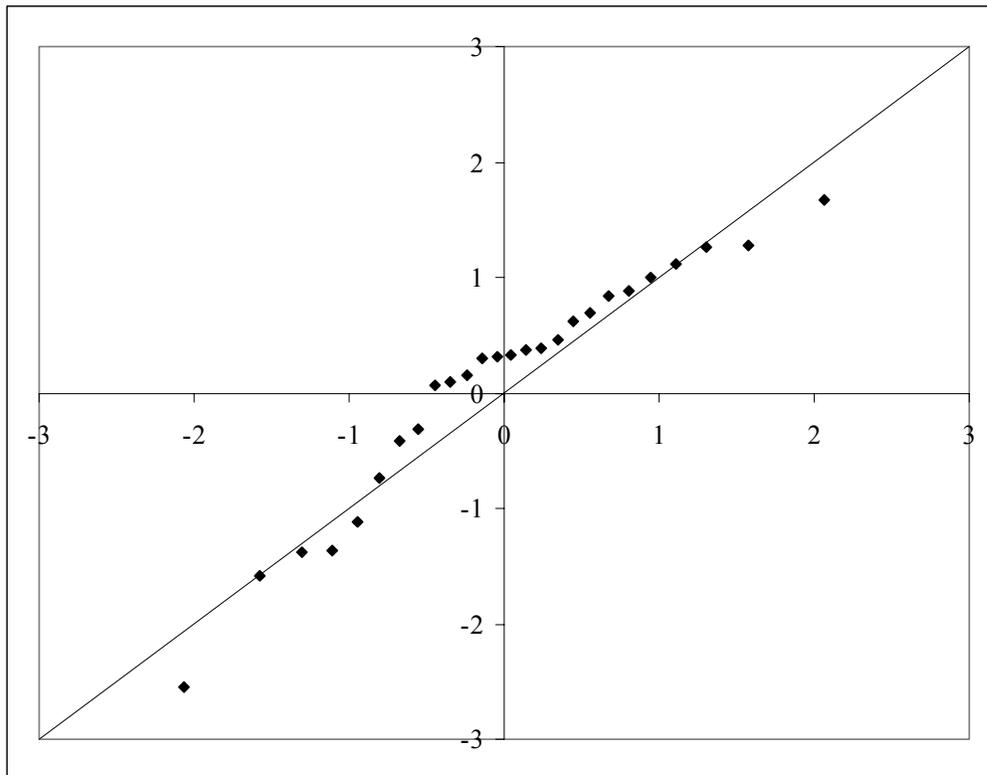


Figure 13. ARCH model Q-Q plot

4.5 SUMMARY

Table 8 summarises the selection criteria of the models. According to the Akaike information criterion, the regime-switching model is considerably superior to the other models. It does imply a substantial bias, though, and this is reflected in a relatively high market price of risk. As shown in Figures 4, 11, 12 and 13, the Q–Q plots of the respective models suggest that, while the assumption of a normal distribution is credible in the case of regime 0 in the regime-switching model, it is questionable in the case of the basic model. In the case of regime 1 in the regime-switching model the Q–Q plot reflects the bias referred to above. The Q–Q plot for the ARCH model reflects an improvement over the basic model.

Table 8. Summary of selection criteria

Criterion	Model		
	basic	regime-switching	ARCH
<i>A</i>	–37.08	–48.73	–38.68
<i>R</i>	0.226	0.802	0.176
<i>B</i>	–0.003	–0.028	–0.003

On the basis of Table 8, the regime-switching model is selected, but the issue of bias needs to be addressed. This is addressed in the next section.

5. THE USE OF THE MODEL FOR PREDICTIVE PURPOSES

As explained in section 1, the purpose of the development of the descriptive models in this paper is to derive ex-post estimates of ex-ante parameters. The reason for this is that, for predictive purposes, ex-ante parameters are required. As also explained there, it is envisaged that the predictive model will be parameterised so that the user will not be able to outperform the market on a risk-adjusted basis.

The question that arises in using the descriptive model to inform the development of a predictive model, is whether and to what extent the biases of the past will persist in the future. A different way of posing the question is: While the market did not follow the ex-ante estimates of the model in the past, is it reasonable to assume that it will do so in the future? The first presupposes that the ex-ante estimates were wrong, while the second presupposes that the sample observed was fortuitously different from the estimates. The question is particularly pertinent in relation to the regime-switching model, which implied a large bias.

The bias in the regime-switching model arises from regime 1, where the implied bias was –0.063.

In the first place, the assumption that the regime is known to market participants at the start of the year is for convenience only. In practice, that will not be true. On the other hand, it must be borne in mind that the information set available to market participants includes more than the history of the return on the market portfolio, so that market participants are likely to be more certain about the regime than the estimated ex-ante probabilities imply.

Secondly, to the extent that this model reflects information available to market participants, it may be assumed that they will have recognised the bias.

The net effect of these factors may be considered in terms of their effect on prices. Suppose that, at a particular time, in the absence of explicit information about the regime at that time, the price of the market portfolio is P . If it is known that the process is in regime 1, then the price of the market portfolio will be reduced to:

$$P_1 = P \frac{e^{\mu_{M1}}}{e^{\mu_M}};$$

where:

μ_M is the expected return on the market portfolio during a particular year in the absence of explicit information about the regime at time the start of that year; and

$\mu_{M|s}$ is the expected return on the market portfolio during a particular year if it is known at the start of that year that the process is in regime s during that year.

Similarly, if it is known that the process is in regime 0, then the price of the market portfolio will be increased to:

$$P_0 = P \frac{e^{\mu_{M0}}}{e^{\mu_M}}.$$

This means that the unconditional expected return during the preceding year is enhanced by:

$$\pi_0 \ln\left(\frac{P_0}{P}\right) + \pi_1 \ln\left(\frac{P_1}{P}\right) = \pi_0 (\mu_{M|0} - \mu_M) + \pi_1 (\mu_{M|1} - \mu_M).$$

In the absence of bias, this should be zero, so that:

$$\mu_{M|0} = \frac{\mu_M - \pi_1 \mu_{M|1}}{\pi_0}. \quad (11)$$

For the purposes of equation (11) an unbiased ex-ante estimate of μ is required. This is not necessarily best estimated by ex-post maximum likelihood. As mentioned in section 2, the problem of unbiased estimation of ex-ante expected returns has been addressed by numerous authors. Amongst these are the following estimates of the market price of risk on equities (all expressed as annual rates):

- Wilkie (1995a): 3% long-term median on U.K. equities;
- Derrig & Orr (2004): 4% to 5% on U.S. equities;
- Campbell (unpublished): 3.8% on world equities.

If we take the estimate for equity at 3.8%, this converts to an annual force of 3.7%. Risk premiums on bond returns are generally lower. With a beta of 0.5 and a weighting of bonds to equities of 0.2:0.8 (Thomson & Gott, unpublished b), the risk premium on the market portfolio may be taken at 3.3% a year.

From the results of the parameterisation, we have:

$$\mu_{M|1} = 0.034;$$

$$\begin{aligned} \mu_M &= \bar{\delta}_l + 0.033 \\ &= 0.031 + 0.033 \\ &= 0.064; \end{aligned}$$

$$\pi_0 = 0.55; \text{ and}$$

$$\pi_1 = 0.45;$$

so that:

$$\mu_{M|0} = 0.089.$$

Retaining $g_0 = 1.2$ and noting that the mean of the risk-free rate in regime 0 is:

$$\bar{\delta}_{l|0} = 0.034;$$

we have:

$$\mu_{M|0} = g_0 \bar{\delta}_{I|0} + h_0;$$

so that:

$$\begin{aligned} h_0 &= \mu_{M|0} - g_0 \bar{\delta}_{I|0} \\ &= 0.048. \end{aligned} \quad (12)$$

This is outside of the confidence limits of h_0 , suggesting that the model should be reparameterised with this value as a constraint. The results are shown in Table 9 as the ‘first iteration’.

Table 9. Reparameterisation of the regime-switching model: local maxima; $g_s = 1.2$;

Parameter	First iteration	Second iteration
p_{00}	0.69	0.70
p_{10}	0.44	0.43
g_0	1.2	1.2
h_0	0.048	0.043
σ_0	0.044	0.048
g_1	1.2	1.2
h_1	0.0	0.0
σ_1	0.152	0.152
k	4	4
l	25.28	24.48
A	-42.56	-40.96
R_0	1.238	1.030
R_1	0.039	0.039
R	0.333	0.302
B_0	0.022	0.025
B_1	-0.059	-0.057
B	-0.012	-0.009

Those results show changed values of the transition probabilities, which means that equation (9) will now give a different value of $\mu_{M|0}$. From the results of the reparameterisation, we have:

$$\mu_{M|1} = 0.036;$$

$$\begin{aligned} \mu_M &= \bar{\delta}_I + 0.033 \\ &= 0.031 + 0.033 \\ &= 0.064; \end{aligned}$$

$$\pi_0 = 0.59; \text{ and}$$

$$\pi_1 = 0.41;$$

so that:

$$\mu_{M|0} = 0.083.$$

Also:

$$\bar{\delta}_{t|0} = 0.032 ;$$

so that, from equation (12):

$$h_0 = 0.045 .$$

This gives the reparameterisation shown as the ‘second iteration’ in Table 9. No further iterations are required. By comparison with Table 8, it may be seen from the reparameterisation that, while the AIC is now considerably increased, it is still lower than for the basic and ARCH models. (If either of those models were to be pursued, it would be necessary to adjust them for predictive purposes too, which would increase their AICs too.)

Figures 14, 15 and 16 show the results.

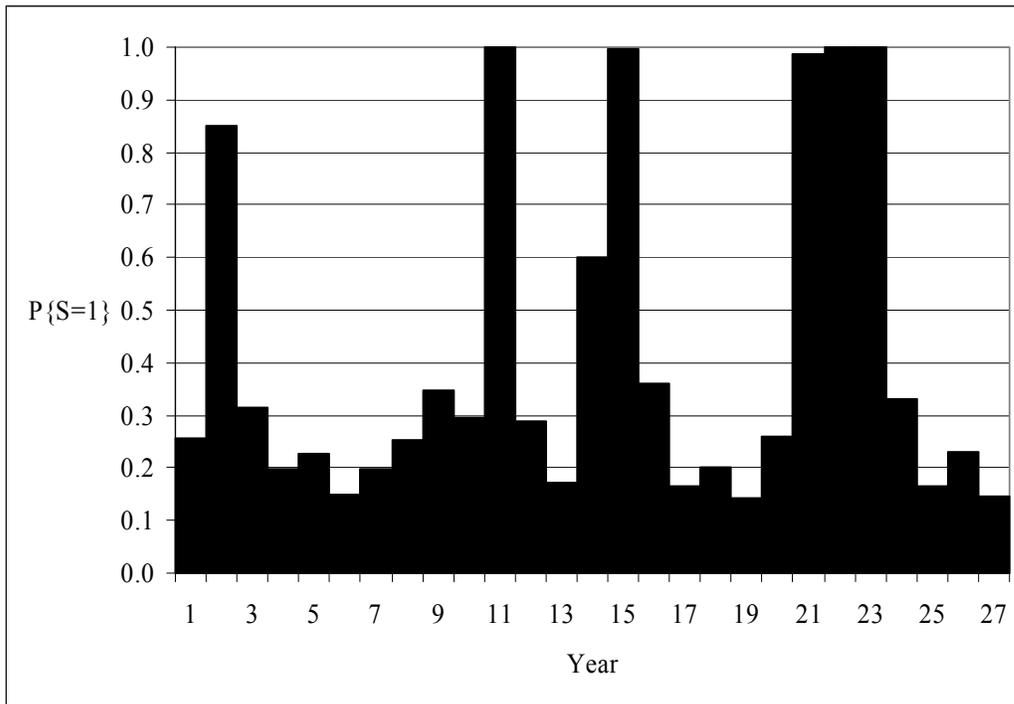


Figure 14. Ex-ante regime-switching model: $P\{S_t = 1 | \delta_{M,t}, \dots, \delta_{M,1}\}$

It may be noted from Figure 14 that the probability of regime 1 is now smaller and there are more years in which the probabilities are in the mid-range than in Figure 8. This is more in keeping with the usual intention behind regime-switching models discussed above.

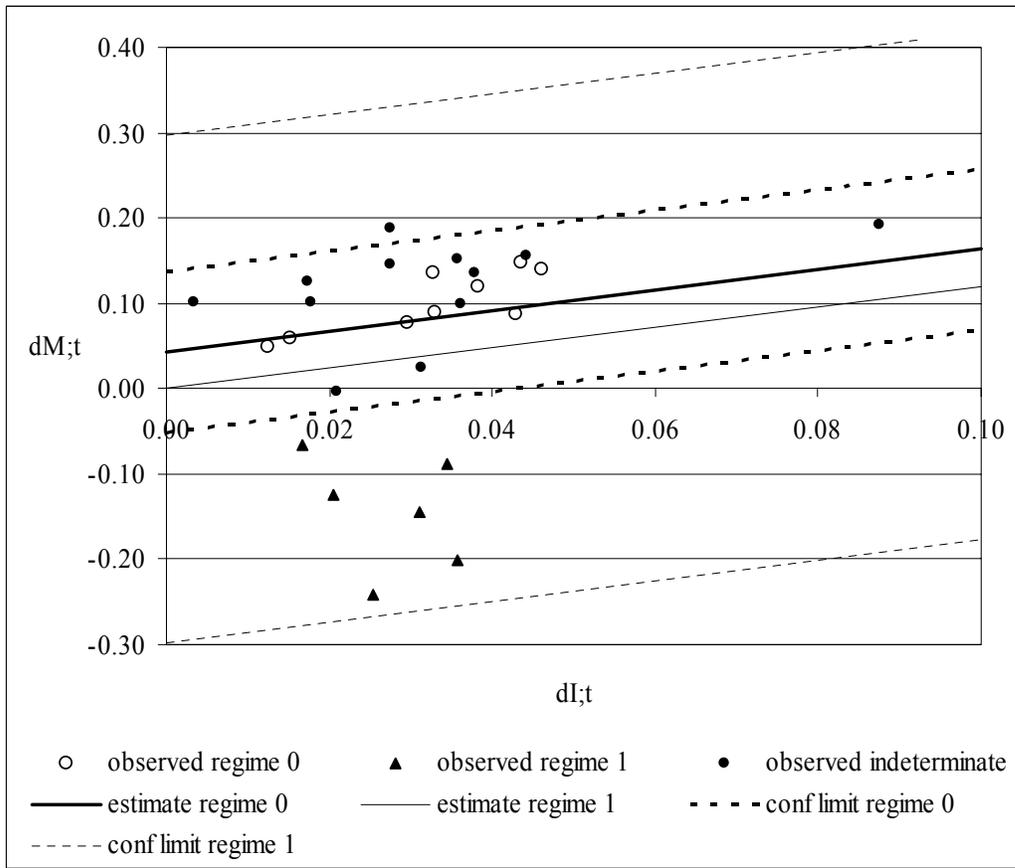


Figure 15. Ex-ante regime switching model: $\delta_{M,t}$ against $\delta_{I,t}$

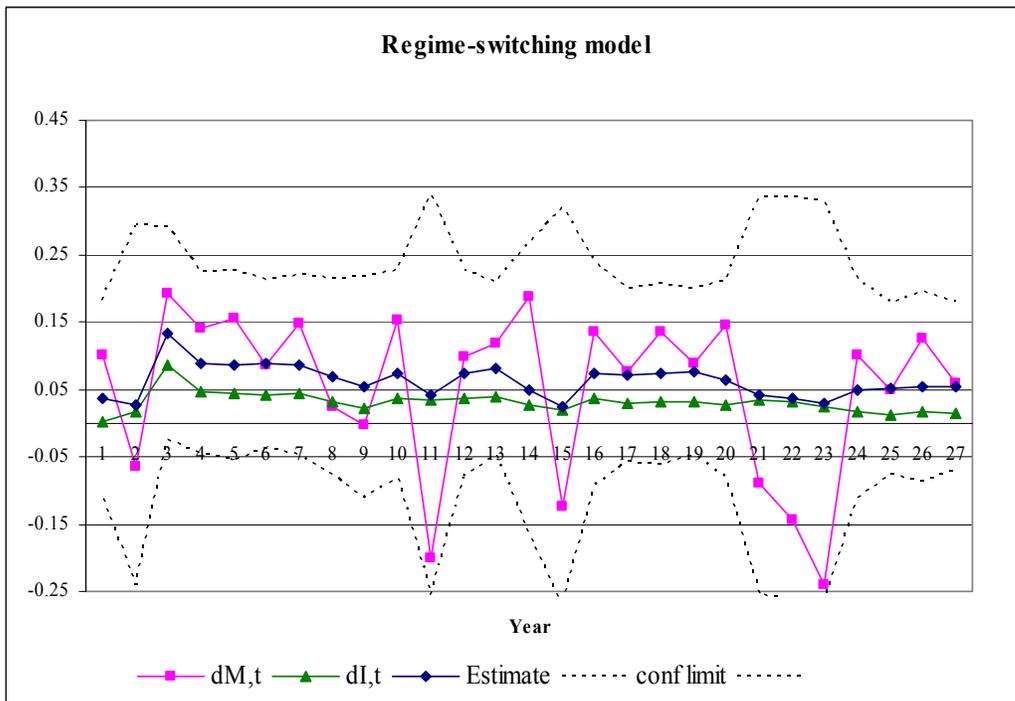


Figure 16. Ex-ante regime switching model: time series

Figure 15 shows, in comparison with Figure 9, the relatively wide confidence limits for regime 0 and identifies the indeterminate observations corresponding to posterior probabilities from 20% to 80%.

In comparison with Figure 10, Figure 16 shows the lower estimates and confidence limits that the ex-ante model produces in some years (these being the years in which the probability of regime 0 is greater).

6. SUMMARY AND CONCLUSION

In this paper, the development of descriptive models of the U.K. market portfolio is described. The models have the attributes specified in section 1. The best of those models was found to be a regime-switching model.

For the purposes of predictive modelling, however, it was decided to use ex-ante estimates of expected returns. This implied bias in the observed mean returns, which negates the REH. In the light of the literature on the subject, this is considered acceptable for these purposes.

The model is defined as:

$$\delta_{M;t} = g\delta_{I;t} + h_{S_t} + \sigma_{S_t}\varepsilon_t; \quad (\text{equation (3)})$$

where:

$$S_t \in \{0,1\};$$

$$\Pr\{S_t = 0 \mid S_{t-1} = 0\} = p_{00} = 0.70;$$

$$\Pr\{S_t = 0 \mid S_{t-1} = 1\} = p_{10} = 0.43;$$

$$g = 1.2;$$

$$h_0 = 0.043;$$

$$h_1 = 0;$$

$$\sigma_0 = 0.048;$$

$$\sigma_1 = 0.152;$$

and $\varepsilon_t \sim N(0,1)$ is serially independent

The model is suitable for use with the equilibrium model developed in Thomson & Gott (unpublished b) and in the pricing of the liabilities in an incomplete market as proposed by Thomson (2005).

As time goes by it will be necessary to revisit the parameterisation of the model. Code in the programming language R is available free of charge from the author for that purpose.

ACKNOWLEDGEMENTS

The assistance of Mr D.V. Gott, particularly with regard to the data used, is acknowledged. The financial assistance of the Actuarial Society of South Africa is acknowledged. Opinions expressed and conclusions drawn are those of the author and are not to be attributed to the Society.

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