

Limited liabilities within a (re-)insurance group

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Abstract: The economic state of a (re-)insurance group can be characterized by a collection of balance sheets, entangled by ownership relations and obligations between group companies. In the case of financial distress the relative seniority of obligations as well as the limited liability towards a subsidiary are important considerations that regulators require to be adequately reflected in an internal risk model. This paper shows how to address these requirements in the context of a stochastic model. In this way an appropriate treatment of the liabilities towards counterparties within or outside the group, as well as the correct assessment of the participation value of a subsidiary is ensured.

Keywords: Group solvency, regulation, intragroup transactions, internal model, risk management.

1. INTRODUCTION

A major (re-)insurance company is structured as a group of several legal entities. Besides through participations the balance sheets of these individual companies might contain, as a result of intragroup transactions like risk transfers and loans, items whose value is dependent on the solvency of other entities in the group.

A consolidated view would neglect the restricted fungibility of capital by netting those intragroup transactions among each other and thus compare only assets and liabilities with respect to group-external counterparties. Such a view entails the willingness and the ability of the group to eventually assist a distressed entity in excess of its legal obligations towards it.

There are valid reasons to believe in this recapitalization intent; after all, it is usually even reflected in the credit ratings of subsidiaries that are deemed core interests for the group. Nevertheless, there are equally valid reasons to assume that in a situation of distress the ability of the group to shift assets between entities is diminished. In particular, under the Swiss Solvency Test (SST) the regulator requires to fully account for the option to default on the obligations of a subsidiary; only legally binding intragroup transactions are admitted for determining capital adequacy.

The following notes describe how one can formalize these relationships between the balance sheets of the individual companies in a (re-)insurance group in such a way that those requirements can be met. We shall work with an economic valuation metric. The underlying stochastic model is assumed to describe the joint development of the economic values of balance sheet positions over a one-year time horizon. In this article we take the stochastic model as a given and build up on stochastic scenarios in which we know the economic value of all positions at the end of the assessment period.

2. AN IDEALIZED SET-UP

We want to assess the risk of a group of n insurance companies (which we refer to as $1,2,3,\dots,n$). Here, “risk” will be understood as the uncertainty of the economic net worth of those companies at the end $t = 1$ of some period (typically, one year).

That is, starting from the known economic balance sheets at the beginning $t = 0$ of the period under consideration, we have to model the random values of all balance sheet positions at $t = 1$, or, to stress that we do not consider any new business that might be written in the next period, at $t = 1 - \varepsilon$. The required granularity is driven by the need to evaluate the limited liabilities of the interlinked balance sheets of the group. We have to distinguish between the obligations of different seniority and between the different group counterparties.

Accordingly, we collect the output of the stochastic model for the group into a collection B of balance sheet entries (excluding intragroup participations and net worth positions) describing the group at the end of the period:

$$B \equiv \{B_{i,j}^s : 1 \leq i \leq n, \quad 0 \leq j \leq n, \quad 0 \leq s \leq S\}$$

Here, each item has to reside on some balance sheet; we denote this by i . Also, as we want to account for the intragroup relations in detail, we have to specify the counterparty j for the item; group-external business partners being denoted by 0. The different seniority of obligations is accounted for by grouping the positions into a seniority class s (for formal reasons, we assign a seniority class of 0 to assets).

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We express the group structure by introducing the *participation matrix* $\Gamma \in [0,1]^{n \times n}$:

$$\Gamma = \begin{pmatrix} 0 & \Gamma_{1,2} & \cdots & \Gamma_{1,n} \\ \Gamma_{2,1} & 0 & \cdots & \Gamma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{n,1} & \Gamma_{n,2} & \cdots & 0 \end{pmatrix},$$

where $\Gamma_{i,j}$ is the share company i holds in company j . Note that the diagonal elements of this participation matrix vanish.

To express the relations between these balance sheet items we structure them into the following substructures. We start with putting the assets with group-external counterparties into a n -dimensional column vector:

$$A = \begin{pmatrix} B_{1,0}^0 \\ B_{2,0}^0 \\ \vdots \\ B_{n,0}^0 \end{pmatrix} > 0.$$

Next we consider the liabilities against group-external counterparties; here we have S vectors, since we have to differentiate the liabilities by their seniority class s :

$$L^{(s)} = \begin{pmatrix} B_{1,0}^s \\ B_{2,0}^s \\ \vdots \\ B_{n,0}^s \end{pmatrix} \leq 0, \quad 1 \leq s \leq S.$$

The remaining items $B_{i,j}^s$ describe intragroup transactions. Note that an intragroup asset on one balance sheet must correspond to some intragroup liability reported by its business partner; in this case we assign the seniority class of that liability to the corresponding asset on the other balance sheet as well. We structure these positions into antisymmetric $n \times n$ -matrices, one per seniority class:

$$R^{(s)} = \begin{pmatrix} 0 & B_{1,2}^s & \cdots & B_{1,n}^s \\ -B_{1,2}^s & 0 & \cdots & B_{2,n}^s \\ \vdots & \vdots & \ddots & \vdots \\ -B_{1,n}^s & -B_{2,n}^s & \cdots & 0 \end{pmatrix}, \quad 1 \leq s \leq S.$$

Here the matrix element $R_{i,j}^{(s)} \equiv B_{i,j}^s = -B_{j,i}^s$ quantifies the liability (if the value is negative) or the asset (if positive) of entity i against entity j .

This notation allows us to formulate an equation for the economic net worth of the group companies. This net worth G is the sum of the participation values, the group-external assets, the group-external liabilities and the intragroup transactions:

$$\begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{pmatrix} = \begin{pmatrix} 0 & \Gamma_{1,2} & \cdots & \Gamma_{1,n} \\ \Gamma_{2,1} & 0 & \cdots & \Gamma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{n,1} & \Gamma_{n,2} & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{pmatrix} + \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} + \sum_{s=1}^S \left(\begin{pmatrix} L_1^{(s)} \\ L_2^{(s)} \\ \vdots \\ L_n^{(s)} \end{pmatrix} + \begin{pmatrix} 0 & R_{1,2}^{(s)} & \cdots & R_{1,n}^{(s)} \\ -R_{1,2}^{(s)} & 0 & \cdots & R_{2,n}^{(s)} \\ \vdots & \vdots & \ddots & \vdots \\ -R_{1,n}^{(s)} & -R_{2,n}^{(s)} & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right),$$

which can be written more compactly as (u is a column of n ones that helps us to designate the row sum of the intragroup transaction matrix):

$$G = \Gamma \cdot G + A + \sum_{s=1}^S (L^{(s)} + R^{(s)} \cdot u)$$

3. PROPAGATING INSOLVENCIES THROUGH THE GROUP

When we look at the above equation we recognize that it is only valid if the net worth values are all positive, ie. $G \geq 0$. Otherwise we would potentially have negative participation values – which would violate the assumption of limited liabilities. Also, the value $R_{i,j}^{(s)} > 0$ of a claim against an insolvent balance sheet cannot be met in entirety and has to be adjusted.

To enforce these conditions we introduce *insolvency scalings* $0 \leq \theta_i^{(s)} \leq 1$. After a loss of the full economic net worth, the company will not be able to satisfy all obligations. The obligations with the lowest seniority will suffer first, more senior liabilities are reduced subsequently via applying scalings to the liabilities in order to account for the reduced value of claims in a seniority class $1 \leq s \leq S$ against an insolvent entity i . Only after complete exhaustion of some class s (i.e. $\theta_i^{(s)} = 1$) the next class $s - 1$ gets reduced.

Thus, introducing column vectors of those scalings per seniority class,

$$\theta^{(s)} = \begin{pmatrix} \theta_1^{(s)} \\ \theta_2^{(s)} \\ \vdots \\ \theta_n^{(s)} \end{pmatrix},$$

we adapt our equation to the case of insolvencies:

$$G = \Gamma \cdot G + A + \sum_{s=1}^S (\text{diag}[L^{(s)} + R_-^{(s)} \cdot u] + R_+^{(s)}) \cdot (1 - \theta^{(s)}),$$

where $\text{diag}[\dots]$ denotes the construction of a diagonal matrix out of a list of values, R_- and R_+ denote the negative and positive elements of R , respectively. This rewriting of terms allowed us to attach the relevant insolvency scaling (namely that of the entity having the liability) to both liabilities and intragroup assets.

Using these (yet undetermined) insolvency scalings we can enforce non-negativity of the participation values and the correct adjustment of the value of claims against insolvent entities. Now, as our enhanced equation is just a system of linear equations we can solve immediately for the column of net worth values:

$$G = (1 - \Gamma)^{-1} \cdot A + \sum_{s=1}^S (1 - \Gamma)^{-1} \cdot (\text{diag}[L^{(s)} + R_-^{(s)} \cdot u] + R_+^{(s)}) \cdot (1 - \theta^{(s)})$$

4. OBTAINING A NUMERICAL SOLUTION

So far, this is a formal solution only, as the insolvency scalings are yet unknown. Of course, once we have a solution we can check whether it is valid: any insolvent entity has to have zero net worth, and the seniority classes have to be exhausted in the correct order.

In the case of a single seniority class we can use *Linear Programming* to solve for the insolvency scalings. However, in the more relevant case of several seniority classes, we have to resort to some iterative solution algorithm, the basic idea of which we are going to describe now.

We start by considering the above expression as a function:

$$\begin{aligned} G : [0,1]^{n \times S} &\mapsto \mathfrak{R}^n \\ \theta &\rightarrow G(\theta) \end{aligned}$$

Now, for no insolvencies, ie. $\mathcal{G}_{(1)} = 0$, we compute $G^{(1)} = G(\mathcal{G}_{(1)})$. If there are any negative elements in this vector, we *individually* adjust the insolvency scalings for those entities (keeping in each the insolvency scalings over all *other* entities fixed). The updated set $\mathcal{G}_{(2)}$ then is used to compute the next iteration step, $G^{(2)} = G(\mathcal{G}_{(2)})$. We repeat until the scalings converge, resulting in the non-negative solution $G(\mathcal{G}_{(n)}) \geq 0$.

It is perhaps worthwhile to note that for sufficiently complex participation structures and networks of intragroup transactions this basic algorithm sometimes has to be enhanced, using some technical tricks, to achieve convergence.

5. ILLUSTRATION

To explore the ideas presented so far, let's look at the balance sheets of a group built out of three balance sheets. Here, we assumed a participation matrix

$$\Gamma = \begin{pmatrix} 0 & 1 & 0.5 \\ 0 & 0 & 0.5 \\ 0 & 0 & 0 \end{pmatrix}$$

i.e. 1 holds 2 and 50% of 3, and 2 holds 50% of 3 as well.

Now we define a small set of essential balance sheet items to describe this toy group; we list the balance sheet (b/s) an item is on, the business partner (b/p – 0 being group-external), the

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seniority class (s/c) and the value (ω is a [positive] parameter that will allow us to explore different situations):

b/s	b/p	s/c	value	description
1	0	0	100	external assets of 1
1	0	1	-20ω	external liabilities of 1
1	0	2	-30ω	external liabilities of 1
1	2	1	-12ω	i/g liability of 1 toward 2
1	3	1	-10ω	i/g liability 1 toward 3
2	0	0	70	external assets of 2
2	0	1	-10ω	external liabilities of 2
2	0	2	-20ω	i/g liability of 1 toward 2
2	1	1	12ω	i/g asset of 2 with 1
2	3	1	2ω	i/g asset of 2 with 3
3	0	0	20	external assets of 3
3	0	1	-5ω	external liabilities of 3
3	0	2	-10ω	external liabilities of 3
3	1	1	10ω	i/g asset of 3 with 1
3	2	1	-2ω	i/g liability of 3 toward 2

Assuming that there are no insolvencies (ie. $0 \leq \omega \leq 2$) we can immediately solve to obtain the net worth for the three entities:

b/s	net worth
1	$95(2 - \omega)$
2	$80 - \frac{39}{2}\omega$
3	$20 - 7\omega$

As an example, we show the balance sheet of entity 1 in the case of $\omega = 1.8$.

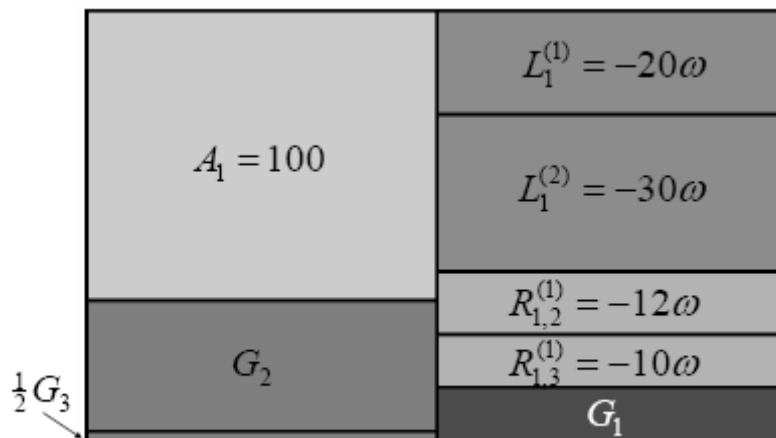


Illustration: Balance sheet of entity 1 (for $\omega=1.8$)

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To explore the behaviour in cases of insolvencies we start by introducing the (yet unknown) insolvency scalings:

b/s	b/p	s/c	value
1	0	0	100
1	0	1	$-20\omega(1-\theta_1^{(1)})$
1	0	2	$-30\omega(1-\theta_1^{(2)})$
1	2	1	$-12\omega(1-\theta_1^{(1)})$
1	3	1	$-10\omega(1-\theta_1^{(1)})$
2	0	0	70
2	0	1	$-10\omega(1-\theta_2^{(1)})$
2	0	2	$-20\omega(1-\theta_2^{(2)})$
2	1	1	$12\omega(1-\theta_1^{(1)})$
2	3	1	$2\omega(1-\theta_3^{(1)})$
3	0	0	20
3	0	1	$-5\omega(1-\theta_3^{(1)})$
3	0	2	$-10\omega(1-\theta_3^{(2)})$
3	1	1	$10\omega(1-\theta_1^{(1)})$
3	2	1	$-2\omega(1-\theta_3^{(1)})$

Again, we immediately obtain the formal solution

b/s	net worth
1	$95(2-\omega) + \omega(20\theta_1^{(1)} + 30\theta_1^{(2)} + 10\theta_2^{(1)} + 20\theta_2^{(2)} + 5\theta_3^{(1)} + 10\theta_3^{(2)})$
2	$80 - \frac{39}{2}\omega - 17\omega\theta_1^{(1)} + 10\omega\theta_2^{(1)} + 20\omega\theta_2^{(2)} + \frac{3}{2}\omega\theta_3^{(1)} + 5\omega\theta_3^{(2)}$
3	$20 - 7\omega - 10\omega\theta_1^{(1)} + 7\omega\theta_3^{(1)} + 10\omega\theta_3^{(2)}$

which for $0 \leq \omega \leq 2$ reproduces with zero insolvency scalings the result obtained above.

Let's now study the case of an insolvency; we choose $\omega = 3.5$. By going through the algorithm outlined in the previous section we arrive at the following result for the insolvency scalings

$$\begin{pmatrix} \theta_1^{(1)} & \theta_1^{(2)} \\ \theta_2^{(1)} & \theta_2^{(2)} \\ \theta_3^{(1)} & \theta_3^{(2)} \end{pmatrix} = \begin{pmatrix} 0.3143 & 1 \\ 0 & 0 \\ 0 & 0.4429 \end{pmatrix},$$

and thus at the following net worth values:

$$\begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.8 \\ 0 \end{pmatrix}$$

6. CONCLUSION

We have introduced a formalism that allows for a simple formal description of the interactions incurring in the network of economic balance sheets of a (re-) insurance group. In contrast to the assessment of risk in a consolidated view, for which models have been developed and used for more than a decade, the construction of a full balance-sheet entity model reflects the legal relationships within a group.

This formalism allows to respect limited liabilities in the network of legal entities. An algorithm has been outlined that is capable of efficiently providing numerical solutions, also for large groups and a substantial number of different seniority classes for the obligations; the method can therefore be integrated into an internal simulation model.

Clearly, there is a certain amount of boldness involved in assuming that modelling economic values only represents defaults adequately. More generally, one might point out that liquidity concerns will have to be addressed in a satisfactory way when developing this approach further.

Quantitative results obviously depend heavily on the concrete input parameters used, such as the group structure and the capitalisation of individual entities. It is difficult to come up with general statements, but obviously, in a well-capitalized group, annual outcomes leading to insolvencies should be rather infrequent. The overall picture will then not be too different from the consolidated view, the value of the default option being moderate.