Valuation and Hedging of Participating Life-Insurance Policies under Management Discretion

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Abstract

The valuation and hedging of participating life insurance policies, also known as with-profits policies, is considered. Such policies can be seen as European path-dependent contingent claims whose underlying security is the investment portfolio of the insurance company that sold the policy. The fair valuation of these policies is studied under the assumption that the insurance company has the right to modify the investment strategy of the underlying portfolio at any time. Furthermore, it is assumed that the issuer of the policy does not setup a separate portfolio to hedge the risk associated with the policy. Instead, the issuer will use its discretion about the investment strategy of the underlying portfolio to hedge shortfall risks. In that sense, the insurer's investment portfolio serves simultaneously as the underlying security and as the hedge portfolio. This means that the hedging problem can not be separated from the valuation problem. We investigate the relationship between risk-neutral valuation and hedging of these policies in complete and incomplete financial markets.

Keywords: Participating Life Insurance Policy, With-Profits contract, Profit-Sharing, Risk-Neutral Valuation, Hedging,

1 Introduction and Literature Review

Participating Life insurance contracts, also called with-profits contracts, have been issued over the past decades by many insurance companies throughout the world. Although the details of the contracts vary significantly between insurer's there are some common features that all with-profits contracts share. An overview about common forms of with-profits contracts in Europe and the United States is provided by Cummins, Miltersen & Persson (2004).We will consider with-profits contracts which are typical in the UK.

In March 2002 the Financial Services Authority in the UK published an issue paper related to the with-profits Review undertaken by the FSA (FSA 2002) in which some of the important features of with-profits contracts are summarized. In particular, they say that

> With-profits products are sold as long duration products and have certain features which normally include:
>
> - policyholder premiums are held in a pooled fund that is invested in a range of assets, a significant proportion of which are usually in the form of equities and property;
> - certain guarantees, which usually increase over the lifetime of the policy. For example, the payment of a guaranteed amount at maturity or retirement, or on death. The guaranteed amount may build through the duration of the contract by the addition of regular bonuses. A final bonus, which does not form part of this guaranteed amount, may be added at the end of the contract;

The FSA mentions a number of other features (including smoothing of guarantees and allowing the pooled fund to share in profits or losses of the insurers business). For the purpose of this paper we ignore these features.

We want to concentrate on the financial risk arising from the fact that "policyholder premiums are held in a pooled fund that is invested in a range of assets" and, we therefore ignore mortality. Instead, we assume that all contracts reach maturity. Furthermore, we assume that their is only one premium to be paid by the policyholder at the time the contract is issued.

Since we ignore mortality, buying a with-profits contract can be seen as an investment by the policyholder into the with-profits fund managed by the insurance company, but in contrast to standard investment funds, with-profits contracts provide some protection against low or negative returns. Instead of receiving the final value of the fund, the policyholder receives certain maturity benefits which are guaranteed by the insurer. We assume that the guaranteed maturity benefits increase during the lifetime of the contract and that the rate of increase depends on the performance of the with-profits fund.

In the literature we find two approaches to calculate fair market-consistent values of with-profit contracts and to derive hedging strategies that the insurance company can apply to protect itself against the risk associated with the given guarantees.

In the first approach authors treat the with-profits fund as a fixed investment portfolio, often called the reference portfolio. The price process of the fund is therefore a given stochastic process and the payoff to the policyholder at maturity is a deterministic function of the performance of the with-profits fund. The payoff is therefore a path-dependent European contingent claim. This approach allows for the direct application of methods known from financial mathematics. The first authors to use these methods to price life insurance contracts were Brennan & Schwartz (1976, 1979). These authors considered unit-linked contract for which the payoff to the policyholder is indeed a contingent claim with a payoff depending on the price of a unit of an externally given reference portfolio. Since then, Market-consistent valuation of participating insurance contracts has been studied by a number of authors. Among these are Persson & Aase (1997), Miltersen & Persson (1999), Miltersen & Persson (2003), Ballotta (2005). Overviews about the available literature can be found in Kleinow & Willder (2007) and Bauer, Kiesel, Kling & Ruß (2005) and the references therein. A very detailed discussion about the approaches and the results of different authors was carried out by Willder (2004).

In contrast to the assumption of a reference portfolio, a second approach to price and hedge with-profits contracts is based on the assumption that the management of the insurance company has full discretion about the composition of the with-profits fund. In particular, the insurer can use this discretion to reduce or increase the riskiness of the with-profits fund by changing the proportion of money invested into equity shares and the proportion invested into fixed-income securities. In this approach, maturity benefits are still increased according to the performance of the with-profits fund. Since the insurer has control about the with-profits fund any change in the composition of the fund will result in a change of the value of the with-profits insurance contract.

Hibbert & Turnbull (2003) where the first to address this issue. They consider an insurance company in which the management have limited discretion in choosing the assets by applying a fixed rule to increase or decrease the equity exposer of the with-profits fund depending on the value of the insurer's assets and the maturity guarantees already declared. They calculate the fair value of the with-profits contract for these fixed rules.

Kleinow & Willder (2007) consider a more realistic setting by assuming that the management of the insurer has the right to change their investment strategy whenever and however they want to. Any change in this portfolio strategy will lead to a change in the underlying price process that is used to calculate the growth rate of the guaranteed maturity benefits. The insurer is using this discretion to hedge the maturity benefits by making sure that the final value of the with-profits fund is equal to the maturity benefits declared during the lifetime of the contract. In that sense the with-profits fund serves simultaneously as the hedge portfolio and the underlying price process for the payoff at maturity. Kleinow & Willder (2007) have used binomial trees to model the financial market.

The purpose of this paper is to generalize the results by Kleinow & Willder (2007). We will particularly emphasis the relationship between hedging and valuation. Mathematically, we are faced with the following problem. We are given a function H, a random variable Γ and a stochastic process S_0. We want to find a stochastic process V such that its final value $V(T)$ is equal to

$$V(T) = \Gamma \prod_{t=0}^{T-1} H(V(t+1)/V(t))$$

and V/S_0 is a martingale with respect to a given filtration. This is a non-standard problem in financial mathematics since the contingent claim $\Gamma \prod_{t=0}^{T-1} H(V(t+1)/V(t))$ is not given but dependents on the "hedge-portfolio" V.

We start in Section 2 with setting the scene by introducing the financial market model used in the remainder of the paper. The with-profits contract is then described in detail in Section 3. The pricing and hedging problem is described in section 4. In Section 5 we discuss the existence of a self-sufficient with-profits fund that is used to solve the hedging and valuation problem, and show some of its properties. We continue in sections 6 and 7 with an investigation of the relationship between risk-neutral valuation and hedging. Finally, we provide an example in section 8 and conclude the paper in sections 9.

2 The Financial Market Model

Let $(\Omega, \mathcal{F}, \mathrm{P})$ denote a probability space, and let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ for $T \in \mathbb{N}$ be a right-continuous and complete filtration defined on this space. We assume that $\mathcal{F}_0$ is the trivial σ-field.

The tradable assets are a bank account and $d \in \mathbb{N}$ further risky assets. We assume that the value process S_0 of the bank account is adapted to $\mathbb{F}$, $S_0(0) = 1$ and S_0 has finite variation.

The price processes of the d risky assets are denoted by $S_1, \ldots, S_d$. We assume that these are semimartingales with respect to $\mathbb{F}$ under P.

The value of any portfolio consisting of the above assets at any time $t \in [0, T]$ is given by

$$V_\xi(t) = \sum_{i=0}^{d} \xi_i(t) S_i(t) \tag{1}$$

where $\xi = (\xi_0, \xi_1, \ldots, \xi_d)$ is predictable with respect to $\mathbb{F}$. The $(d+1)$-dimensional process ξ is called the portfolio strategy. $\xi_0(t)$ represents the number of units of the bank account and $\xi_i(t)$ for $i = 1, \ldots, d$ is the number of risky assets that belong to the portfolio at time t.

A portfolio strategy ξ is self-financing if

$$\begin{aligned} V_\xi(t) &= V_\xi(0) + \int_0^t \xi(u) dS(u) \\ &= V_\xi(0) + \sum_{i=0}^{d} \int_0^t \xi_i(u) dS_i(u) \qquad \forall\, t \in [0, T]. \end{aligned}$$

We assume that our model is arbitrage-free, and therefore the set of equivalent martingale measures

$$\mathcal{Q} = \{ \mathrm{Q} \sim \mathrm{P} : \ S_i/S_0 \text{ is a Q-martingale } \forall\, i = 1, \ldots, d \} \, . \tag{2}$$

in not empty. The price of any risky asset is therefore given as

$$S(u) = \mathrm{E}_{\mathrm{Q}} \left[\frac{S_0(u)}{S_0(t)} S(t) \Big| \mathcal{F}_u \right] \text{ for } u \leq t, \ \forall\, \mathrm{Q} \in \mathcal{Q} \, .$$

If a portfolio strategy ξ is self-financing during a time period $[u, t]$, we obtain that V_ξ / S_0 is also a martingale with respect to all $\mathrm{Q} \in \mathcal{Q}$ during $[u, t]$.

3 The Contract

We consider an insurance contract with maturity T. As mentioned in the introduction the payoff to the policyholder at maturity depends on the success of the insurer's investment strategy in the with-profits fund and some exogenously given contingent claim.

We introduce the with-profits fund in terms of its price process.

Definition 1 *A financial instrument with price process* $V = \{V(t), t \in [0,T]\}$ *is called* with-profits fund *if* V *is a* $\mathbb{F}$*-adapted stochastic process such that the discounted price process* V/S_0 *is a Q-martingale for at least one* $Q \in \mathcal{Q}$.

For a with-profits fund with value process V *we define the set*

$$\mathcal{Q}_V = \{Q \in \mathcal{Q} \,:\, V/S_0 \text{ is a Q-martingale}\}$$

and call it the with V *associated set of martingale measures.*

Since a with-profits fund is completely characterized by its value process V we do not distinguish between the two and sometimes call V the with-profits fund.

A with-profits fund V might be the value process of a self-financing portfolio ξ in which case $V = V_\xi$ is a Q-martingale for all Q $\in \mathcal{Q}$ and, therefore, $\mathcal{Q}_V = \mathcal{Q}$. However, definition 1 is general enough to include other financial instruments. Economically, this means that the insurer can invest into particular financial instruments which are not publicly traded, for example over-the-counter contingent claims. Many insurance companies are also large financial institution who can issue new financial derivatives. However, definition 1 ensures that no arbitrage opportunity is introduced to the financial market by the introduction of a with-profits fund.

To define a with-profits contract related to a with-profits fund V we consider a discrete-time process $\{X(t), t = 0, \ldots, T\}$ given by $X(0) = x_0$ and

$$\begin{aligned} X(t+1) &= X(t) H\left(\frac{V(t+1)}{V(t)}\right) \\ &= x_0 \prod_{s=0}^{t} H\left(\frac{V(s+1)}{V(s)}\right) \quad \text{for } t = 0, \ldots, T-1 \end{aligned} \tag{3}$$

for a positive real valued function $H : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and a constant $x_0 \in \mathbb{R}_+$.

Economically, we think of X as the maturity benefits to the policyholder declared at time t but payable at maturity T. Alternatively we can interpret $X(t)$ as the balance at time t of an account that the insurance company holds on behalf of the policyholder, but which the policyholder can access at maturity only. Since $X(T)$ is $\mathcal{F}_T$-measurable, it can be seen as a European path-dependent contingent claim where the underlying security is the with-profits fund V. For all $t < T$, $X(t)$ can be interpreted as the payoff to the policyholder if the contract were to mature at time t instead of T. Therefore, $X(t)$ should not be interpreted as a value in an economic sense for any $t < T$. Instead, it should be seen as the "intrinsic value" at time t of a European contingent claim with maturity T and payoff $X(T)$.

We call the function H contract function or bonus distribution mechanism since H describes how X changes depending on the performance of the with-profits fund V.

To generalize the payoff to the policyholder we assume that the policyholder receives a further derivative at maturity. A typical example for such a derivative is a guaranteed annuity option that allows the policyholder to use the final balance $X(T)$ of his account to buy an annuity for a price which was fixed at time 0, see for example Pelsser (2003). In general, the inclusion of such a derivative means that the final payoff to the policyholder is

$$X(T)\Gamma \text{ where } \Gamma \text{ is a } \mathcal{F}_T\text{-measurable random variable.}$$

To summarize these arguments we define a participating life insurance policy in the following way.

Definition 2 *Consider a constant $x_0 > 0$, a function $H : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and a $\mathcal{F}_T$-measurable random variable Γ. We call the triplet (x_0, H, Γ) a participating life insurance policy or with-profits contract for the with-profits fund V if the payoff to the policyholder at maturity is $X(T)\Gamma$ where $X(T)$ is the final value of the process X defined in (3). The constant x_0 is called nominal value, H is called contract function, and Γ is called terminal option of the contract.*

We make the following assumptions about the contract function H and the terminal option Γ

(A1) H is continuous and non-decreasing,

(A2) $H(0) > 0$ and $\Gamma \geq \varepsilon > 0$ almost surely,

(A3) $y/H(y)$ is strictly increasing and $\lim_{y\to\infty} y/H(y) = \infty$.

Assumption (A1) means that the policyholder participates in the success of the investment strategy applied to the with-profits fund, that is the higher the return on the with-profits fund in a particular year, the larger is the growth rate of the policyholder's account in that year, and therefore, the larger are the maturity benefits declared up to time t.

Since we also assume (A2), the policyholder is guaranteed to obtain a positive payout at maturity. The lower boundary for the payoff at maturity is given by $X(0)H(0)^T\varepsilon$. In most realistic cases $\varepsilon \geq 1$, that is, the payoff to the policyholder can only be increased through Γ but is at least equal to the final balance $X(T)$ of the policyholder's account.

Assumption (A3) is needed to prove the results in the following sections. However, there is also an economic interpretation of (A3). Assuming an initial value $V(t)$ at the beginning of the period $[t, t+1]$ and denoting the return on V during that period by $r_V(t) = \log V(t+1) - \log V(t)$ we find that $H(\exp r_V(t))/\exp r_V(t)$ is the fraction of the growth of V that is distributed to policyholders in the form of maturity benefits at the end of the period. Assumption (A3) means that this fraction is decreasing in r_V. Therefore, the better the return on V the higher is the proportion of the return that is used to provide protection against possible low returns in the future or to compensate for low returns in the past.

Therefore, policyholders participate strongly in the return of V if the return is low. But for very high returns the policyholders' participation is relatively low and therefore the insurer keeps a larger fraction of the profits. This should give an incentive to the management of the insurance company to achieve a high return during each period.

Remark 1: A more general participating contract could have the form

$$X(T) = \mathcal{H}(V(t),\ t \in [0,T])$$

where $\mathcal{H}$ is a functional mapping the space of functions on $[0,T]$ into $\mathbb{R}$. In this paper we will not consider such general contracts. □

Remark 2: A typical example for a contract function H considered by Kleinow & Willder (2007) and others is

$$H(y) = \max\left\{e^{\gamma}, y^{\delta}\right\} \tag{4}$$

where γ and $\delta \in (0,1)$ are constant. The growth rate of X during $[t, t+1]$ is then

$$r_X(t+1) = \log X(t+1) - \log X(t) = \max\{\gamma, \delta r_V(t+1)\}$$

where $r_V(t+1) = \log V(t+1) - \log V(t)$ is the return on the with-profits fund V during the same period. The growth rate $r_X(t+1)$ of X is therefore the maximum of a guaranteed rate γ and the return $r_V(t+1)$ of the portfolio V during this period multiplied by a participation rate δ. This particular example is considered in more detail in section 8. □

Since we want to concentrate on the financial risk we assume that the policyholder does not have a surrender option and there is no mortality risk. Therefore, the policyholder will receive $X(T)\Gamma$ at maturity T. Some remarks about a surrender option can be found in section 6.

4 The Valuation and Hedging Problem

Since $X(T)\Gamma$ is $\mathcal{F}_T$-measurable, the fair price of the contract is given by $\mathrm{E}_{\mathrm{Q}}[S_0(T)^{-1}X(T)\Gamma|\mathcal{F}_0]$ where $\mathrm{Q} \in \mathcal{Q}$ is the martingale measure chosen by the insurance company to price financial assets. However, the calculation of this expectation requires knowledge about the with-profits fund V.

There are two ways to proceed. If we assume that the with-profits fund is an exogenously given financial security, for example the value process of a known and fixed portfolio consisting of the assets $S_0, \ldots, S_d$, the with-profits contract can be priced by directly applying risk-neutral valuation methods. This approach has been applied by Bauer et al. (2005) and Ballotta (2005) among others. These authors assume that both the with-profits contract (x_0, H, Γ) and the with-profits fund V, usually referred to as the reference portfolio, are given at outset. With this approach it is possible to hedge the risk associated with the with-profits contract if $X(T)\Gamma$ is an attainable contingent claim, that is, there exists a self-financing portfolio strategy ξ such that the final value of that portfolio $V_\xi(T) = X(T)\Gamma$. In this case $V_\xi(0)$ will be equal to the risk-neutral price of the contract. In general $V_\xi(0)$ will not be equal to the initial value of the with-profits fund $V(0)$.

However, for most with-profits contracts the with-profits fund is not a fixed reference portfolio. Instead, the insurer's management has often full discretion about the investment strategy applied to the with-profits fund. Empirical results found by a working party of the UK actuarial profession, suggest that many UK insurance companies are ready to exercise their right to change the investment strategy in with-profits funds, see Pike (2006). In particular, many insurers will decrease the equity exposer of their with-profits funds to zero if they think this is necessary. The right of the insurer's management to change the investment strategy in the with-profits fund at any time $t \in [0, T]$ is a feature of with-profits contracts that substantially distinguishes with-profits contracts from other financial derivatives and makes it virtually impossible to apply risk-neutral valuation methods directly.

We will incorporate this particular feature of with-profits contracts and assume in the following that the insurance company has the right to choose the with-profits fund and to rebalance it at any time $t \in [0, T]$ in a self-financing way. This means that the insurer cannot inject extra cash into or withdraw money from the fund at any time $t > 0$. In that sense the insurance company can choose the underlying financial security of the with-profits contract. However, we will assume that the with-profits contract (x_0, H, Γ) is fixed.

The insurance company wishes to hedge the risk associated with the contract (x_0, H, Γ), i.e. the European contingent claim $X(T)\Gamma$. Since the insurance company has complete discretion about investment decisions in the with-profits fund we are interested in whether the insurance company can actually choose the with-profits fund in way such that the final value of the with-profits fund is equal to the liabilities from the with-profits contract. In that case there is no need to invest some of the premium paid by the policyholder into an external hedge portfolio. Instead the insurer can invest the full premium into the with-profits fund. In contrast to the hedging of an option with a payoff depending on a given stochastic process, the portfolio V serves simultaneously as underlying process and hedge-portfolio.

This motivates the following definition of self-sufficiency.

Definition 3 *A with-profits fund V is called:*

1. self-sufficient *for the with-profits contract* (x_0, H, Γ) *if*

$$P[V(T) = X(T)\Gamma] = 1 \qquad \text{for} \qquad X(T) = x_0 \prod_{t=1}^{T} H\left(\frac{V(t)}{V(t-1)}\right)$$

2. *attainable if there exist a portfolio strategy ξ such that*

$$V(t) = V(0) + \int_0^t \xi(u)dS(u) \; .$$

3. *If V is a with-profits fund and (x_0, H, Γ) is a with-profits contract we call the process C defined by*

$$C(t) = V(t)/X(t) \qquad \forall\, t = 0, 1, \dots, T$$

the relative value of the with-profits fund V with respect to (x_0, H, Γ)..

The aim for the remainder of the paper is to investigate the existence and structure of a self-sufficient with-profits fund, and to discuss its properties.

5 The Self-Sufficient With-Profits Fund

We start with the last period $[T-1, T]$ and obtain that $V(T) = X(T)\Gamma$ if and only if $\Gamma = C(T) = V(T)/X(T)$ and therefore

$$C(T) = \frac{V(T)}{X(T)} = \frac{V(T)}{X(T-1)H\big(V(T)/V(T-1)\big)} \tag{5}$$

$$= g\big(X(T-1), V(T-1), V(T)\big) \tag{6}$$

with
$$g(x, v_0, v) = \frac{v}{xH\big(v/v_0\big)}. \tag{7}$$
Note that g has the property
$$g(x, v_0, v) = g(\alpha x, \alpha v_0, \alpha v) \qquad \forall\, \alpha \neq 0. \tag{8}$$
We obtain from assumption (A3) that for any fixed x and $v_0 \neq 0$ there exists the inverse function $g^{-1}(x, v_0, .)$ of g with
$$c = g(x, v_0, v) \iff v = g^{-1}(x, v_0, c). \tag{9}$$
It follows that (5) holds almost surely if and only if
$$V(T) = g^{-1}(X(T-1), V(T-1), C(T)) \text{ a.s.} \tag{10}$$
From (8) we obtain that $g^{-1}(x, v_0, .)$ has the property
$$\alpha g^{-1}(x, v_0, c) = g^{-1}(\alpha x, \alpha v_0, c) \qquad \forall\, \alpha \neq 0 \tag{11}$$
and, in particular, for $x \neq 0$ and $\alpha = 1/x$, g satisfies
$$\frac{1}{x} g^{-1}(x, v_0, c) = g^{-1}\left(1, \frac{v_0}{x}, c\right).$$

Using Definition 3 we obtain for a self-sufficient with-profits fund V and its relative value process C that $C(T) = V(T)/X(T) = \Gamma$ almost surely and
$$\begin{aligned} C(T-1) &= \frac{V(T-1)}{X(T-1)} \qquad\qquad (12)\\ &= \mathrm{E}_{\mathrm{Q}}\left[\frac{S_0(T-1)}{S_0(T)}\frac{V(T)}{X(T-1)} \;\Big|\; \mathcal{F}_{T-1}\right] \\ &= \mathrm{E}_{\mathrm{Q}}\left[\frac{S_0(T-1)}{S_0(T)}\frac{g^{-1}\left(X(T-1), V(T-1), C(T)\right)}{X(T-1)} \;\Big|\; \mathcal{F}_{T-1}\right] \\ &= \mathrm{E}_{\mathrm{Q}}\left[\frac{S_0(T-1)}{S_0(T)} g^{-1}\left(1, \frac{V(T-1)}{X(T-1)}, C(T)\right) \;\Big|\; \mathcal{F}_{T-1}\right] \end{aligned}$$
for every $\mathrm{Q} \in \mathcal{Q}_V$. Using Definition 3 again we conclude that the relative value $C(T-1) = V(T-1)/X(T-1)$ of a self-sufficient with-profits fund with respect to (x_0, H, Γ) at time $T-1$ must be the solution of the equation
$$C(T-1) = \mathrm{E}_{\mathrm{Q}}\left[\frac{S_0(T-1)}{S_0(T)} g^{-1}\Big(1, C(T-1), C(T)\Big) \;\Big|\; \mathcal{F}_{T-1}\right] \qquad \forall\, \mathrm{Q} \in \mathcal{Q}_V$$

with $C(T) = \Gamma$. The value $V(T-1)$ of a self-sufficient with-profits fund for the policy (x_0, H, Γ) at time $T-1$ is $V(T-1) = X(T-1)C(T-1)$. By multiplying (12) with $X(T-1)$ and using (11) we find that $V(T-1)$ then fulfills the equation

$$V(T-1) = \mathrm{E}_{\mathrm{Q}}\left[\frac{S_0(T-1)}{S_0(T)} g^{-1}\Big(X(T-1), V(T-1), C(T)\Big) \,\Big|\, \mathcal{F}_{T-1}\right]$$

for all $\mathrm{Q} \in \mathcal{Q}_V$.

Using backward induction we obtain the following theorem that characterizes a self-sufficient with-profits fund in terms of its relative value process C.

Theorem 1 *We assume that assumptions (A1) – (A3) are fulfilled. If a stochastic process V is the value process of a self-sufficient with-profits fund for a policy (x_0, H, Γ) then the corresponding relative value process $C(t) = V(t)/X(t)$ is an $\mathbb{F}$-adapted process for which*

(C1) $C(T) = \Gamma$ *a.s.*

and for all $Q \in \mathcal{Q}_V$ holds

(C2) $C(t) = E_Q\left[\frac{S_0(t)}{S_0(t+1)} g^{-1}\Big(1, C(t), C(t+1)\Big) \,\Big|\, \mathcal{F}_t\right] \;\forall\, t = 0, \ldots, T-1.$

Proof of Theorem 1: $V(T) = X(T)\Gamma \iff C(T) = \Gamma$ by definition.

Since V/S_0 is a martingale and $C(T) = \Gamma$ we obtain from (10) and the properties of g that for $t = T$ and $Q \in \mathcal{Q}_V$ holds

$$\begin{aligned}
C(t-1) &= \frac{V(t-1)}{X(t-1)} \\
&= \mathrm{E}_{\mathrm{Q}}\left[\frac{S_0(t-1)}{S_0(t)} \frac{V(t)}{X(t-1)} \,\Big|\, \mathcal{F}_{t-1}\right] \\
&= \mathrm{E}_{\mathrm{Q}}\left[\frac{S_0(t-1)}{S_0(t)} \frac{1}{X(t-1)} g^{-1}\left(X(t-1), V(t-1), C(t)\right) \,\Big|\, \mathcal{F}_{t-1}\right] \\
&= \mathrm{E}_{\mathrm{Q}}\left[\frac{S_0(t-1)}{S_0(t)} g^{-1}\left(1, \frac{V(t-1)}{X(t-1)}, C(t)\right) \,\Big|\, \mathcal{F}_{t-1}\right] \\
&= \mathrm{E}_{\mathrm{Q}}\left[\frac{S_0(t-1)}{S_0(t)} g^{-1}\left(1, C(t-1), C(t)\right) \,\Big|\, \mathcal{F}_{t-1}\right]
\end{aligned}$$

This shows that (C2) is fulfilled at time $T-1$. Using the same arguments and backward induction proves the assertion. $\square$

In the next step we show how we can construct a self-sufficient fund.

Theorem 2 *We consider a with-profits policy (x_0, H, Γ) and a financial market modelled by a probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$. We assume that assumptions (A1) – (A3) are fulfilled. If a stochastic process C with properties (C1) and (C2) in Theorem 1 exists for a measure $Q \in \mathcal{Q}$ then there exists a self-sufficient with-profits fund V for the policy (x_0, H, Γ) and $Q \in \mathcal{Q}_V$. The value process V for this fund is given by:*

1. $V(0) = x_0 C(0)$,
2. $V(t) = g^{-1}(X(t-1), V(t-1), C(t))$ *and* $X(t) = X(t-1)H(V(t)/V(t-1))$ *for all* $t = 1, \ldots, T$, *and*
3. $V(s) = E_Q\left[\frac{S_0(s)}{S_0(t)} V(t) \mid \mathcal{F}_s\right]$ *for all* $s \in (t-1, t]$ *and all* $t = 1, \ldots, T$.

Proof of Theorem 2: From the construction of V and X in part 2 of the theorem and the definition of g and g^{-1} we obtain immediately that

$$C(t) = g(X(t-1), V(t-1), V(t)) = \frac{V(t)}{X(t-1)H\big(V(t)/V(t-1)\big)} = \frac{V(t)}{X(t)}$$

Given a measure $\mathrm{Q} \in \mathcal{Q}$ and a process C that has properties (C1) and (C2) in Theorem 1 for Q we have to show that V/S_0 is a Q-martingale. Using property (C2) in Theorem 1, (11) and the construction of V in part 2 of the theorem we obtain

$$\begin{aligned} V(t-1) &= X(t-1)C(t-1) \\ &= \mathrm{E_Q}\left[\frac{S_0(t-1)}{S_0(t)} X(t-1) g^{-1}(1, C(t-1), C(t)) \,\Big|\, \mathcal{F}_{t-1}\right] \\ &= \mathrm{E_Q}\left[\frac{S_0(t-1)}{S_0(t)} g^{-1}(X(t-1), V(t-1), C(t)) \,\Big|\, \mathcal{F}_{t-1}\right] \\ &= \mathrm{E_Q}\left[\frac{S_0(t-1)}{S_0(t)} V(t) \,\Big|\, \mathcal{F}_{t-1}\right] \end{aligned}$$

□

The next lemma is required to prove the existence of a self-sufficient with-profits fund.

Lemma 1 *We consider a with-profits policy (x_0, H, Γ) and a financial market modelled by a probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$. If assumptions (A1) – (A3) are fulfilled, then there exists for each $Q \in \mathcal{Q}$ a stochastic process $C = C_Q$ that has the properties (C1) and (C2) in Theorem 1.*

Proof of Lemma 1: We fix a measure $\mathrm{Q} \in \mathcal{Q}$. We consider any $t \in \{1, 2, \dots, T\}$ and define

$$\mathrm{Q}_t[A] = \mathrm{E}_{\mathrm{Q}}\left[\mathbb{I}_A \,\middle|\mathcal{F}_{t-1}\right] \qquad \forall\, A \in \mathcal{F}_t$$

Given $\mathcal{F}_{t-1}$, Q_t is a probability measure. To prove the assertion it is sufficient to show that there exists a $c_0 \in \mathbb{R}_+$ such that

$$c_0 = \mathrm{E}_{\mathrm{Q}_t}\left[Dg^{-1}(1, c_0, C)\right] \tag{13}$$

where the discount factor $D = S_0(t)/S_0(t-1)$ and C are $\mathcal{F}_t$ measurable random variables. Equation (13) holds if and only if

$$1 = \mathrm{E}_{\mathrm{Q}_t}\left[D\frac{1}{c_0}g^{-1}(1, c_0, C)\right] = \mathrm{E}_{\mathrm{Q}_t}\left[Dg^{-1}\left(\frac{1}{c_0}, 1, C\right)\right] \tag{14}$$

Using the notation

$$G(v) = \frac{v}{H(v)}$$

we get

$$g\left(\frac{1}{c_0}, 1, v\right) = c_0 G(v) \text{ and } g^{-1}\left(\frac{1}{c_0}, 1, C\right) = G^{-1}(C/c_0)$$

where G^{-1} is the inverse function of G and g^{-1} was defined in (9). It follows from Assumption (A3) that $G^{-1} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is strictly increasing with

$$\lim_{c\to\infty} G^{-1}(c) = \infty \text{ and } \lim_{c\to 0} G^{-1}(c) = 0\ .$$

For a strictly increasing sequence $\{c_n\}_{n=1,2,\dots}$ of positive real numbers with $c_n \to \infty$ we conclude with the monotone convergence theorem that

$$\begin{aligned}\lim_{c_n\to\infty} \mathrm{E}_{\mathrm{Q}_t}\left[Dg^{-1}\left(\frac{1}{c_n}, 1, C\right)\right] &= \lim_{c_n\to\infty} \mathrm{E}_{\mathrm{Q}_t}\left[DG^{-1}(C/c_n)\right] \\ &= \mathrm{E}_{\mathrm{Q}_t}\left[D \lim_{c_n\to\infty} G^{-1}(C/c_n)\right] = 0\end{aligned}$$

Similarly, for a strictly decreasing sequence $\{c_n\}_{n=1,2,\dots}$ of positive real numbers with $c_n \to 0$ we find that

$$\begin{aligned}\lim_{c_n\to 0} \mathrm{E}_{\mathrm{Q}_t}\left[Dg^{-1}\left(\frac{1}{c_n}, 1, C\right)\right] &= \lim_{c_n\to 0} \mathrm{E}_{\mathrm{Q}_t}\left[DG^{-1}(C/c_n)\right] \\ &= \mathrm{E}_{\mathrm{Q}_t}\left[D \lim_{c_n\to 0} G^{-1}(C/c_n)\right] = \infty\end{aligned}$$

Since G^{-1} is continuous, $\mathrm{E}_{\mathrm{Q}_t}\left[Dg^{-1}\left(\frac{1}{c_n}, 1, C\right)\right]$ is continuous in c_0 and an application of the intermediate value theorem completes the proof. $\square$

Theorem 3 *We consider a with-profits policy* (x_0, H, Γ) *and a financial market modelled by a probability space* $(\Omega, \mathcal{F}, P, \mathbb{F})$. *If assumptions (A1) – (A3) are fulfilled, then there exists for each* $Q \in \mathcal{Q}$ *a self-sufficient with-profits fund* $V = V_Q$.

Proof of Theorem 3: The assertion is a direct consequence of Lemma 1 and Theorem 2. □

Theorem 3 ensures the existence of a with-profits fund for any martingale measure $Q \in \mathcal{Q}$. However, it is not clear from the theorem whether the insurer can actually invest into this with-profits fund. We can have two possible scenarios: firstly, the insurer can invest into the fund because V is attainable or the insurer can buy an over-the-counter security with price process V from another financial institution, or, secondly the insurer cannot invest into the with-profits fund V.

6 The Self-Sufficient Fund is Available

If the self-sufficient with-profits fund is an available investment opportunity for the insurance company, the insurer should invest into the fund V. The required initial investment is $V(0)$ which should be the premium for the policy (x_0, H, Γ). In this case we call V the fair value process of (x_0, H, Γ).

Definition 4 *A self-sufficient fund* V *for the with-profits policy* (x_0, H, Γ) *is called the fair value process of* (x_0, H, Γ) *if* V *forms an investment opportunity for the insurance company, that is,* V *is either attainable or the insurer can buy a financial instrument with value process* V *from a third party.*

The process $C = V/X$ *is then called the fair relative value process.*

To call the V the fair value process is justified in this case for the following reasons. Let us assume the policyholder pays a premium $\bar{V}(0) > V(0)$ at time 0. Then there exists an arbitrage opportunity for the insurance company, since the insurance company can invest $\bar{V}(0)$ into the self-sufficient fund and the value of the fund will be $\bar{V}(t)$ at any time t. The return on the fund is independent of its size and therefore, the fund with value process $\bar{V}$ has the same return as the fund with value process V and therefore the insurance company will make a profit $\bar{V}(T) - V(T)$ at time T since $\bar{V}(T) > V(T) = X(T)$. We therefore, conclude that the initial value $V(0)$ of the self-sufficient with-profits fund is an upper limit for the price of the policy.

On the other hand, since the policyholder has no discretion about the investment decisions by the insurance company, an initial price of less than

$V(0)$ does not necessarily lead to an arbitrage opportunity for the policyholder. However, if the initial premium is less than $V(0)$ the insurer faces a shortfall risk without an adequate compensation (risk premium). With an initial capital less than $V(0)$ the insurance company can not make the required investment into the with-profits fund V. The final value of the insurer's portfolio at maturity might therefore not be sufficient to pay for the maturity benefits $X(T)\Gamma$. Therefore, $V(0)$ is the minimum amount required at time 0 to successfully hedge the payoff of the policy.

Let us consider the existence of a surrender option for the policyholder. This option gives the policyholder the right to "sell back" the insurance contract to the insurance company at any time $t \leq T$ before maturity. To make this transaction fair, the price for which the contract can be sold back should be equal to the fair value $V(\tau)$ of the policy at the time of surrender τ. Therefore, the policyholder should receive $V(\tau)$, his "asset-share" at the time of surrender τ. Economically, this can be compared to standard European options which are traded for their market price but not for their "intrinsic value".

Let us make some more remarks about the structure of the fair value process V. Given the fair relative value process C and the fair value process V the payoff $X(T)\Gamma$ of the policy (x_0, H, Γ) can be replicated by a sequence of contingent claims. At any time $t \in \{0, \dots, T-1\}$ the replicating portfolio consists of a contingent claim that matures at time $t+1$ and has the payoff $g^{-1}(X(t), V(t), C(t+1))$. The fair value of this claim at time t is $V(t)$. If V is attainable, there exists a portfolio strategy ξ such that

$$\begin{aligned} g^{-1}(X(t), V(t), C(t+1)) &= V(t+1) \\ &= V(t) + \int_t^{t+1} \xi(u) dS(u) \\ &= g^{-1}(X(t-1), V(t-1), C(t)) + \int_t^{t+1} \xi(u) dS(u) \end{aligned}$$

for all $t = 1, \dots, T-1$ with $X(0) = x_0$. The hedging problem is therefore reduced to the hedging of the contingent claim $g^{-1}(X(t), V(t), C(t+1))$ during each of the periods $[t, t+1]$ for $t = 0, \dots, T-1$. The hedging strategies applied in each period are then combined to a hedging strategy ξ of the policy (x_0, H, Γ).

If V is not attainable, definition 4 means that the insurer is able to buy the contingent claims with payoff $g^{-1}(X(t), V(t), C(t+1))$ at any time $t = 0, \dots, T-1$ from a third party. The financial risk involved in the with-profits policy is therefore transferred to another financial institution.

7 The Self-Sufficient Fund is Not Available

If the insurance company can not invest into a portfolio or a financial derivative that has the same value process as the required self-sufficient fund V, this process V can not be the underlying value process of the payoff $X(T)\Gamma$. Instead, the insurer will have to invest the premiums paid by the policyholder into a portfolio ξ with value process V_ξ with

$$V_\xi(t) = V_\xi(0) + \int_0^t \xi(u)dS(u) \qquad \forall\, t \in [0,T].$$

Given this portfolio, the fair value at time 0 of the payoff

$$X_\xi(T)\Gamma$$

with

$$X_\xi(T) = x_0 \prod_{t=1}^{T} H\left(\frac{V_\xi(t)}{V_\xi(t-1)}\right) \tag{15}$$

should be calculated as the expected discounted value of the payoff under the martingale measure $\mathrm{Q} \in \mathcal{Q}$ that the insurer has chosen for valuation purposes. There are several ways to choose a measure for the risk-neutral valuation of contingent claims. However, we do not discuss the choice of a particular martingale measure here.

Since $V \neq V_\xi$, the expected discounted value of the payoff is, in general, not equal to $V(0)$, i.e.

$$\mathrm{E}_{\mathrm{Q}}[V_\xi(T)/S_0(T)] \neq \mathrm{E}_{\mathrm{Q}}[V(T)/S_0(T)] = V(0).$$

Therefore, $V(0)$ is not the fair value of the policy (x_0, H, Γ) at time 0. A similar argument holds for all times $t = 0, \ldots, T-1$.

Comparing these arguments to the situation in section 6, we see that the existence of a hedging strategy is closely related to risk-neutral valuation because only if the self-sufficient fund is an available investment opportunity for the insurer its initial value is the risk-neutral price of the with-profits policy.

However, even if V is not the fair value process, it is a lower bound for the value of a super-hedging strategy.

Lemma 2 *Let (x_0, H, Γ) be a participating policy and let V be a self-sufficient with-profits fund for (x_0, H, Γ). Let V_ξ be the price process of a self-financing portfolio ξ, i.e.*

$$V_\xi(t) = V_\xi(0) + \int_0^t \xi(u)dS(u) \qquad \forall\, t \in [0,T]\ .$$

If

$$P[V_\xi(T) \geq X_\xi(T)\Gamma] \;=\; 1 \tag{16}$$

where $X_\xi(T)$ is defined in (15) then $V_\xi(0) \geq V(0)$.
If in addition

$$P[V_\xi(T) > X_\xi(T)\Gamma] \;>\; 0 \tag{17}$$

then $V_\xi(0) > V(0)$.

Proof of Lemma 2: We only prove the result for a policy with maturity $T = 1$ and $x_0 = 1$. The result for policies with maturity $T > 1$ follows with a backward induction argument.
Firstly note that it follows from assumption (A1) that $g(1, v_0, v) = v/H(v/v_0)$ is a non-decreasing function in v_0. Assume now that $V_\xi(0) < V(0)$.

On $\{V_\xi(1) \geq X_\xi(1)\Gamma\}$ we have

$$\Gamma \leq \frac{V_\xi(1)}{X_\xi(1)} = g(1, V_\xi(0), V_\xi(1))$$

and, since V is self-sufficient for $(1, H, \Gamma)$,

$$g(1, V(0), V(1)) = \Gamma \leq g(1, V_\xi(0), V_\xi(1)) \leq g(1, V(0), V_\xi(1))$$

It now follows from assumption (A3) and (16) that for any $\mathrm{Q} \in \mathcal{Q}$ holds

$$\mathrm{Q}[V(1) \leq V_\xi(1)] = 1 \tag{18}$$

which is a contradiction to the assumption $V_\xi(0) < V(0)$ since V_ξ/S_0 and V/S_0 are martingales wrt. Q.

On $\{V_\xi(1) > X_\xi(1)\Gamma\}$ we have

$$\Gamma < g(1, V_\xi(0), V_\xi(1))$$

and

$$g(1, V(0), V(1)) = \Gamma < g(1, V_\xi(0), V_\xi(1)) \leq g(1, V(0), V_\xi(1))$$

It now follows from assumption (A3) that

$$\mathrm{Q}[V(1) < V_\xi(1)] > 0. \tag{19}$$

(19) and (18), the martingale property of V/S_0 and V_ξ/S_0 and the equivalence of Q and P prove the assertion. $\square$

The following theorem about the initial value of a super-hedging strategy follows directly from Lemma 2.

Theorem 4 *We consider a with-profits policy (x_0, H, Γ) and a financial market modelled by a probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$. We assume that assumptions (A1) – (A3) are fulfilled.*

Let V_ξ be the price process of a self-financing portfolio ξ, i.e.

$$V_\xi(t) = V_\xi(0) + \int_0^t \xi(u) dS(u) \qquad \forall\, t \in [0, T],$$

and let $X_\xi(T)$ be as defined in (15). We define the set $\mathcal{V}$ of all self-sufficient funds for the policy (x_0, H, Γ) as

$$\mathcal{V} = \Big\{ V \ : \quad \begin{array}{l} V/S_0 \text{ is a } Q\text{-martingale for some } Q \in \mathcal{Q} \text{ and} \\ V(T) = x_0 \prod_{t=1}^{T} H\left(\frac{V(t)}{V(t-1)}\right)\Gamma \end{array} \Big\}$$

If $P[V_\xi(T) \geq X_\xi(T)\Gamma] = 1$ then

$$V_\xi(0) \geq \sup_{V \in \mathcal{V}} V(0).$$

Since a perfect hedge of the participating policy might not exist in an incomplete financial market we suggest to use the following definition for the risk-neutral value of a participating policy.

Definition 5 *We consider a with-profits policy (x_0, H, Γ).*

1. *The process $V = \{V(t), t \in [0, T]\}$ is called risk-neutral value process of the policy (x_0, H, Γ) under the measure $Q \in \mathcal{Q}$ if*

   (a) there exists a self-financing portfolio strategy ξ with value process V_ξ such that

   $$V(t) = V_\xi(t) = V(0) + \int_0^t \xi(u) dS(u) \qquad \forall\, t \in [0, T]$$

   and

   (b) $E_Q\left[\frac{1}{S_0} X(T)\Gamma\right] = E_Q\left[\frac{1}{S_0} V(T)\right] = V(0)$

   with $X(T) = x_0 \prod_{t=1}^{T} H\left(\frac{V(t)}{V(t-1)}\right)$

2. *If $V(t)$ is the risk-neutral value of (x_0, H, Γ) then $C(t) = V(t)/X(t)$ is called the risk-neutral relative value of (x_0, H, Γ).*

The study of the existence and uniqueness of a risk-neutral value process is left for further research. However, if for a given measure $\mathrm{Q} \in \mathcal{Q}$ a risk-neutral value process exists we can consider two situations.

Firstly, if we find a risk-neutral value process for which $\mathrm{Q}[V(T) = X(T)\Gamma] = 1$, then $V(T)$ is equal to $X(T)\Gamma$ P-almost surely since Q is equivalent to P, This means we are back in the situation of a complete market.

If $\mathrm{Q}[V(T) = X(T)\Gamma] \neq 1$ then $\mathrm{Q}[V(T) < X(T)\Gamma] > 0$ since

$$\mathrm{E}_{\mathrm{Q}}\left[(1/S_0)(V(T) - X(T)\Gamma)\right] = 0.$$

This means that the insurer faces the risk of making a loss. To avoid the loss in this situation without investing into a super-hedging strategy, the insurer could sell the risk to a third party. This could be done by buying a financial instrument with a value process V and $V(T) = X(T)C$ from a third party. This approach is discussed in section 8 for the particular contract studied there. However, this approach means that the hedging problem is transferred from the insurance company to a third party. In other words, the market is made complete by the introduction of these derivatives.

8 Example

We consider the with-profits policy $(1, H, 1)$ with

$$H(v) = \max\left\{e^{\gamma}, v^{\delta}\right\}$$

for $\delta \in (0,1)$ and $\gamma \in \mathbb{R}$. Note that the terminal option in this example is $\Gamma \equiv 1$ and therefore the payoff to the policyholder is $X(T)$.

In this example, the maturity benefits $X(t)$ declared up to time t are given by

$$\begin{aligned} X(t) &= X(t-1)H\left(\frac{V(t)}{V(t-1)}\right) \\ &= X(t-1)\max\left\{e^{\gamma}, \left(\frac{V(t)}{V(t-1)}\right)^{\delta}\right\} \\ &= X(t-1)e^{r_X(t)} \quad \text{with} \quad r_X(t) = \max\left\{\gamma, \delta r_V(t)\right\} \qquad (20) \end{aligned}$$

where

$$r_V(t) = \log V(t) - \log V(t-1)$$

denotes the growth rate of the with-profits fund V. Therefore, the growth rate of X during the period $[t-1, t]$

$$r_X(t) = \log X(t) - \log X(t-1)$$

is given in (20) as the maximum of a guaranteed rate γ and the return of the with-profits fund during each period multiplied by a participation rate δ.

For the particular function H defined above the function g defined in (7) is explicitly given by

$$g(x, v_0, v) = \frac{v}{xH(v/v_0)} = \frac{1}{x}\begin{cases} e^{-\gamma}v & \text{for} \quad v < e^{\gamma/\delta}v_0 \\ v^{1-\delta}v_0^{-\delta} & \text{for} \quad v \geq e^{\gamma/\delta}v_0 \end{cases} .$$

For every $x > 0$ and $v_0 > 0$ we find that $g(x, v_0, v)$ is a continuous and strictly increasing function of v and the inverse function g^{-1} as defined in (9) is given by

$$g^{-1}(x, v_0, c) = \begin{cases} e^{\gamma}xc & \text{for} \quad c < \frac{v_0}{x}e^{\gamma/\delta-\gamma} \\ (v_0^{-\delta}xc)^{\frac{1}{1-\delta}} & \text{for} \quad c \geq \frac{v_0}{x}e^{\gamma/\delta-\gamma} \end{cases} . \tag{21}$$

Note that (11) holds indeed for this particular form of $g^{-1}(x, v_0, c)$.

Let us first consider the last period $(T-1, T]$. As in section 5 we require for a perfect hedge, $V(T) = X(T)$, that

$$V(T) = g^{-1}(X(T-1), V(T-1), 1)$$

and, therefore, $V(T)$ must be $\mathcal{F}_{T-1}$-measurable, that is, the final value of the fund V must already be known at time $T-1$. To achieve this the insurer would invest at time $T-1$ into zero-coupon bonds that mature at time T. We assume that such zero-coupon bonds exists in the considered financial market and denote the bond price at any time t by $P(t)$ with $P(T) = 1$.

The number of bonds required at time $T-1$ is $V(T)$ since $P(T) = 1$. The amount $V(T-1)$ required to buy the bonds at time $T-1$ is $V(T-1) = V(T)P(T-1)$. Without using the relative value C of the self-sufficient fund we immedeately find that

$$\begin{aligned} V(T) &= X(T) \\ &= X(T-1)H(V(T)/V(T-1)) \\ &= X(T-1)H\left(\frac{V(T)}{V(T)P(T-1)}\right) \\ &= X(T-1)H\left(\frac{1}{P(T-1)}\right) \\ &= X(T-1)\max\left\{e^{\gamma}, P(T-1)^{-\delta}\right\} . \end{aligned}$$

Since $V(T)$ solves the equation

$$V(T) = g^{-1}\big(X(T-1), V(T)P(T-1), 1\big)$$

and the discounted zero-coupon bond price is a martingale for any $\mathrm{Q} \in \mathcal{Q}$ we obtain for any such Q that

$$\begin{aligned} V(T-1) &= \mathrm{E}_{\mathrm{Q}}\left[\frac{S_0(T-1)}{S_0(T)}V(T) \,\middle|\, \mathcal{F}_{T-1}\right] \\ &= \mathrm{E}_{\mathrm{Q}}\left[\frac{S_0(T-1)}{S_0(T)}g^{-1}\big(X(T-1), V(T-1), 1\big) \,\middle|\, \mathcal{F}_{T-1}\right] \end{aligned}$$

and, therefore, $C(T-1) = V(T-1)/X(T-1)$ has indeed property (C2) in theorem 1 for $t = T-1$.

A similar result was already obtained by Brennan & Schwartz (1976). As mentioned above, the payoff to the policyholder is already known at time $T-1$. Furthermore, the insurance company does not invest into any assets except the zero-coupon bond with maturity T. This might contradict the expectation of the policyholder to participate in a portfolio that includes some equity risk. However, the insurer can only invest into equity if he is ready to take some shortfall risk or if there is more money available at time $T-1$. The possibility to invest into equities was studied in detail by Kleinow & Willder (2007) for a simple financial market model in which interest rates and equities are modelled by binomial trees. As we will here concentrate on a perfect hedge, $V(T) = X(T)$ almost surely, we leave the consideration of investments into equity for further research.

To shed further light on the structure of the with-profits fund in this particular example we rewrite the above result using the relative fund value C. Here we make the dependence of C on the zero-coupon bond price explicit. We obtain

$$V(T-1) = X(T-1)C[T-1, P(T-1)] \tag{22}$$

with

$$C[T-1, P(T-1)] = \max\left\{e^{\gamma}, P(T-1)^{-\delta}\right\} P(T-1)\ . \tag{23}$$

This provides an explicit relationship between the balance of the policyholder's account $X(T-1)$ and the value of the assets $V(T-1)$ of the insurance company at time $T-1$, one year before maturity. This shows that the relative value $C(T-1)$ of the fund at time $T-1$ is an interest rate derivative. The payoff of that derivative is needed to provide sufficient funds at time $T-1$ to hedge the payoff at time T.

Therefore, at time $T-2$ the insurer should invest into such interest rate derivatives that mature at time $T-1$ either by buying them from a third party or by replicating their payoff with a self-financing portfolio. The number of derivatives needed at $T-2$ is $X(T-1)$ which is not known at

$T-2$. However, using theorems 1, 2 and 3 we know that the total price of all required derivatives at $T-2$ is $V(T-2) = X(T-2)C(T-2)$ where $C(T-2)$ solves (C2) in theorem 1 and $X(T-2)$ is known at time $T-2$.

However, $V(T-2) = X(T-2)C(T-2)$ might not be an interest rate derivative. In particular, in a complex financial market model interest rates and equity prices might not be independent. Nevertheless, $V(T-2)$ and $C(T-2)$ are $\mathcal{F}_{T-2}$-measurable random variables and therefore, the required fund value $V(T-2)$ can be considered as the payoff of a contingent claim maturing at time $T-2$ that the insurer should buy at time $T-3$ for the price $V(T-3)$ given by theorem 2. This argument can now be used for any other period before $T-2$ to give the required value $V(0)$ of the self-sufficient fund at the time the policy $(1, H, 1)$ is issued.

9 Conclusion

Most hedging and valuation problems in mathematical finance deal with given contingent claims. We have considered a different type of problem where only the payoff function is specified but the underlying security can be chosen by the issuer of the claim. In this situation we have constructed an underlying security price process for a given payoff function such that the security itself provides a perfect hedge for the payoff.

We have also shown that the proposed pricing algorithm can only be applied if the issuer of the contract can actually trade in a security who's price process is the process V constructed in section 5. Therefore, there is a strong relationship between hedging and valuation in the sense that risk-neutral valuation only leads to meaningful results if a security with process V is available.

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