OPTIMAL TRADE CREDIT REINSURANCE PROGRAMS WITH SOLVENCY REQUIREMENTS

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ABSTRACT
In this paper we study the design of the optimal strategy of reinsurance for a particular line of heavy-tailed non-life type of insurance, namely trade credit insurance. Optimality is achieved by maximizing shareholders expected rate of return over a reisurance program composed of a proportional quota share treaty and a complex multi-layer excess-of-loss treaty with multiple reinstatements.

The work participates to the debate on insurance firm risk management within the Solvency II framework by considering, in particular, the constraint imposed by the solvency capital requirement and by assuming, both for the insurer and the reinsurer, a pricing principle based on a cost-of-capital approach.

Due to the nature of this type of insurance, we use a loss model that takes into account correlations between defaults. To cope with the complex payoff of the non-proportional treaty, the computation of the loss distribution is performed within a Monte Carlo framework equipped with an additional importance sampling variance reduction technique. Finally, we discuss numerical results obtained for a realistic credit insurance portfolio.

KEYWORDS
Solvency, reinsurance, excess-of-loss, credit insurance.

1 Introduction

In this paper we study the design of the optimal strategy of reinsurance for a particular line of heavy-tailed non-life type of insurance, namely trade credit insurance. The reinsurance program is formed by the combination of a simple proportional treaty (*quota share* reinsurance) and a complex multi-layer *excess-of-loss* reinsurance treaty with multiple reinstatements. This combination is often present in the practice of credit insurance as an alternative to the combination of a proportional treaty and a *stop-loss* treaty. While proportional reinsurance is generally conceived to reduce the capital requirement to levels judged acceptable by the insurance firm management, the second type of treaties (either excess-of-loss or stop-loss) is designed to protect the insurer against extreme ("catastrophic") losses. In this work, similarly to [18] and [29], the optimality of the reinsurance program will be inspired to the principle of economic capital allocation from the shareholders point of view, taking into account the constraint imposed by the solvency requirement. Differently from past works, a new cost-of-capital inspired pricing principle will be adopted.

The investigation is inspired by the debate on insurance firm risk management, and in particular by the guidelines of the solvency capital requirements assessed by the Solvency II framework [6].

Trade credit insurance protects sellers from the risk of a buyer nonpayment, either due to commercial or political risks. Given the nature of the risks involved, the measurement of risk capital requires the use of a model accounting for the presence of correlations between defaults, that produce heavy tails in the loss distribution. In this sense modeling issues are similar to those involved in the development of the Basel II framework for the adequacy of banking capital requirements [2].

Among the large variety of credit risk models accounting for correlations in default, either explicitly or implicitly (for a review common references are *e.g.* [9] and [26]) the model used in our analysis belongs the class of the so-called factor models, *i.e.* those models based on conditional independence between defaults. One of the main advantages of the model is that it allows the calculation of the distribution of the insurance company aggregate loss in a fast semi-analytical way by numerical inversion of the probability generating function (PGF) of the loss distribution.

When the complex payoff of excess-of-loss contracts is taken into account the semi-analytical tractability of the model is spoiled, so that the computation of the loss distribution has to be performed within a Monte Carlo framework. In principle this is not a problem, but there could be computational issues when the confidence levels required in the valuation are large, as those used in the calculation of risk capitals and solvency margins. For example in the Solvency II framework the solvency capital requirement shall correspond to the 99.5% confidence level Value-at-Risk over one year period (see [6], art. 100), while it is common practice for insurance companies to use a confidence level defined of the basis of their credit reputation, which may easily require values of the confidence level larger than 99.9%.

To decrease computing time we improve on "plain" Monte Carlo results by the use of an importance sampling technique, originally proposed by Glasserman and Li [21], tailored for the application of the model. Timing performance are relevant in the process of optimizing the large number of parameters of the reinsurance treaties, since the solution of the optimization problem will also be found numerically, with a well established [25, 28] random search technique known as *simulated annealing*.

In the existing literature the valuation of excess-of-loss reinsurance treaties has been considered by several other authors, in particular by Sundt [27] and, more recently, by Mata [19], Wahlin and Paris [30] and Hürlimann [13]. However, all these authors have adopted an approach based on the collective model of risk theory (see *e.g.* [16] p. 45) which is based on a series of hypotheses the most important being the assumption that claims should be independent and identically distributed.

The rest of paper is organized as follows: in the first section we introduce trade credit insurance and the computation of the aggregate loss gross of reinsurance; we then discuss the reinsurance contracts, we formulate the optimization problem, describe the model used for the loss computation and finally provide a realistic numerical example.

2 Loss distribution for trade insurance

In the following we consider a trade credit insurance company over a single period $[0, T]$. In order to follow Solvency II indications and without loss of generality, we will later assume $T = 1$ year. At time $t = 0$ the company holds a portfolio $\mathcal{P}$ of polices protecting the insured parties against the risk of losses they may suffer due to default of contracts with deferred payments. The nature of the event determining the default is contractually specified and can encompass commercial events as well as political events or natural disasters. Thus, the company is exposed to the default of the N customers, hereafter buyers, of the insured parties, where N is typically very large.

Trade credit companies generally offer coverage either for single buyer or for the whole set of buyers trading with the insured party during the coverage period, even if they are still unknown at the moment the policy is issued. Since insurance conditions (premium rate, eventual riders such as benefit participation, etc.) have to be contractually specified *ex-ante*, trade insurance companies maintain the right to cap dynamically the amount granted in case of the default of each buyer to a maximum value, $M_i(t)$, with $t \in [0, T]$ and $i = 1, \ldots, N$. Therefore, unlike a conventional loan or bond, for which there is a fixed face amount that cannot be changed once the loan has been made or the bond purchased, credit insurance is based on limits of coverage that might be varied as risk perception changes. For the same reason one expects the value a single name trade insurance policy to be smaller than the otherwise financially equivalent credit default swap contract.

We define $E_i(t)$ as the exposure of the company towards the default of the i-th buyer, by subtracting from $M_i(t)$ the fraction of the loss retained by the insured. In practice, this is done by multiplying $M_i(t)$ by the (contractually specified) coverage fraction ρ_i

$$E_i(t) := \rho_i \, M_i(t). \tag{1}$$

Typical values of ρ_i ranges from 80% to 90% and generally depend on the geographical area and economic sector to which the buyer belongs.

For simplicity sake we shall further assume that the buyers referring to one given policy are distinct from those referring to any other policy. If this were not the case than $E_i(t)$ should have been understood as the aggregate exposure at time t. Moreover, we assume that default is an absorbing state for the reference period $[0, T]$, so that a buyer cannot default more than once. Let Y_i be the buyers default indicators over the reference time period and τ_i the corresponding default time, that is

$$Y_i = \mathbb{1}_{\{\tau_i \leq T\}} = \begin{cases} 1 & \text{if} \quad \tau_i \leq T; \\ 0 & \text{otherwise}; \end{cases} \tag{2}$$

and let $c_i \in \Big(0, E_i(\tau_i^-)\Big]$ be a random variable representing the claim amount arising from the insolvency of buyer i for whom there was a pre-default maximum credit allowance $E_i(\tau_i^-)$; the total loss suffered by the company over the reference period is then

$$L = L[0, T] = \sum_{i=1}^{N} c_i Y_i. \tag{3}$$

As pointed out by the Credit & Surety PML Working Group [5] while for loans and bonds it is possible to define a loss severity rate, a similar quantity does not exist in trade credit insurance since the exposure is changed dynamically. Possibly, a reference indicator might be provided by the random variable $c_i/E_i(\tau_i^- - 1)$, with time measured in years. However, the are no publicly available statistics for $c_i/E_i(\tau_i^- - 1)$ or similar ratios. On the European market, typical values for the loss severity ranges between 55% and 70% (see *e.g.* [31], Exhibit 15). Later on we shall assume that

$$c_i = E_i(0) \quad i = 1, \dots N. \tag{4}$$

Let now $F(x), x \geq 0$ be the distribution function of L defined in the probability space $(\Omega, \mathbf{P}, \mathcal{F})$. In insurance practice companies often employ internal models for which there is no closed-form expression for $F(x)$, that –therefore– has to be determined in a numerical way. Similarly, we shall use in our numerical example a realistic loss model able to account for the granularity of the insured portfolio $\mathcal{P}$. This model will later be described in section 5.

From $F(x)$ it is then possible to compute the statistics of the distribution used as indicators of risk. These encompass moment-based variables, such as

a) the expected loss $EL = \mathbf{E}[L]$;

b) the variance $V(L) = \mathbf{V}[L]$;

c) the standard deviation $\mathbf{SD}(L) = \mathbf{V}[L]^{1/2}$;

and quantile-based variables such as

d) the *Value-at-Risk* VaR_α at the confidence level α, with $\alpha \in (0, 1)$,

$$\mathrm{VaR}_\alpha = \inf\{\ell \in \mathbb{R}, \mathbf{P}\Big[L > \ell\Big] \leq 1 - \alpha\} = \inf\{\ell \in \mathbb{R}, \mathbf{P}\Big[L \leq \ell\Big] \geq \alpha\}; \tag{5}$$

e) the *expected shortfall* at confidence level α (ES_α), defined as in [10],

$$ES_\alpha = \mathbf{E}[L|L \geq \text{VaR}_\alpha] = \frac{1}{1-\alpha}\mathbf{E}[L\, \mathbb{1}_{\{L \geq \text{VaR}_\alpha\}}], \tag{6}$$

f) the *generalized expected shortfall* at confidence level α (GES_α)

$$GES_\alpha = \frac{1}{1-\alpha}\Big\{\mathbf{E}[L\, \mathbb{1}_{\{L \geq \text{VaR}_\alpha\}}] + \text{VaR}_\alpha(L)\Big(1 - \alpha - \mathbf{P}[L \geq \text{VaR}_\alpha(L)]\Big)\Big\}, \tag{7}$$

which is the most popular coherent risk measure in the sense of Artzner *et al.* [1];

g) the risk capital at confidence level α

$$RC_\alpha(L) = \text{VaR}_\alpha(L) - EL(L). \tag{8}$$

Remark. Notice that $\text{VaR}_\alpha(L)$ is simply the α-quantile of the loss distribution

$$\text{VaR}_\alpha(L) = F^{\leftarrow}(\alpha) := \inf\{\ell \in \mathbb{R}_+ : F_L(\ell) \geq \alpha\} \tag{9}$$

where for continuous distributions the generalized inverse function $F^{\leftarrow}(x)$ (see, *e*.g., [11]) coincides with the ordinary inverse $F^{-1}(x)$. ∎

Remark. As already stated, $\text{VaR}_\alpha(L)$ is not a coherent measure of risk; in particular it is not sub-additive, so that the relation

$$\text{VaR}_\alpha(L_1 + L_2) \leq \text{VaR}_\alpha(L_1) + \text{VaR}_\alpha(L_2)$$

is not automatically satisfied for all distribution of aggregate loss $L = L_1 + L_2$ $(L_1, L_2 > 0)$. As a consequence, also the risk capital $RC_\alpha(L)$ is not sub-additive. ∎

3 Reinsurance

Classical reinsurance can be classified into two types: the proportional and the non-proportional. These two types are often combined in what is defined a *reinsurance program*. Real reinsurance programs might be fairly complex. In the following we shall study a program where the proportional treaty is extremely simple (pure “quota share”) while the non-proportional part is more similar to market practice.

3.1 Proportional reinsurance

In this case all the characteristics of each risk (exposures, claims, premia, etc.) are proportionally shared between insurer and reinsurer, with the fraction of cession per risk equal to

λ_i, ($\lambda_i \in [0,1]$) and consequentially a retention per risk equal to $1-\lambda_i$. In market practice, there exist two forms of proportional insurance: a "strictly" proportional type of *quota share* reinsurance for which $\lambda_i = \lambda$ $(i = 1, \ldots N)$, and a *surplus* type of reinsurance where the claims are redistributed in such a way that

$$\lambda_i = \frac{\max\{c_i - \bar{c}_i, 0\}}{c_i} = \begin{cases} 0 & \text{if} \quad c_i < \bar{c}_i \\ \dfrac{c_i - \bar{c}_i}{c_i} & \text{if} \quad c_i > \bar{c}_i \end{cases}$$

where $\bar{c}_i > 0$ is the so-called "line" amount of retained loss by the cedant. In is also common practice of reinsurers to deviate from the proportionality rule by applying a *sliding scale* commission, that is by increasing the reinsurance premium with increasing suffered losses. Noticeably, the problem of the optimal program for proportional reinsurance was addressed by de Finetti [7] already in 1940 using a mean-variance approach.

3.2 Non-proportional reinsurance

In this case the insurer seeks cover for claims judged "too high". Non-proportional forms of insurance include (see, *e.g.*, [20])

1. *stop-loss* reinsurance where the reinsured will retain aggregate losses up to a maximum amount $\bar{C}$, so that the net loss is

$$L_{sl} = L - \max\{L - \bar{D}, 0\} = \min\{L, \bar{D}\}$$

   where $\bar{D} > 0$ is called the "retention level" of the ceding company;

2. *largest claims* reinsurance where the reinsurer guarantees, at time $t = 0$, that the k largest claims originated by the portfolio $\mathcal{P}$ will be covered;

3. *excess-of-loss* reinsurance where the reinsured retains only a fraction of each claim according to

$$L_{xl} = L - \sum_{i=1}^{N} Y_i \max\{c_i - \bar{d}, 0\} = \sum_{i=1}^{N} Y_i \min\{c_i, \bar{d}\}$$

   where $\bar{d} > 0$ is called – in different insurance branches – the "priority" or the "deductible" .

3.3 The reinsurance program

In this work we consider a trade credit reinsurance program composed by a quota share treaty followed by a multi-layer excess-of-loss treaty with multiple reinstatements. The latter are treaties formed by J protection layers, each defined by the *limit* m_j, the *deductible* ℓ_j, and by a number of *reinstatements* r_j $(j = 1, \ldots J)$. It is customary to refer to the layers with the short notation m_j xs ℓ_j. For each claim amount c_i the layer j will provide a coverage $Z_i^{(j)}$ defined as

$$Z_i^{(j)} = \min\Big\{\max\{c_i - \ell_j, 0\}, m_j\Big\}, \tag{10}$$

as far as the aggregate loss $X_j = \sum_{i=1}^n Z_i^{(j)}$ is smaller than $(r_j+1)m_j$. The contract might also include an aggregate deductible L_j, so that the effective coverage C_j provided by each layer is

$$C_j = \min\Big\{\max\{X_j - L_j, 0\}, (r_j+1)\, m_j\Big\}. \tag{11}$$

For a fixed number J of adjacent layers, *i.e.* layers for which $\ell_{j+1} = \ell_j + m_j$ $(j = 1, \dots J - 1)$, the parameters characterizing the treaty can be grouped in a $(2J+1)$-dimensional vector $\boldsymbol{\nu}$ of components

$$\boldsymbol{\nu} = \{\ell_1, m_1, \dots m_J, r_1, \dots r_J\}$$

The loss W suffered by the re insurer and the loss L_{XL} suffered by the cedant are respectively

$$\begin{aligned} W &= \textstyle\sum_{j=1}^J C_j, \\ L_{XL} &= \textstyle\sum_{i=1}^n c_i\, Y_i - W. \end{aligned} \tag{12}$$

Remark: the layer are reinstated in the sense that every time a claim hits a layer there is an extra premium charged to the cedant at a pre-determined rate φ_j, usually pro rata to the claim size. If $\Pi_0^{(j)}$ is the initial premium paid by the cedant for the layer j, the extra premium $\Pi_i^{(j)}$ is given by

$$\Pi_i^{(j)} = \varphi_j \frac{Z_i^{(j)}}{m_j} \Pi_0^{(j)} \quad i = 1, \dots n. \tag{13}$$

Typical values of the rates φ_j are 100 % and 50%, while for $\varphi_j = 0$ one speaks of free reinstatements. When $r_j \geq 1$ the total premium paid by the cedant to the reinsurer for the j-th layer is therefore a random variable

$$\Pi^{(j)} = \Pi_0^{(j)} \left(1 + \frac{\varphi_j}{m_j} \sum_{k=1}^{r_j} C_j^{(k)}\right), \tag{14}$$

with

$$C_j^{(k)} = \min\Big\{\max\{X_j - L_j - k\, m_j, 0\}, m_j\Big\}. \tag{15}$$

■

Remark: Notice that the non-proportional form of reinsurance can be mapped to financial credit derivatives (see, *e.g.* [3]). In this sense stop-loss reinsurance can be exactly replicated by writing a call option on the loss generated by portfolio $\mathcal{P}$ while largest-claim reinsurance can be replicated by a basket of first-to-default and n-to-default swaps. Similarly, the payoff structure of the excess-of-loss treaties considered here finds a counterpart in the structure of collateralized debt obligation (CDO's), albeit for the absence of the reinstatement mechanism in the latter. ■

4 Optimization strategy

Similarly to the works of [18] and [29] we consider the problem of optimizing the reinsurance program as a problem of efficient allocation of the shareholders capital. Thus, in our approach reinsurance is not expected to minimize ruin probability, but only to maintain risks below a target level, defined – at least – by the solvency requirement. Differently from the above mentioned works we consider a pricing principle based on the cost of capital, under which it will turn out that it might be difficult for the cedant to obtain a benefit from reinsurance in terms of expected rate of return on the economic capital of the shareholder.

The rate of return the shareholders expect on their capital depends both on the net operating profit (after taxes) and the constraints fixed by the Regulators. Thus, we need to specify a liability generating portfolio, a loss model, a reinsurance program, and a pricing principle both for the insurer and the reinsurer. On the contrary, we shall neglect taxes and other "frictional effects" such as double taxation, financial distress or agency costs.

We shall further assume that at time $t = 0$ the shareholders of the insurance company provide a *initial capital* u that is invested in risk-free assets with maturity T. Thus at time $t = T$ the *surplus* U_T is

$$U_T = \frac{u}{v} + \Pi_T - L \tag{16}$$

where $v = v(0, T)$ is the risk-free discount factor over the period $[0, T]$, Π_T is the value at time T of the aggregate premium and L is the aggregate loss. From eq. (16) it immediately follows that the ruin probability is given by $\mathbf{P}[U_T < 0]$.

In credit trade insurance premia are usually linked to the turnover of the insured; thus the value at time t of the aggregate premium Π_t is – in general – a random variable up to time T. For simplicity we shall assume that Π_t is deterministic, that all payments are made in advance, and that the insurance company invest the premium in risk-free assets with maturity T. We further define $\Pi := \Pi_0 = v\,\Pi_T$ as the value of the aggregate premium at time $t = 0$. Later, we shall further assume that also the reinsurance premia are deterministic, so that the only source of incertitude is the loss generated by the portfolio $\mathcal{P}$.

The Solvency II capital requirement implies that

$$\mathbf{P}[(u_T + \Pi_T - L) > 0] \geq \underline{\alpha} \tag{17}$$

with $\underline{\alpha} = 0.995$. Similarly, a strategic "reputational requirement" would imply that

$$\mathbf{P}[(u_T + \Pi_T - L) > 0] \geq \alpha_R \tag{18}$$

where $\alpha_R \in (0,1)$ is a confidence level fixed on the basis of the insurer target reputation. Therefore we can assume

$$\mathbf{P}[(u_T + \Pi_T - L) > 0] \geq \alpha = \max\{\alpha_R, \underline{\alpha}\} \tag{19}$$

and define $\underline{u}$, the *minimum capital*, as the value of u that satisfies

$$\mathbf{P}[(\underline{u}_T + \Pi_T - L) > 0] = \alpha \tag{20}$$

and the *excess of initial capital* δ_u as

$$\delta u = u - \underline{u} \tag{21}$$

From eq. (19) it follows immediately that value of the minimum capital at time T, u_T, should be

$$\underline{u}_T = \text{VaR}_\alpha(L) - \Pi_T \tag{22}$$

so that the value of the minimum capital provided by the shareholders is

$$\underline{u} = v\Big[\text{VaR}_\alpha(L) - \Pi_T\Big] \tag{23}$$

Remark. Strictly speaking, nothing prevents $\text{VaR}_\alpha(L)$ to be smaller than Π_T, so we make the additional assumption that $\underline{u} > 0$. ■

Remark. Notice that under the hypothesis that $\delta u > 0$ the probability $\beta(u)$ the company is solvent is larger than α

$$\beta(u) := \mathbf{P}[(u_T + \Pi_T - L) > 0] > \mathbf{P}[(\underline{u}_T + \Pi_T - L) > 0] = \alpha \tag{24}$$

■

The expected rate of return $\rho(u)$ for the capital u is (assuming limited liability)

$$\rho(u) = \frac{\mathbf{E}[\max\{u_T + \Pi_T - L, 0\}]}{u} - 1 \tag{25}$$

For continuous distributions the first term in (25) can be rewritten

$$\begin{aligned}\frac{\mathbf{E}[(u_T + \Pi_T - L)\,\mathbb{1}_{\{L \leq L_{\beta(u)}\}}]}{u} &= \frac{1}{u}\left[\beta(u)\,(u_T + \Pi_T) - \int_0^{L_{\beta(u)}} x\,dF(x)\right] = \\ = \frac{1}{u}&\left[\beta(u)\,(u_T + \Pi_T) - \int_0^{L_{\beta(u)}} x\,dF(x) \pm \int_{L_{\beta(u)}}^{\bar{L}} x\,dF(x)\right] = \\ &= \frac{1}{u}\left[\beta(u)\,(u_T + \Pi_T) - \mathbf{E}[L] + \Big(1 - \beta(u)\Big)\,ES_{\beta(u)}\right]\end{aligned}$$

where $L_{\beta(u)} = F^{-1}\Big(\beta(u)\Big)$. As a consequence eq. (25) becomes

$$\rho(u) = \frac{1}{u}\left[\beta(u)\,(u_T + \Pi_T) - \mathbf{E}[L] + \Big(1 - \beta(u)\Big)\,ES_{\beta(u)}\right] - 1 \tag{26}$$

Since in practice $\beta(u) \geq \alpha \geq 99.5\%$, we shall assume $\beta(u) \approx 1$, so that

$$\rho(u) \simeq \rho^\star(u) := \frac{1}{u}\left[\delta u_T + RC_\alpha(L)\right] - 1 \tag{27}$$

with $\delta u_T = u_T - \underline{u}_T$. The rate $\rho^\star(u)$ can be decomposed *as if* the shareholders had invested in a risk-free asset and a risky fund with expected return $i_{\mathcal{P}}$

$$\rho^\star(u) = \frac{1}{u}\left[\frac{\delta u}{v} + RC_\alpha(L)\right] - \left[\frac{\underline{u} + \delta u}{u}\right] = i\frac{\delta_u}{u} + \left[\frac{RC_\alpha(L)}{\underline{u}} - 1\right]\frac{\underline{u}}{u} \tag{28}$$

$$i_{\mathcal{P}} = \frac{RC_\alpha(L)}{\underline{u}} - 1 = \frac{RC_\alpha(L)}{v\Big[\mathrm{VaR}_\alpha(L) - \Pi_T\Big]} - 1 \tag{29}$$

A well established link between the expected rate of return and economic capital is provided by the use of the EVA®, or *Economic Value Added* (EVA® is a registered trademark of the US consulting firm Stern Stewart). In its simple form:

$$\mathrm{EVA}^{®} = \mathrm{NP} - h \times u \tag{30}$$

where NP is net operating profit after tax (adjusted for various accounting items), h (the *hurdle rate*) is the weighted average cost of capital (WACC) required by the shareholders, and u is economic capital. The hurdle rate h_0 that makes EVA® vanish is $h_0 = \rho^\star(u) - i$. Thus, for the minimum capital requirement one has that the expected value of EVA® is:

$$\mathcal{E}(\underline{u}) = \Pi_T - \mathbf{E}[L] - h\Big(\mathrm{VaR}_\alpha(L) - \Pi_T) \tag{31}$$

For reason that will be clear in the following we prefer the use of the expected rate of return to that of EVA® in the statement of the optimization problem.

We shall now discuss how different pricing principles affect the values of the expected rate of return $\rho^\star(u)$. In particular, we consider two principles of the form $\Pi_T = \mathbf{E}[L] + \delta\Pi_T(L)$, namely

1. **(PP1)** a "traditional" *expected value* principle

$$\Pi_T^{(1)} = (1 + \theta)\,\mathbf{E}[L] \tag{32}$$

   where $\theta > 0$ is a constant, and

2. (**PP2**) a *Cost-of-Capital* (CoC) inspired principle

$$\Pi_T^{(2)} = \mathbf{E}[L] + (\mu - i)\left(\text{VaR}_\alpha(L) - \mathbf{E}[L]\right) = \mathbf{E}[L] + (\mu - i)\, RC_\alpha(L)$$

   where $i = i(0,T)$ is the risk-free interest rate over the period $[0,T]$ and $\mu \geq i$ is an exogenous constant having the dimensions of a rate of return over the same time period.

Under the first principle

$$\underline{u} = v\, \underline{u}_T = v\left[RC_\alpha(L) - \theta\, \mathbf{E}[L]\right] \tag{33}$$

and

$$\rho^\star(u) = i\frac{\delta_u}{u} + \left(\frac{i\, RC_\alpha(L) + \theta\, \mathbf{E}[L]}{RC_\alpha(L) - \theta\, \mathbf{E}[L]}\right)\frac{\underline{u}}{u} \tag{34}$$

$$i_{\mathcal{P}} = i_{\mathcal{P}}(\zeta) = \frac{i\,\zeta + \theta}{\zeta - \theta} \qquad \zeta = \frac{RC_\alpha(L)}{\mathbf{E}[L]} \tag{35}$$

$$\mathcal{E} = h\, v\, \mathbf{E}[L]\left[\theta\frac{1 + v\, h}{v\, h} - \zeta\right] \tag{36}$$

Notice that $i_{\mathcal{P}}(\zeta)$ is monotonically decreasing in ζ. For "typical" values of $i = 5\%$, $\theta = 1/3$, $\alpha = 99.5\%$ the value $i_{\mathcal{P}}(\zeta)$ that would be obtained with a log-normal distribution of mean 1 and variance 2 is obtained for $\zeta^\star \simeq 7.6$, and thus $i_{\mathcal{P}}(\zeta^\star) \simeq 10\%$. Under the first principle the shareholder is likely to prefer portfolios with a low value of ζ ("low tails") and a high value of $\mathbf{E}[L]$.

Under the second principle the situation is quite different. In fact, one has that

$$\underline{u} = v\, \underline{u}_T = v\left[1 - (\mu - i)\right] RC_\alpha = \left[1 - v\,\mu\right] RC_\alpha \tag{37}$$

and

$$\rho^\star(u) = i\frac{\delta_u}{u} + \left(\frac{v\,\mu}{1 - v\,\mu}\right)\frac{\underline{u}}{u} \tag{38}$$

$$i_{\mathcal{P}}(\mu) = \frac{v\,\mu}{1 - v\,\mu} \approx \mu \tag{39}$$

$$\mathcal{E}(\underline{u}) = RC_\alpha(L)\left[(\mu - i) - h\left(1 - v\,\mu\right)\right] \tag{40}$$

Equations (39) and (40) imply that shareholders expected return is identical for portfolios having different loss distributions but the same risk capital $RC_\alpha(L)$. Moreover, since the expected value of EVA® increases with increasing risk capital, shareholders would prefer "heavy tails" since the risk is properly remunerated by the premium paid by the insured.

Remark. Eq. (35) implies that if $\theta = 0$ then $i_{\mathcal{P}}(\zeta) = i$ for any loss distribution; similarly (39) implies that for $\mu = i$ then $i_{\mathcal{P}}(i) = i$ for any loss distribution.

Remark. It is to verify that the two couples of equations, respectively eqs. (35) and (36), eqs. (39) and (40), correctly satisfy the relation $h_0 = i_{\mathcal{P}} - i$.

We now consider the effects of reinsurance. For simplicity we assume that the expected value of the reinsurance premium in $t = T$ is Π_T^R and that the reinsured loss relief in $t = T$ is $\bar{L}^R$. The shareholders expected return is then

$$\rho(u) = \frac{\mathbf{E}[\max\{u_T + \Pi_T - L - \Pi_T^R + \bar{L}^R, 0\}]}{u} - 1 \tag{41}$$

Under "strict" proportional reinsurance only, with a ceded fraction $\lambda \in (0, 1)$,

$$\Pi_T^R = \lambda\, \Pi_T \quad \bar{L}^R = \lambda\, L \tag{42}$$

while the cedant retains a fraction $(1-\lambda)$ of the premium and of the loss. It easy is to check that the new minimum capital requirement $\underline{u}_T^{QS}$ is equal to $(1-\lambda)\underline{u}_T$; moreover $\rho^\star(u)$ can be written as

$$\rho_\lambda^{QS}(u) = i\left[\frac{\delta_u}{u} + \lambda\frac{\underline{u}}{u}\right] + i_{\mathcal{P}}\left[(1-\lambda)\,\frac{\underline{u}}{u}\right] = \rho^\star(u) - (i_{\mathcal{P}} - i)\frac{\underline{u}}{u} \tag{43}$$

where the superscript QS indicates that a quota share treaty has been applied and the subscript λ refers to the parameter of the treaty.

Similarly, when excess-of-loss reinsurance after quota share is applied eq. (41) is changed into

$$\rho(u) = \frac{\mathbf{E}\Big[\max\{u_T + (1-\lambda)\Pi_T - (1-\lambda)L - \Pi_T^{XL}(\boldsymbol{\nu}) + L_{\boldsymbol{\nu}}^{XL}, 0\}\Big]}{u} - 1 \tag{44}$$

where $\Pi_T^{XL}(\boldsymbol{\nu})$ and $L_{\boldsymbol{\nu}}^{XL}$ are respectively the premium for the excess-of-loss treaty and the corresponding loss relief for a given set of excess-of-loss reinsurance treaty parameters $\boldsymbol{\nu}$. The total loss suffered by the insurer is now

$$\tilde{L}_{\lambda,\boldsymbol{\nu}} = (1-\lambda)\, L - L_{\boldsymbol{\nu}}^{XL} \tag{45}$$

The new minimum requirement and expected rate of return are respectively

$$\underline{u}_T^{XL}(\lambda, \boldsymbol{\nu}) = \text{VaR}_\alpha(\tilde{L}_{\lambda,\boldsymbol{\nu}}) - (1-\lambda)\Pi_T + \Pi_T^{XL}(\boldsymbol{\nu}) \tag{46}$$

$$\rho_{\lambda,\boldsymbol{\nu}}^{XL}(u) = i\left[\frac{\delta_u}{u} + \frac{u - \underline{u}_T^{XL}}{u}\right] + i_{\mathcal{P}}^{XL}(\lambda, \boldsymbol{\nu})\,\frac{\underline{u}_T^{XL}}{u} \tag{47}$$

$$i_{\mathcal{P}}^{XL}(\lambda, \boldsymbol{\nu}) = \frac{RC_\alpha(\tilde{L}_{\lambda,\boldsymbol{\nu}})}{v\,\underline{u}_T^{XL}} - 1 \tag{48}$$

4.1 A first optimality problem

We shall now adopt as first optimality criterion the maximization of the expected rate of return $\rho_{\mathcal{P}}^{XL}(\lambda,\boldsymbol{\nu})$ with respect to the set of parameters $\{\lambda,\boldsymbol{\nu}\}$ under the constraint $\underline{u}_T^{XL}(\lambda,\boldsymbol{\nu}) = \kappa\,\underline{u}_T$, where $\kappa \in (0,1)$ is a target *reduction factor* of the initial minimum capital requirement. Thus, we define problem PB1 as

$$\begin{aligned} \textbf{PB1} \qquad\qquad & \max_{\lambda,\boldsymbol{\nu}} \quad \rho_{\lambda,\boldsymbol{\nu}}^{XL}(u) \\ \text{subject to:} \quad & \underline{u}_T^{XL}(\lambda,\boldsymbol{\nu}) = \kappa\,\underline{u}_T \end{aligned} \tag{49}$$

Remark. The fact that the released capital is left in the investment portfolio and contributes to the expected rate of return $\rho_{\lambda,\boldsymbol{\nu}}^{XL}(u)$ at the risk free rate i acts as a barrier against the possibility of full reinsurance. ■

The solution of PB1 depends on *a)* the choice of a pricing principle for the cedant and the reinsurer and *b)* of a loss model. If, following the indications of the Solvency II framework, we use the pricing principle PP2 for both the cedant and the reinsurer, we then have what we define as problem PB1a. In this case eq. (46) and eq. (48) become

$$\underline{u}_T^{XL}(\lambda,\boldsymbol{\nu}) = RC_\alpha(\tilde{L}_{\lambda,\boldsymbol{\nu}}) - (1-\lambda)(\mu-i)RC_\alpha\big(L\big) + (\mu_R-i)RC_\gamma\big(L_{\boldsymbol{\nu}}^{XL}\big) \tag{50}$$

$$i_{\mathcal{P}}^{XL}(\lambda,\boldsymbol{\nu}) = \frac{RC_\alpha(\tilde{L}_{\lambda,\boldsymbol{\nu}})}{v\Big[RC_\alpha(\tilde{L}_{\lambda,\boldsymbol{\nu}}) - (1-\lambda)(\mu-i)RC_\alpha\big(L\big) + (\mu_R-i)RC_\gamma\big(L_{\boldsymbol{\nu}}^{XL}\big)\Big]} - 1 \tag{51}$$

where $\mu_R > i$ and $\gamma \in (\underline{\alpha},1)$ are constants fixed by the reinsurer on the basis of its risk aversion and target reputation. In the next section we shall introduce the loss model. Before that, notice that $i_{\mathcal{P}}^{XL}(\lambda,\boldsymbol{\nu}) > i_{\mathcal{P}}$ iif

$$\begin{aligned} (1-\lambda)(\mu-i)RC_\alpha\big(L\big) - (\mu_R-i)RC_\gamma\big(L_{\boldsymbol{\nu}}^{XL}\big) &> (\mu-i)RC_\alpha(\tilde{L}_{\lambda,\boldsymbol{\nu}}) \\ (1-\lambda)RC_\alpha\big(L\big) &> RC_\alpha(\tilde{L}_{\lambda,\boldsymbol{\nu}}) + \frac{\mu_R-i}{\mu-i}RC_\gamma\big(L_{\boldsymbol{\nu}}^{XL}\big) \\ RC_\alpha\Big((1-\lambda)L\Big) &> RC_\alpha\Big((1-\lambda)L - L_{\boldsymbol{\nu}}^{XL}\Big) + \frac{\mu_R-i}{\mu-i}RC_\gamma\big(L_{\boldsymbol{\nu}}^{XL}\big) \end{aligned} \tag{52}$$

Eq. (52) shows that in the case $\mu_R = \mu$ and $\gamma = \alpha$ there could be an increase in the expected rate of return if the risk capital were not sub-additive, that –although possible– is quite unlike. Later on we shall numerically investigate PB1a in this setting.

Differently, if we chose principle PP2 for the cedant and PP1 for the reinsurer we obtain problem PB1b. In this case $i_{\mathcal{P}}^{XL}(\lambda, \boldsymbol{\nu}) > i_{\mathcal{P}}$ iif

$$
\begin{aligned}
(1-\lambda)(\mu - i) RC_\alpha\Big(L\Big) - \theta_R\,\mathbf{E}\Big[L_{\boldsymbol{\nu}}^{XL}\Big] &> (\mu - i) RC_\alpha(\tilde{L}_{\lambda,\boldsymbol{\nu}}) \\
(1-\lambda) RC_\alpha\Big(L\Big) &> RC_\alpha(\tilde{L}_{\lambda,\boldsymbol{\nu}}) + \frac{\theta_R}{\mu - i}\mathbf{E}\Big[L_{\boldsymbol{\nu}}^{XL}\Big] \\
RC_\alpha\Bigg((1-\lambda)L\Bigg) &> RC_\alpha\Bigg((1-\lambda)L - L_{\boldsymbol{\nu}}^{XL}\Bigg) + \frac{\theta_R}{\mu - i}\,\mathbf{E}\Big[L_{\boldsymbol{\nu}}^{XL}\Big] \\
RC_\alpha\Bigg((1-\lambda)L\Bigg) &> RC_\alpha\Bigg((1-\lambda)L - L_{\boldsymbol{\nu}}^{XL}\Bigg) + \frac{\theta_R}{\mu - i}\,\frac{\mathbf{E}\Big[L_{\boldsymbol{\nu}}^{XL}\Big]}{RC_\alpha\Big(L_{\boldsymbol{\nu}}^{XL}\Big)} RC_\alpha\Big(L_{\boldsymbol{\nu}}^{XL}\Big)
\end{aligned}
\tag{53}
$$

where θ_R is the risk loading parameter of the reinsurer. In this case it is possible for the cedant to increase the expected rate of return by choosing $\boldsymbol{\nu}$ in order to have a "heavy" tailed distribution of $L_{\boldsymbol{\nu}}^{XL}$. Later we shall numerically investigate PB1b in this setting by assuming $\theta_R = \Pi_T/\mathbf{E}[L] - 1$.

4.2 A second optimality problem

Whenever it is not possible to enhance the expected rate of return through reinsurance, it is still significant to investigate if there is a trade off between a reduction in the expected rate of return and the reduction in risk, where for the latter we have to specify a risk measure, *e.g.* the standard deviation of the loss distribution . We can thus define a second problem

$$
\begin{aligned}
\mathbf{PB2} \qquad\qquad & \min_{\lambda,\boldsymbol{\nu}} \;\; \mathbf{SD}(\tilde{L}_{\lambda,\boldsymbol{\nu}}) \\
\text{subject to:} \quad & \underline{u}_T^{XL}(\lambda, \boldsymbol{\nu}) = \kappa\, \underline{u}_T \\
& \rho_{\lambda,\boldsymbol{\nu}}^{XL}(u) = \text{constant}
\end{aligned}
\tag{54}
$$

This problem is very similar to that solved in the pioneering work of de Finetti [7], except for the presence of non-proportional reinsurance, the solvency constraint, the correlations between risks and –possibly– for the use of a quantile-based pricing principle. However, PB2 is a not as promising as it could seem at first glance. In fact, whilst the constraint on the capital requirement (which is essentially a constraint on the quantile of the loss distribution) is not fixing the standard deviation of the distribution, it is nevertheless very likely to restrict strongly the accessible range of values. Later on we shall numerically investigate this issue by determining the smallest possible value of the standard deviation by solving problem PB2b defined as

$$\textbf{PB2b} \qquad \min_{\lambda,\boldsymbol{\nu}} \ \textbf{SD}(\tilde{L}_{\lambda,\boldsymbol{\nu}})$$
$$\text{subject to:} \quad \underline{u}_T^{XL}(\lambda,\boldsymbol{\nu}) = \kappa\, \underline{u}_T \tag{55}$$

5 The loss model

The loss model adopted in our study is based on the CreditRisk$^+$ methodology [4] and is detailed in [23]. It is a credit portfolio loss model developed for financial applications that implements computational techniques derived from actuarial loss models and is characterized by the fact that the distribution of the aggregate loss can be obtained via a semi-analytical procedure based on a recurrence equation, or –alternatively– on numerical procedures such as the fast Fourier transform. Extensions of the original model by several authors are collected in [22].

In the model defaults are driven by stochastic common risk factors, that are independent Gamma-distributed random variables $\mathbf{s} = (s_1, \dots, s_K)$, with mean 1 and variances $\sigma = (\sigma_1^2, ..., \sigma_K^2)$. Conditional on these random variables, each default indicator Y_i is assumed to be Poisson distributed with mean p_i

$$p_i = p_i(\mathbf{s}) = \bar{p}_i\Big(\omega_{i0} + \omega_{i1}\, s_1 + \cdots + \omega_{iK}\, s_K\Big), \quad i = 1, \dots N, \tag{56}$$

for some positive coefficients (weights) $\omega_{i0}, \dots, \omega_{iK}$ for which $\sum_{k=0}^{K} \omega_{ik} = 1$, and a positive parameter $\bar{p}_i$, corresponding to the unconditional mean $\bar{p}_i = \mathbf{E}[p_i(\mathbf{s})]$. The choice of the Poisson distribution, the so-called “Poisson approximation”, is done in the sake of the analytical tractability of the model and correspond to an approximation of a Bernoulli random variable. In the model the Y_i may be viewed as a Poisson random variable with a random mean. Numerical effects of this approximation are discussed in [22] p. 289, and in [23].

Differently, claim amounts c_i are assumed to be deterministic. As already stated, the cedant loss, gross of reinsurance, is then given by

$$L = \sum_{i=1}^{N} c_i\, Y_i \tag{57}$$

where N is the number of exposures in the cedant portfolio.

The risk factor are interpreted as representing different market segments, or “sectors” in the model jargon. Using Gamma random variables for the risk drivers allows calculation of the distribution of L through numerical inversion of its probability generating function, as shown in [4]. Moreover it turns out that the distribution of the number of defaults in the portfolio is equal to the distribution of a sum of independent negative Binomial random variables.

5.1 Monte Carlo simulation

The analytical tractability of the model is lost when the complex payoff of multi-layer excess-of-loss treaties with fixed reinstatements have to be considered. In this case to compute the loss distribution one can revert to Monte Carlo techniques.

Monte Carlo simulation without importance sampling is particularly simple. In each replication, it is sufficient to generate the common risk factors s_k from the distributions $\Gamma(y; \alpha_k, \beta_k)$, $k = 1, ..., K$, with

$$\alpha_k = \frac{1}{\sigma_k^2} \quad \beta_k = \sigma_k^2 \quad k = 1, ..., K. \tag{58}$$

In this way each s_k has mean 1 and variance σ_k^2. Then one has to generate the value of the default indicators Y_i from a Poisson distribution with mean p_i, where p_i is calculated according to eq. (56). From the $Y_i's$ and the exposures c_i the portfolio loss is obtained by using eq. (57).

The use of importance sampling is slightly more complex. Our implementation of importance sampling is based on an *exponential twisting* approach, originally proposed by Glasserman and Li [21] and fully described in [24]. For the specific case of the CreditRisk$^+$ setting, exponential twisting is used to distort first the default indicators $Y_i (i = 1, \dots N)$ and successively the risk factors $s_k (k = 1, \dots K)$. Clearly better sampling in the region of large losses is achieved when default probabilities are increased. This is done by using a one-parameter family of the form

$$\bar{p}_i(\theta) = \frac{\bar{p}_i e^{\theta c_i}}{\bar{p}_i e^{\theta c_i} + (1 - \bar{p}_i)} = \frac{\bar{p}_i e^{\theta c_i}}{1 + \bar{p}_i \left(e^{\theta c_i} - 1\right)}, \tag{59}$$

which is ineffective for $\theta = 0$ and ensures that the tilted defaulted probabilities $p_i(\theta)$ are monotonically increasing with $\theta > 0$.

The second step of the importance sampling procedure is composed of a similar twist on the s_k by some twisting parameters τ_k. For a particular choice of the τ_k parameters (see [24]) the likelihood ratio for this two-step change of distribution depends only on θ and is given by

$$\mathcal{L} = \exp\left\{-\theta\, L + \psi_L(\theta)\right\} \tag{60}$$

where

$$\psi_L(\theta) = \sum_{i=1}^{N} \bar{p}_i\, \omega_{i0} \left(e^{\theta c_i} - 1\right) - \sum_{k=1}^{K} \alpha_k \log\left(1 - \beta_k \sum_{i=1}^{N} \bar{p}_i\, \omega_{ik} \left(e^{\theta c_i} - 1\right)\right) \tag{61}$$

At this point, under the θ measure, $e^{-\theta L + \psi_L(\theta)} \mathbb{1}_{\{L > x\}}$ is an unbiased estimator of $\mathbf{P}[L > x]$. To achieve efficient importance sampling θ is chosen to minimize the upper bound on the second moment of this estimator

$$M_2(\theta, x) = \mathbf{E}_\theta\left[e^{-2\theta L + 2\psi_L(\theta)} \mathbb{1}_{\{L>x\}}\right] - \mathbf{P}[L > x]^2 \leq e^{-2\theta L + 2\psi_L(\theta)} - \mathbf{P}[L > x]^2 \tag{62}$$

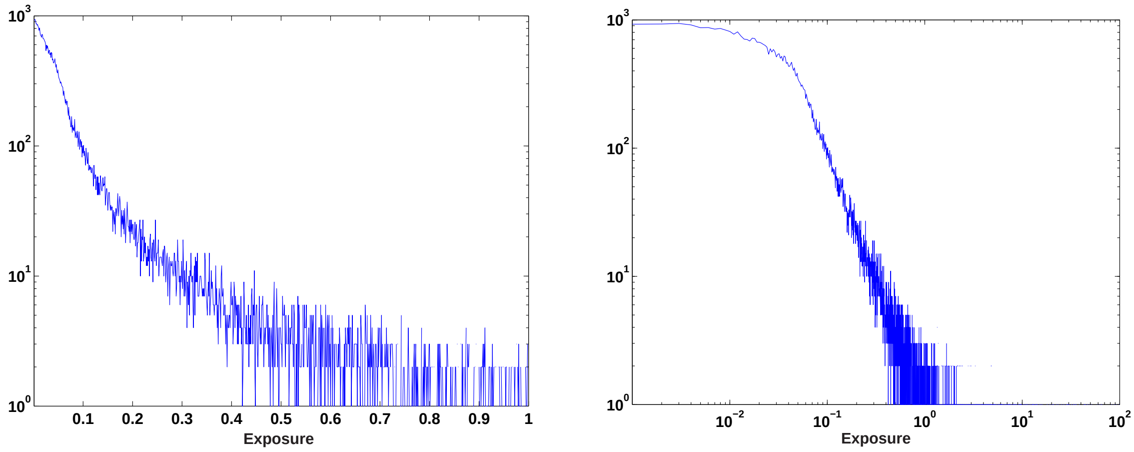

Figure 1: Distribution of the exposures (in millions of currency units) in the portfolio used for the numerical evaluation. The left plot is in linear-logarithmic scale and shows exposures smaller than one million; the right plot is in logarithmic-logarithmic scale to show the behavior of the high side of the tail.

that is by (numerically) solving the equation

$$\psi'_L(\theta_x) = x. \tag{63}$$

Eq. (63) can be reinterpreted by saying that the change of measure that optimizes the measurement of $\mathbf{P}[L > x]$ is that particular change which makes x to be the expected value of the loss under the distorted measure. Practical use of this importance sampling technique requires to provide the value of x. In [24] it is shown that an initial guess of x obtained with the PGF inversion technique is sufficient to improve in accuracy by a factor of about 5 with respect to “plain” Monte Carlo.

6 A numerical evaluation

We consider a realistic portfolio of polices with $N = 50000$ distinct buyers, over a risk horizon of $T = 1$ year. The input data for the calculation of the loss distribution are the exposures of the buyers, from which we determine the claim size assuming the hypothesis of eq. (4), *i.e.* $c_i = E_i(0)$ $(i = 1, \dots N)$, the buyers unconditional default probabilities $\bar{p}_i$, the number of sectors K, the parameters α_k, β_k $(k = 1, \dots K)$ of the Gamma distributions of the risk drivers and the weights $\omega_{i,k}$ $(i = 1, \dots N, k = 0, \dots K)$. For this analysis we assume an economy with a single macroeconomic sector $(K = 1)$ and an idiosyncratic component of 50 % for all buyers (examples for other cases are given in [24]), so that $\omega_{i,0} = 0.5$ and $\omega_{i,1} = 0.5$ for $i = 1, \dots N$.

The credit quality of the buyers are computed using the values reported by Dietsch and Petey [8] which in turn are based on a very large data sets provided by the trade credit insurance companies Coface and Creditreform. Default probabilities are assigned to each buyer on the basis of an internal indicator of *risk class* and an indicator of *turnover size* using the values reported in table 3, p. 779 of [8]. Following the analysis done in [23], we assume the coefficient of variation cv_i of each buyer

$$cv_i = \frac{\mathbf{V}[p_i(\mathbf{s})]^{1/2}}{\mathbf{E}[p_i(\mathbf{s})]}, \quad i = 1, \ldots N, \tag{64}$$

to be 1.5. The variances of the conditional default probabilities $\mathbf{V}[p_i(\mathbf{s})]$ can be deduced from the above equation and the knowledge of the default probabilities, and used to determine the parameters of the Gamma distribution of the risk driver. In a more general framework, see *e.g.* [22] pag. 249, a multivariate analysis is required to identify the number of sectors, the Gamma parameters and the buyers weights.

The composition of the portfolio is summarized in tables 1 and 2. The total exposure of the portfolio is about 9000 M currency units. The fraction of large firms is 0.04% (22 exposures larger than 40 M) with the largest exposure being about 230 M. The distribution of exposures is reported in linear-log and log-log scale in fig. 1.

The calculation of the loss distribution gross of reinsurance is performed using three methods: the numerical inversion of the PGF with the standard Panjer algorithm [4], the "plain" Monte Carlo method and the Monte Carlo method with importance sampling. The semi-analytical method provides a benchmark for the other two. Moreover, it provides the value of $x = \text{VaR}_{99.5\%}$, which is used as input for the optimization of the importance sampling technique. The calculations are implemented in standard ANSI C code complied with gcc 4.1.3 in a Unix environment. Random number generation is performed using the algorithms of the GSL libraries [12].

The results obtained in this way are summarized in Table 4. In general there is a remarkable agreement between the three methods. As discussed in [24] importance sampling increase the accuracy by a factor of about 5. The shape of loss distribution is shown in fig. 2; notice that it is a heavy tailed distribution typical of credit risks. The agreement between the numerical techniques is further shown in fig. 3 where the distribution obtained with the PGF technique and the importance sampling technique are compared. The figure is in log-scale to allow a better comparison on the right tail.

The reinsurance program implemented in the evaluation is composed of a fixed number $J = 3$ of consecutive layers. The parameters vector is then

$$\boldsymbol{\nu} = (\lambda, \ell_1, m_1, m_2, m_3, r_1, r_2, r_3) \tag{65}$$

where $r_i \in [0, 5]$ for $i = 1, 2, 3$; no other bound are applied. Tables 5, 6 and 7 report the results for the problems PB1a, PB2b and PB1b respectively. In the first case we have assumed a one-year risk-free rate of 4% and –following Solvency II indications – a cost of capital 6% higher than the risk-free rate, thus $\mu = \mu_R = 10\%$. We have also assumed equal confidence levels for the determination of the risk capital of the cedant and the reinsurer, $\alpha = \gamma = 99.5\%$. For the second problem we have assumed that $\theta_R \simeq 25\%$ by asking θ_R to solve the equation $\Pi_T = (1 + \theta_R)\mathbf{E}[L]$, which –in principle– would hold for the cedant; in practice we have assumed that the cedant and the reinsurer had the same risk loading θ. For the last case we have used the same assumptions of the first case.

All the three optimization problems have been solved numerically using the *simulated annealing* technique [17], in the fast adaptive version [14] provided by Ingberg [15]. Simulated annealing is theoretically well understood [25, 28] random-search optimization technique which is particularly suited to solve integer-valued non-convex problems. For each iteration of the search procedure the loss distribution is computed using 1000 simulations with importance sampling.

The results reported in Table 5 confirm that, when the cedant and the insurer use the same pricing principle PP2 with an identical cost of capital, adding an excess-of-loss treaty to a quota share one does not bring a benefit to the insurer in terms of expected rate of return since the same coverage could have obtained with a different "pure" quota share treaty. Differently, Table 7 shows that a roughly 2% average increase in the expected rate of return can be obtained by the insurer if the reinsurer uses pricing principle PP1. This result is in qualitative agreement with what reported in the literature, albeit for different loss models. Finally, Table 6 shows that once the minimal capital requirement has been fixed, there is very little room for a reduction of the standard deviation of the net loss distribution. As a consequence, in this case, the efficient frontier in the plane expected rate of return *vs* loss standard deviation compressed in a very small range, with possible improvements of less then 10%.

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Size	(turnover in M)	firms	%	Exposure	firms	%
1	(0,15 to 1)	31.605	63.21	$<$ 1 M	49.086	98.17
2	(1-7)	15.062	30.12	1-7 M	768	1.54
3	(7-40)	3.333	6.67	7-40 M	124	0.25
4		-	-	> 40 M	22	0.04
Total		50.000	100		50.000	100

Table 1: The size distribution for firms in the portfolio used in the numerical evaluation.

Size	Risk classes (from low risk – class 1 to high risk – class 8)							
	1	2	3	4	5	6	7	8
Total	3.21	40.88	12.18	13.78	13.64	12.41	2.23	1.67

Table 2: The risk distribution for firms in the portfolio used in the numerical evaluation.

Risk classes	Size classes		
	size 1	size 2	size 3
	$<$1M	1-7M	7-40M
1 (low)	0.33	0.24	0.15
2	0.41	0.27	0.19
3	0.90	0.68	0.48
4	1.64	1.35	0.84
5	2.79	2.61	1.53
6	4.94	4.51	2.44
7	9.99	9.44	5.49
8 (high)	14.89	16.24	13.28
Total	2.63	1.74	0.79

Table 3: Average annual default probabilities (in %) used in the numerical evaluation.

50% idiosyncratic component						
	PGF inversion		plain MC		IS MC	
	millions	%	millions	%	millions	%
EL	132.22	1.46	131.38	1.45	132.71	1.46
$RC_{99.5\%}$	565.22	6.24	563.02	6.21	563.74	6.22

Table 4: Expected loss and risk capital at the $\alpha = 99.5\%$ confidence level gross of reinsurance with $N_s = 50000$ simulations. Values are expressed both in millions of currency units and as fraction of the total exposure.

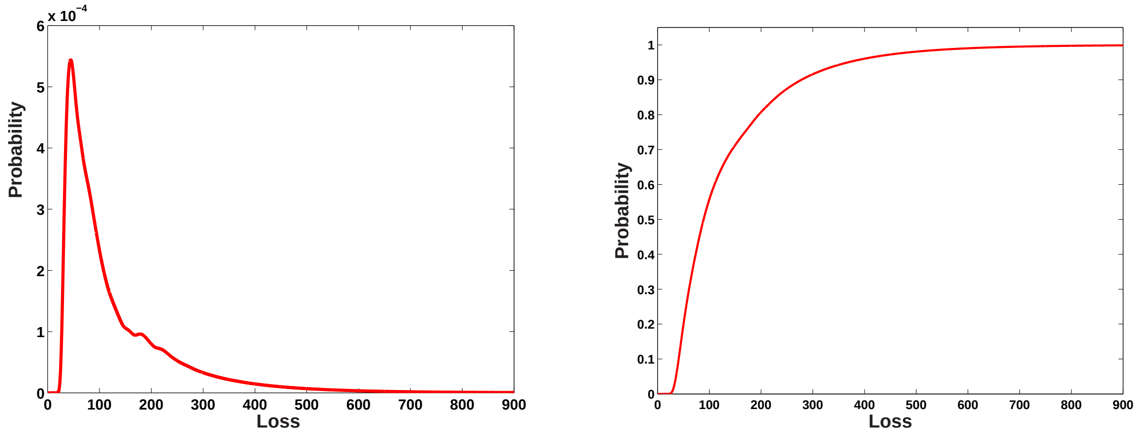

Figure 2: Loss distribution (in millions of currency units) gross of reinsurance for $K = 1$ and 50% idiosyncratic risk.

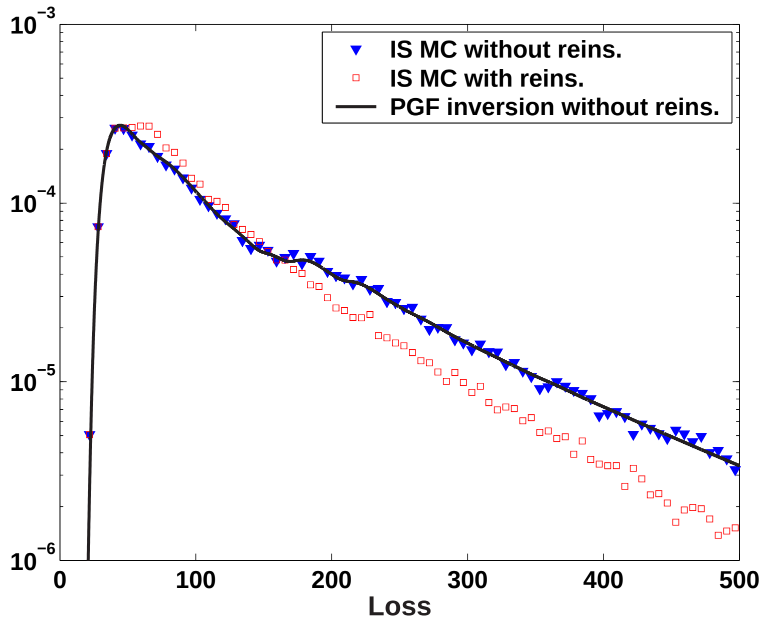

Figure 3: Loss distribution (in millions of currency units) with and without reinsurance for $K = 1$ and 50% idiosyncratic risk. Importance sampling MC results gross of reinsurance are plotted with triangles; they are in good agreement with the continuous line that is obtained with PGF inversion. An example of importance sampling MC result after reinsurance is also plotted with empty squared shaped symbols; the reinsurance parameters are the same as in [24].

Required reduction κ	0.5	0.6	0.7	0.8	0.9
parameters					
λ	69	63	73	85	96
ℓ_1	10	61	72	78	87
m_1	54	25	58	81	99
m_2	148	77	103	91	189
m_3	160	90	94	162	185
r_1	1	5	3	3	3
r_2	5	3	0	3	4
r_3	1	2	5	1	2
gross					
Expected loss	134.09	134.09	134.09	134.09	134.09
Standard dev.	123.96	123.96	123.96	123.96	123.96
Risk Capital	550.58	550.58	550.58	550.58	550.58
Expected Shortfall	834.83	834.83	834.83	834.83	834.83
net after quota share only					
Expected loss	68.40	83.15	105.95	116.67	130.08
Standard dev.	63.22	76.86	97.93	107.85	120.24
Risk Capital	280.80	341.35	434.95	479.00	534.07
Expected Shortfall	425.77	517.61	659.53	726.31	809.79
after quota share and excess-of-loss					
Expected loss	68.40	81.68	97.90	109.79	122.81
Standard dev.	63.22	73.95	83.41	95.11	106.73
Risk Capital	280.80	331.67	385.37	441.18	494.06
Expected Shortfall	425.77	506.45	571.68	665.76	745.32
excess-of-loss only					
Expected loss	0.40	1.47	8.05	6.88	7.27
Standard dev.	2.62	6.46	22.72	20.44	21.37
Risk Capital	8.15	33.53	116.65	111.87	117.00
Expected Shortfall	30.89	53.34	140.47	129.35	117.05
results					
Expected Premium	85.14	100.14	116.98	131.81	147.82
Expected surplus	16.85	18.47	19.10	22.03	25.02
Minimum capital requirement	251.08	301.15	352.19	403.03	451.00
Expected rate of return (%)	7.39	7.71	7.84	8.43	9.03
comparison with pure quota share					
Pure QS* return (%)	7.32	7.98	8.65	9.31	9.97
Expected return reduction (%)	99.57	96.60	90.64	90.50	90.51
St. Dev. reduction (%)	100.61	99.42	96.12	95.90	95.67
ES reduction (%)	101.32	101.11	97.83	99.69	99.20

Table 5: Results for PB1a. The parameters are obtained by maximizing the expected rate of return under a constraint on the fraction of capital requirement reduction.

Required reduction κ	0.5	0.6	0.7	0.8	0.9
parameters					
λ	69	63	73	85	96
ℓ_1	10	61	72	78	87
m_1	54	25	58	81	99
m_2	148	77	103	91	189
m_3	160	90	94	162	185
r_1	1	5	3	3	3
r_2	5	3	0	3	4
r_3	1	2	5	1	2
gross distribution					
Expected loss	134.09	134.09	134.09	134.09	134.09
Standard dev.	123.96	123.96	123.96	123.96	123.96
Risk Capital	550.58	550.58	550.58	550.58	550.58
Expected Shortfall	834.83	834.83	834.83	834.83	834.83
distribution after quota share					
Expected loss	92.54	84.49	97.90	113.99	128.74
Standard dev.	85.53	78.10	90.49	105.37	119.00
Risk Capital	379.91	346.86	401.93	467.98	528.56
Expected Shortfall	576.04	525.95	609.44	709.61	801.45
distribution after quota share and excess-of-loss					
Expected loss	69.16	79.76	92.63	106.94	120.59
Standard dev.	55.45	69.29	80.64	92.35	103.99
Risk Capital	266.71	322.69	376.97	431.31	485.14
Expected Shortfall	417.40	484.35	555.26	630.00	709.77
excess-of-loss only					
Expected loss	23.38	4.73	5.27	7.06	8.15
Standard dev.	41.04	14.20	15.96	20.79	23.91
Risk Capital	177.32	78.02	87.98	111.62	126.97
Expected Shortfall	213.87	91.24	103.55	127.35	146.14
results					
Expected Premium	81.30	95.88	111.45	128.31	144.67
Expected surplus	12.15	16.13	18.84	21.38	24.10
Minimum capital requirement	244.77	294.77	344.36	394.16	443.31
Expected rate of return (%)	8.97	9.47	9.47	9.42	9.44
comparison with pure quota share					
Pure QS* return (%)	7.32	7.98	8.65	9.31	9.97
Expected return reduction (%)	88.02	90.71	90.04	89.11	88.65
St. Dev. reduction (%)	89.47	93.16	92.93	93.12	93.21
ES reduction (%)	100.00	96.70	95.02	94.33	94.47

Table 6: Results for PB2b. The parameters are obtained by minimizing the standard deviation of the net loss under a constraint on the fraction of capital requirement reduction.

Required reduction κ	0.5	0.6	0.7	0.8	0.9
parameters					
λ	52	71	77	99	100
ℓ_1	74	51	70	30	88
m_1	46	38	58	39	54
m_2	23	99	7	29	5
m_3	74	96	25	69	27
r_1	2	5	2	0	3
r_2	3	4	1	5	4
r_3	2	3	2	4	0
gross distribution					
Expected loss	134.09	134.09	134.09	134.09	134.09
Standard dev.	123.96	123.96	123.96	123.96	123.96
Risk Capital	550.58	550.58	550.58	550.58	550.58
Expected Shortfall	834.83	834.83	834.83	834.83	834.83
distribution after quota share					
Expected loss	69.81	95.31	103.37	132.76	134.09
Standard dev.	64.52	88.10	95.55	122.72	123.96
Risk Capital	286.59	391.31	424.38	545.09	550.58
Expected Shortfall	434.56	593.33	643.47	826.49	834.83
distribution after quota share and excess-of-loss					
Expected loss	68.97	86.89	96.92	112.60	125.63
Standard dev.	62.78	73.20	83.65	93.57	108.43
Risk Capital	280.88	334.74	389.71	444.70	500.07
Expected Shortfall	428.39	503.19	570.27	675.55	738.10
excess-of-loss only					
Expected loss	0.84	8.42	6.45	20.17	8.46
Standard dev.	4.64	23.16	18.77	43.04	23.84
Risk Capital	21.06	116.90	100.10	184.88	127.97
Expected Shortfall	44.61	141.04	108.53	220.30	121.01
results					
Expected Premium	85.94	108.28	120.77	140.32	156.58
Expected surplus	16.99	21.40	23.87	27.74	30.95
Minimum capital requirement	253.74	301.28	351.77	400.93	451.07
Expected rate of return (%)	7.41	8.30	8.80	9.57	10.22
comparison with pure quota share					
Pure QS* return (%)	7.32	7.98	8.65	9.31	9.97
Expected return reduction (%)	101.29	103.98	101.74	102.82	102.46
St. Dev. reduction (%)	101.28	98.41	96.39	94.35	97.19
ES reduction (%)	102.63	100.46	97.59	101.15	98.24

Table 7: Results for PB1b. The parameters are obtained by maximizing the expected rate of return of the net loss under a constraint on the fraction of capital requirement reduction. The reinsurer is assumed to use pricing principle PP1 instead of PP2.