Double risks portfolio optimization problem for pension funds

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Abstract

Well known within Solvency is a preference on the measures of risk for the measures VaR, nevertheless, as shown by Artzner [1] et al. there are some bad qualities of VaR such as non subadditivity. The asset allocation for pension funds is done according to a mean-VaR portfolio optimisation.

Beside this, there are other and additional risks which an asset manager has to consider. Such risks are, that the rate of return of the pension fund is below a priori guarantied rate of return or that the rate of return is not better then the average of the competitors.

We show that one has to apply additional measures and that optimization only by using the VaR will increase these others risks.

1. Introduction

The standard deviation of the return of portfolio is predominant measure of risk. Indeed mean-variance portfolio selection using quadratic optimization, introduced by Markowitz [20], is the standard.

It is well known that the mean-variance portfolio selection paradigm maximizes the expected utility of an investor if the utility is quadratic or if returns a jointly normal, or more generally, obey an elliptically symmetric distribution. It has long been recognized, that there are several conceptual difficualties with using standard deviation.

Alternative downside-risk measure have been proposed and analyzed in the literature.

The general idea of downside risk measures has been discussed in the financial economics literatur starting with the discussion of semi-variance (see Roy [20], Markowitz[20].) A more general notion of lower patial moment (LPM) as a measure of risk was introduced (see Bawa [3], [4], Fishborn [11], Harlow and Rao [13].)

In Recent years it was extensively used quantile-based downside risk measures. Value-at- Risk, or VaR has been increasingly used as risk management tool (see Jorion[16], Dowd[8], Duffie and Pan [9]). VaR measures the worst losses wich can be expected with certain probability. It does not address how large these losses can be expected when the small probability events (“bad events”) occur. To address this issue, the “mean excess function”, from extreme value theory, can be used (Embrechts, McNeil, and Straumann [10], Artzner at al. [1]).

A version of the mean excess function is tail conditional expectation (TailVaR) (see Uryasev und Rockafellar [24].)

The other version of the mean excess function is shortfall (see Dimitris Bertsimas, Geoffrey J. Lauprete and Alexander Samarov [6].)

The structural and mathematical properties and relations to other risk measures have been discussed in Uryasev und Rockafellar [24] und Tasche [23].

We want to propose a methodology for defining, measuring, analyzing, and optimizing double risks, namely downside- risks and the risk, that the rate of return is not better than the average rate of return of competitors (or that the rate of return is below a priori guarantied rate of return.)

2. Downside-risk and the risk, that the rate of return is below a priori guarantied rate of return.

Let r_0 be a priori guarantied rate of return, X be a random variable, $F(x) = Prob(X \leq x)$ the distribution funktion of X and $E = E[X]$ the expected value of X.

Consider an arbitrary probability p within the interval $0 < p < 1$ and let u denote an x- value for which $F(u) = p$.

Such an u is called a p- quantile of the distribution of X (note that is assumed to be continuous at u,), or $u = \inf\{ x \mid Prob(X \leq x) \geq p\}$.

The expected values of X can be split into

$$E[X] = p \cdot E[X \mid X \leq u] + (1 - p) \cdot E[X \mid X > u].$$

Def.1:

$$M(x) = Prob(X \leq x) * E[X \mid X \leq x]$$

is the distribution function of E[X]

The value of the risk, that the rate of return is below a priori guarantied rate of return is equal to the expected value $E[X \mid X \leq r_0]$ and $Prob(X \leq r_0)$.

For this, it is reasonable to find that random variable or process containing random variables with minimal

$$E[X \mid X \leq r_0]$$

and

$$Prob(X \leq r_0)$$

respectively $M(r_0)$ from the set of usable or constructabel processes respectively random variable with

$$E[X]- M(r_0) \geq C_1 \text{ } (C_1 = \text{Constant.})$$

A simple result for the valuation of downside-risk is the largest possible value of $E[X \mid X \leq u]$.

If not u, but as often done only the value of the probability p is defined a priori, one can find that random variable or process with maximal u or maximal $E[X \mid X \leq u]$.

Also one can find M(u) from the set of allowed or construable processes with

$$E[X] \geq C_2 \text{ } (C_2 = \text{Constant.}).$$

The optimization is to find instead of maximal u or $E[X \mid X \leq u]$,

$$\text{minimal } E[X]- u$$

or

$$E[X]-E[X \mid X \leq u].$$

Problem:

It is obvious, that an optimization with respect to minimize the downside- risk can effect an increase of the risk, that the rate of return is below a priori guarantied rate of return.

And vice versa an optimization with minimization of the risk, that the rate of return is below a priori guarantied rate of return can result in, that the downside-risk is not optimized.

We will now discuss several solutions of this problem of optimization with respect to two measures of risk.

Solution 1:

We suggest to minimize

$$r_0 - \mathrm{u} + \mathrm{Prob}(\mathrm{X} \le r_0)$$

With the constraint, that

$$r_0 > \mathrm{u}.$$

Weighting of risks we obtain

$$a(r_0 - \mathrm{u}) + (1 - a)\,\mathrm{Prob}(\mathrm{X} \le \mathrm{r}_0),$$

$$0 \le a \le 1.$$

This solution is very simple, but because on well known reasons not being applied.

Solution 2:

We suggest to minimize:

$$\mathrm{E[X]} - \mathrm{E}[\mathrm{X} \mid \mathrm{X} \le \mathrm{u}] + \mathrm{E}[\mathrm{X} \mid \; < \mathrm{X} < \mathrm{r}_0],$$

or to minimize

$$\mathrm{E[X]} - \mathrm{M(u)} + \mathrm{M}(r_0)),$$

with the constraint that

$$r_0 > \mathrm{u}.$$

Weighting of risks we obtain

$$a\,(\mathrm{E[X]} - \mathrm{E}[\mathrm{X} \mid \mathrm{X} \le \mathrm{u}]) + (1 - a)\,\mathrm{E}[\mathrm{X} \mid \; < \mathrm{X} < r_0],\ 0 \le a \le 1.$$

3. Downside- risk and the risk, that the rate of return is not better than average of the competitors

The estimation of the risk, that the rate of return is not better than the average of the competitors is similar complicated. Let assume the average of the competitors with random variable Y. We suppose that the average of the competitors and $\mathrm{E}[Y] = r_0$.

A simple solution can be obtained for $\mathrm{E}[Y] = r_0$.

And using solution 2. If we don't want this solution, we can do

Solution 3:

Minimize of

$$\mathrm{E[X]} - \mathrm{E}[\mathrm{X} \mid \mathrm{X} \le \mathrm{u}] + \mathrm{E}[\mathrm{E}[\mathrm{X} \mid \; < \mathrm{X} < Y] \mid u_Y < Y],$$

$$u_Y = \text{constant}.$$

For weighting of risks we obtain:

$$a\,(\mathrm{E}[X]-\mathrm{E}[X \mid X \le u]) + (1-a)\,\mathrm{E}[\mathrm{E}[X \mid\ < X < Y] \mid\ u_Y < Y],$$

$$0 \le a \le 1.$$

4. Formulation of Risk Models for Downside- risk and the risk, that the rate of return is below a priori guarantied rate of return.

Let us assume that

$$\boldsymbol{R} = (R_1, \ldots, R_n)$$

is distributed over a finite set of points

$$r_t = (r_{t1}, \ldots, r_{tn}),\ t = 1,2,\ldots,T.$$

These are obtained either directly be historical data or be simulation based upon some sort of stochastic models about *R*.

Let p_t = 1/T be the probability that

$$\boldsymbol{R} \text{ attains } r_t,\ t = 1, 2, \ldots, T,$$

$$\mathrm{F}(u) = \mathrm{Prob}(X \le u) = 1/\mathrm{T}.$$

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be portfolio weights, so that

$$\sum_{j=1,\ldots n} x_j = 1.$$

The **X = R'x** is total random return of the portfolio.

Model 1.:

minimize the following:

$$r_0 - \max_{x\in\Omega} [\min_{t=1,\ldots,T} [\sum_{j=1,\ldots n} x_j * r_{jt}]] + \sum_{t\in T} 1/T$$

subject to

$$\boldsymbol{T} = \{t \mid \sum_{j=1,\ldots n} x_j * r_{jt} \le r_0;\ \boldsymbol{x} \text{ belong } \boldsymbol{\Omega}.\},$$

$$\boldsymbol{\Omega} = \{x \mid \sum_{j=1,\ldots n} x_j = 1.\},$$

For this Model, we follow Solution 1: minimize

$$r_0 - u + \mathrm{Prob}(X \le r_0).$$

Model 2.:

minimize

$$\sum_{t=1,\ldots,T} \sum_{j=1,\ldots,n} x_j * r_{jt} - \mathbf{1/T}* \max_{x\in\Omega} [\min_{t=1,\ldots,T} [\sum_{j=1,\ldots n} x_j * r_{jt}]] + \mathbf{1/T}* \sum_{t\in T} \sum_{j=1,\ldots,n} x_j * r_{jt}$$

subject to

$$\boldsymbol{T} = \{ t \mid \sum_{j=1,\dots n} x_j * r_{jt} \le r_0 ;\ \boldsymbol{x} \text{ belong } \boldsymbol{\Omega} . \},$$

$$\boldsymbol{\Omega} = \{ x \mid \sum_{j=1,\dots n} x_j = 1. \},$$

For this Model, we follow Solution 2:

Minimize

$$E[X] - E[X \mid X \le u] + E[X \mid < X < r_0].$$

5. Formulation of Risk Models for Downside-risk and the risk, that the rate of return is not better the average of the competitors

Let us assume that ***Y*** is distributed over a finite set of points

$$y = (y_1, y_2, \dots, y_m).$$

These are obtaind either directly be historical data or be simulation based upon some sort of stochastic models about ***Y***.

<u>**Model 3.:**</u>

Minimize

$$\sum_{t=1,\dots,T} \sum_{j=1,\dots,n} x_j * r_{jt} \; \mathbf{- 1/T*} \max_{x \in \Omega} \; [\min_{t=1,\dots,T} [\sum_{j=1,\dots n} x_j * r_{jt}]] \; \mathbf{+}$$

$$\mathbf{+\ 1/T*} \sum_{\substack{y_i \le u_y; \\ i=1,\dots,m}} [\sum_{t \in T(y_i)} \sum_{j=1,\dots,n} x_j * r_{jt}] \mathbf{*p(Y=} y_i \mathbf{))}$$

subject to

$$\boldsymbol{T(y_i) = \{ t \mid} \sum_{j=1,\dots n} x_j * r_{jt} \le y_i \boldsymbol{;\ x \text{ belong } \Omega . \},}$$

$$\boldsymbol{\Omega = \{ x \mid} \sum_{j=1,\dots n} x_j \boldsymbol{= 1. \},}$$

For this Model, we follow Solution 3:

Minimize

$$\mathbf{E[X] - E[X \mid X \le u] + E[E[X \mid < X < \mathit{Y}] \mid} \; u_Y \; \mathbf{< \mathit{Y} \,]}$$

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