ON THE OPTIMAL SST INITIAL CAPITAL OF A LIFE CONTRACT

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Abstract

We present a new perspective on the Swiss Solvency Test (SST) for a life insurance contract under the assumption that the insurance company is risk neutral with respect to mortality, that is mortality risk is assumed to be diversifiable. In fact, if the portfolio of an insurance company is not too small, the law of large numbers for mortality risks can be applied and mortality dependent random cash flows can be replaced by their expected cash flows with respect to some life table. In this setting, the driver of the SST is the random assets of the insurance portfolio, which depend on the realized future investment returns on the capital market. The corresponding SST target capital can be viewed as an approximate value of the market risk of a life insurance contract. We show how to determine the SST risk measure and a modified SST coherent risk measure for a life insurance contract. We illustrate with a numerical example for a typical endowment contract. We observe that in all of our examples, which depend on the choice of the expected rate of return and the volatility of return, the expected shortfall of the first year risk-bearing capital is negative. If we assume that at the entry date of the endowment contract there is no liability and set the SST target capital equal to its initial capital, then the defined implicit equation has a unique initial capital as solution. Moreover, the obtained unique solutions correspond either to a vanishing SST risk measure or to a vanishing SST coherent risk measure. A unique solution is either *called optimal SST initial capital* or *optimal SST coherent initial capital.* An immediate reinterpretation yields the following rule of thumb. Under the made assumptions, the risk margin or market value margin of an endowment contract coincides in absolute value with the expected shortfall of the first year risk-bearing capital.

Key words

Swiss Solvency Test, market risk, coherent risk measure, optimal initial capital, market value margin, endowment insurance

1. Introduction

An insurance company needs capital in order to being able to take risks from its policy holders. The appropriate amount of capital is determined according to some solvency regulator rules. For example, the Swiss Solvency Test (SST) is a system to determine the amount of

- the available capital (“risk bearing capital”) and
- the required capital according to the risks (“target capital”)

This system has an economic point of view, which means that all portfolios are not evaluated on a statutory accounting but on an economic basis using market consistent valuation of assets and liabilities (consult Wüthrich et al.(2008) for a recent introduction).

Apart from determining the target capital and the risk bearing capital the SST aims to increase the insurer’s quantitative awareness about risks. To that end the SST does not consist of a defined model but is based on a set of first principles. These principles define a boundary in which insurance companies can develop and run their own risk models (the so-called internal models) to quantitatively evaluate their risks in a manner which is useful to them. Additionally, a publicly available standard risk model is given by the supervisor. An excellent introduction to SST is the White Paper on the Swiss Solvency Test (SST) that is available from the homepage of the Swiss Federal Office of Private Insurance (see FOPI(2004)). Technical guidelines to SST are found in the documentation FOPI(2006).

In the present contribution, we present a new perspective on the SST for a life insurance contract under the assumption that the insurance company is risk neutral with respect to mortality. This means that mortality risk is assumed to be diversifiable. In fact, if the portfolio of an insurance company is not too small, the law of large numbers for mortality risks can be applied and mortality dependent random cash flows can be replaced by their expected cash flows with respect to some life table. In this setting, the driver of the SST is the random assets of the insurance portfolio, which depend on the realized future investment returns on the capital market. The corresponding SST target capital can be viewed as an approximate value of the market risk of a life insurance contract. The paper is organized as follows.

In Section 2 we recall the definition of the SST target capital in terms of a multi-period *SST risk measure* in the mathematical sense. From a methodological point of view, capital risk measures should be coherent risk measures. However, Filipovic and Vogelpoth(2006) have shown by counterexample that the SST risk measure does not satisfy the axiom of monotonicity for coherent multi-period risk measures. These authors have determined the largest coherent risk measure among those dominated by the SST risk measure, here called *SST coherent risk measure*. The latter risk measure, if appropriate, might be used as a valuable alternative to the SST risk measure. Section 3 contains the core of our approach. Under the assumption that the accumulated rates of return on investment in discrete time periods are independent and log-normally distributed, and are independent of the random premium income and stochastic insurance costs, we determine in Theorem 3.1 the mean and variance of the random assets of an arbitrary insurance portfolio at each future time of valuation. In order to apply our result to a life insurance contract, we recall in Section 4 the technical values of a life insurance contract, which are required in a concrete evaluation. Section 5 shows how the SST risk measure is determined and Section 6 concludes with a numerical example for a typical endowment contract. We observe that in all of our examples, which depend on the choice of the expected rate of return and the volatility of return, the expected shortfall of the first year risk-bearing capital is negative. If we assume that at the entry date of the endowment contract there is no liability and set the SST target capital equal to its initial capital, then the defined implicit equation has a unique initial capital as solution. Moreover, the obtained unique solutions correspond either to a vanishing SST risk measure or to a vanishing SST coherent risk measure. A unique solution is either *called optimal SST initial capital* or *optimal SST coherent initial capital*. An immediate reinterpretation yields the following rule of thumb. Under the made assumptions, the risk margin (RM) or market value margin (MvM) of an endowment contract coincides in absolute value with the expected shortfall of the first year risk-bearing capital.

2. <u>The SST risk measure</u>

Risk Bearing Capital (RBC) is defined as the difference between the market consistent values of assets and the best-estimate of liabilities:

Market consistent values of *Assets*	Best-estimate of *Liabilities*
	Risk Bearing Capital

The *Target Capital* (TC) relates the risks incurred by an insurer to a solvency capital requirement. It is defined as the sum of the required *Economic Capital* (EC) and the *Risk Margin* (RM) (also called *Market value Margin* (MvM)):

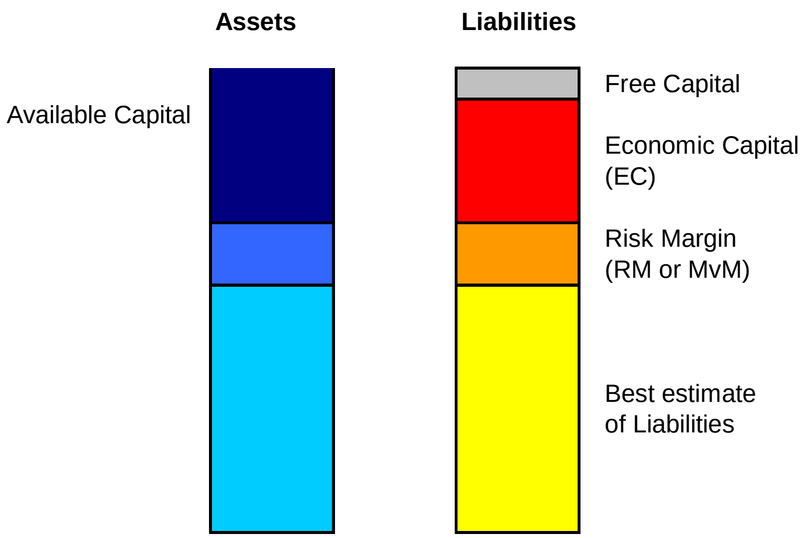
Target Capital	*Economic Capital* = Risk measure of the change of RBC over a given time horizon
	Risk Margin = appropriate amount of capital besides best-estimate of liabilities allocated to cover future liabilities

The SST has adopted the following recommendations made by IAA(2004):

- The total balance sheet approach
- The expected shortfall (ES) (also called conditional value-at-risk (CVaR), tail value-at-risk (TVaR), conditional tail expectation (CTE)) as a risk measure used to calculate EC
- The time horizon of 1 year for calculation of EC
- The explicit definition of the risk margin

Let us illustrate the above concepts within the economic balance sheet for the SST (e.g. Keller(2006)):

Figure 1: the market consistent economic balance sheet



Risk Bearing Capital = Available Capital + Risk Margin
= Market value of Assets - Best estimate of Liabilities
Market value of Liabilities = Best estimate + Risk Margin
Available Capital = Market value of Assets - Market value of Liabilities
Target Capital = Economic Capital + Risk Margin

The modeling of the risk margin concerns the question of what portion of the available funds of a company should be allocated to cover future liabilities (reserves) and hence is not available to fulfill standard capital requirements. In this context *Future Liabilities* is defined as the sum of the best-estimate of future liabilities (=net present value of future liabilities). Within CEIOPS (Committee of European Insurance and Occupational Pensions Supervisors), the EU and the various groups working on Solvency II development, the basic issue is the actual aim of a risk margin for future liabilities covering a period of several years (see Solvency II(2005), Newsletter no. 5).

Currently there are two proposals for defining the risk margin:

Quantile approach (working assumption for Solvency II)

The risk margin is primarily considered as security providing policyholders with a certain probability that they will receive payment pending final settlement. This is the view favored by the European Commission and already officially adopted in Australia and Singapore.

In this situation the risk margin is calculated on the basis of the probability distribution of future liabilities until they are settled. It is the difference between the best estimate of liabilities and a selected quantile of the probability distribution. By December 2005, the European Commission has selected this approach as a working assumption and has set the quantile provisionally at 75%.

Cost of Capital approach (incorporated in SST)

This approach reflects the wish to come closer to the market value approach. In the attempt to define a market value for future liabilities, the distinction must be made between risks tradable in the financial markets and those for which there is no market. Non-tradable risks include for example the mortality risk in life insurance and underwriting risks in P&C insurance. For these risks, a substitute for market value is used as an approximation.

The risk margin is defined as the cost of capital for future regulatory capital needed for the run-off of the portfolio, following financial distress of the company. For the regulator it is imperative that in the case of insolvency, the rightful claimants be protected. Policyholders are best served if a third party can take over the assets and liabilities of their initial insurer. A third party will only be prepared to do this if the cost of setting up the regulatory capital that would be required is covered by the portfolio. As the regulatory capital depends on both assets and liabilities, risks emanating from the asset portfolio enter the calculation of the risk margin. The risk margin is set such that one part of it can be used to pay for the necessary regulatory capital for the current year while the other part is sufficient to set up the risk margin at the end of the current year.

The risk margin is calculated as being the discounted value of the future costs of maintaining the SST target capital level if the insurance portfolio was being run off by third party. For the SST, the cost of capital rate has been set at 6%.

For theoretical and practical reasons it is useful to define the SST target capital in a formal mathematical way as follows.

Notations:

T	: finite time horizon
(Ω, F, P)	: probability space endowed with a filtration $(F_t)_{t=0}^{T}$ such that $F_0 = \{\Omega, \emptyset\}$ and $F_T = F$ (Think of F_t as the information available up to time t)
$L^\infty(F_t)$	: space of essentially bounded random variables on (Ω, F, P)
R^∞	: space of essentially bounded stochastic processes on (Ω, F, P) which are adapted to the filtration $(F_t)_{t=0}^{T}$

Expected shortfall:

The *expected shortfall* (ES) to the confidence level $\alpha \in (0,1)$ of a *loss* $X \in L^\infty(F_t)$ is defined by

$$ES_\alpha[X] := \frac{1}{\varepsilon} \cdot \int_\alpha^1 Q_X(u)du \tag{2.1}$$

where $Q_X(u)$ is a quantile function of X. It represents the average of the $100\varepsilon\%$ worst losses, where $\varepsilon = 1-\alpha$ denotes the loss probability. The expected shortfall is known to coincide with the notion of *conditional value-at-risk* (=conditional expected loss given the loss strictly exceeds its value-at-risk) and satisfies a lot of equivalent formulas of common use in the fields of reliability, actuarial science, finance and economics (see Hürlimann(2003)).

SST target capital:

Let $C \in R^{\infty}$ be the stochastic process of *risk-bearing capital* (RBC) and consider the stochastic process $SC = -C \in R^{\infty}$, which represents the *shortfall risk-bearing capital* (SRBC) (=negative of risk-bearing capital). The determination of target capital (TC) according to the SST approach follows two steps.

Step 1: determination of the economic capital

The *economic capital* (EC) is the 1-year risk capital required to cover the risk of the asset-liability portfolio within a one year time horizon and is given by the expected shortfall of the change in shortfall risk-bearing capital

$$EC := ES_{\alpha}\left[SC_1 - SC_0\right] = C_0 + ES_{\alpha}\left[SC_1\right] \tag{2.2}$$

Step 2: determination of the risk margin

The *risk margin* (RM) assigns a capital requirement to the run-off of the in force asset-liability portfolio and is defined by

$$RM := i_{CoC} \cdot \sum_{t=2}^{T} ES_{\alpha}\left[SC_t - SC_{t-1}\right] \tag{2.3}$$

where the *cost of capital rate* i_{CoC} represents the spread between interest rates at which money can be borrowed and reinvested at no risk.

Summarizing the two steps, one writes

$$\begin{aligned} TC &= EC + RM \\ &= C_0 + ES_{\alpha}\left[SC_1\right] + i_{CoC} \cdot \sum_{t=2}^{T} ES_{\alpha}\left[SC_t - SC_{t-1}\right] = C_0 + R_{\alpha}^{SST}\left[SC\right] \end{aligned} \tag{2.4}$$

where

$$R_{\alpha}^{SST}\left[SC\right] := ES_{\alpha}\left[SC_1\right] + i_{CoC} \cdot \sum_{t=2}^{T} ES_{\alpha}\left[SC_t - SC_{t-1}\right] \tag{2.5}$$

The functional $R_{\alpha}^{SST} : R^{\infty} \to R$ defines the *SST risk measure*.

The SST risk measure is a prototype of a *multi-period risk measure* used in risk assessment within a multi-period framework. The design and properties of multi-period risk measures is not yet well-understood. From a methodological point of view capital risk measures should be coherent risk measures. However, Filipovic and Vogelpoth(2006) have shown by counterexample that the SST risk measure does not satisfy the axiom of monotonicity for coherent multi-period risk measures. If for $X, Y \in R^{\infty}$ one has $X \geq Y$ with probability one, than $R_{\alpha}^{SST}\left[X\right] \geq R_{\alpha}^{SST}\left[Y\right]$ does not necessarily hold, that is a higher risk does not necessarily lead to a higher target capital, which is unsatisfactory. The largest coherent risk measure among those dominated by the SST risk measure, is given by

$$R_{\alpha}^{SST,c}[SC] := (1 - i_{CoC}) \cdot ES_{\alpha}[SC_1] + i_{CoC} \cdot ES_{\alpha}[SC_T], \tag{2.6}$$

and (if appropriate) could be used as a valuable alternative to the SST risk measure. The risk measure (2.6) will be called *SST coherent risk measure*. For a discussion of problems encountered with the application of "coherent risk measures" in a solvency framework, one should consult Dhaene, Goovaerts and Kaas(2003). Moreover and so far, the task of consistent capital assessments across time within Solvency II has only been scarcely discussed. Dynamic time-consistent aspects in the context of the SST are studied in Vogelpoth(2006).

3. Mean and variance of the random assets

Consider the stochastic dynamic evolution of the random asset values of an insurance portfolio in a discrete time setting over a time horizon $[0,T]$. Let A_t be the random value of the *assets* at time $t \in \{1,...,T\}$. In the time period $(t-1,t]$ the cash-in flow consists of random *premiums* of amount P_t at time $t-1$, and the cash-out flow consists of random *insurance costs* of amount X_t at time t. The latter amount includes insurance claims, expenses and bonus payments to the insured. The assets at time t satisfy the recursive equation

$$A_t = (A_{t-1} + P_t) \cdot (1 + I_t) - X_t, \quad t \in \{1,...,T\}, \tag{3.1}$$

where I_t denotes the random *rate of return* on investment in the time period $(t-1,t]$ and A_0 denotes the *initial capital* of the insurance portfolio. It follows that the asset value at time T is given by

$$A_T = A_0 \cdot \prod_{t=1}^{T} (1 + I_t) + \sum_{t=1}^{T} \{P_t \cdot (1 + I_t) - X_t\} \cdot \prod_{j=t+1}^{T} (1 + I_j). \tag{3.2}$$

Our goal is the determination of the mean and variance of the assets under the following model assumptions:

(M1) The random premiums P_t and insurance costs X_t are independent from the returns $I_1, ..., I_T$ and their means and variances are given by $\mu_{P_t} = E[P_t], \sigma_{P_t}^2 = Var[P_t]$ and $\mu_{X_t} = E[X_t], \sigma_{X_t}^2 = Var[X_t]$ respectively.

(M2) The random *accumulated rates of return* in time period $(t-1,t]$ are independent and log-normally distributed such that

$$Z_t = \ln\{1 + I_t\}, \quad t \in \{1,...,T\}, \tag{3.3}$$

is normally distributed with mean μ and standard deviation σ.

To simplify calculations, we consider the random variables $Z_{t,T}$ defined by

$$\exp(Z_{t,T}) = \prod_{j=t}^{T} (1 + I_j), \quad t \in \{1,\dots,T\}, \tag{3.4}$$

and which represent the random accumulated rates of return over the time period $(t-1,T]$. Clearly, the random sum $Z_{t,T} = \sum_{j=t}^{T} Z_j$ is normally distributed with mean and standard deviation

$$\mu_{t,T} = E[Z_{t,T}] = (T-t+1)\cdot \mu, \quad \sigma_{t,T} = \sqrt{Var[Z_{t,T}]} = \sqrt{T-t+1}\cdot \sigma. \tag{3.5}$$

The constant one-period *expected accumulated rate of return* over the time horizon $(t-1,t]$ is denoted and given by

$$r = E[\exp(Z_t)] = \exp\left(\mu + \tfrac{1}{2}\sigma^2\right), \quad t \in \{1,\dots,T\}. \tag{3.6}$$

We are ready for the following result.

<u>Theorem 3.1</u>. Under the model assumptions (M1) and (M2), the mean and variance of the random assets of an insurance portfolio are given by the expressions

$$E[A_T] = r^T \cdot \left\{ A_0 + \sum_{t=1}^{T} r^{-t} \cdot \left(\mu_{P_t} \cdot r - \mu_{X_t}\right) \right\} \tag{3.7}$$

$$Var[A_T] = r^{2T} \cdot \left\{ \begin{array}{l} A_0^2 \cdot \left(e^{T\sigma^2} - 1\right) \\ + A_0 \cdot \sum_{t=1}^{T} r^{-t} \cdot \left\{ \left(\mu_{P_t} r - \mu_{X_t}\right)\left(e^{(T-t)\sigma^2} - 1\right) + \mu_{P_t} r e^{(T-t)\sigma^2}\left(e^{\sigma^2} - 1\right) \right\} \\ + \sum_{t=1}^{T} r^{-t} \cdot \left\{ \begin{array}{l} \left(\mu_{P_t} r - \mu_{X_t}\right)^2 \left(e^{(T-t)\sigma^2} - 1\right) + \left(\mu_{P_t} r\right)^2 e^{(T-t)\sigma^2}\left(e^{\sigma^2} - 1\right) \\ + \sigma_{P_t}^2 r^2 e^{(T-t+1)\sigma^2} + \sigma_{X_t}^2 r^2 e^{(T-t)\sigma^2} - 2Cov[P_t, X_t] \cdot r e^{(T-t)\sigma^2} \end{array} \right\} \\ + 2 \cdot \sum_{1 \le s < t \le T} r^{-(s+t)} \cdot \left\{ \begin{array}{l} \left(\mu_{P_s} r - \mu_{X_s}\right)\left(\mu_{P_t} r - \mu_{X_t}\right)\left(e^{(T-t)\sigma^2} - 1\right) \\ + \left(\mu_{P_s} r - \mu_{X_s}\right)\mu_{P_t} r e^{(T-t)\sigma^2}\left(e^{\sigma^2} - 1\right) \\ + Cov\left[P_s r - X_s, P_t r e^{\sigma^2} - X_t\right] \cdot e^{(T-t)\sigma^2} \end{array} \right\} \end{array} \right\} \tag{3.8}$$

<u>Proof</u>. With the notation (3.4) the expression (3.2) can be rewritten as

$$A_T = A_0 \cdot \exp(Z_{1,T}) + \sum_{t=1}^{T} \left\{ P_t \cdot \exp(Z_{t,T}) - X_t \cdot \exp(Z_{t+1,T}) \right\}, \tag{3.9}$$

from which one gets without difficulty (3.7). To get the expression for the variance, several terms must be calculated. One has

$$Var\left[A_0 \cdot \exp\left(Z_{1,T}\right)\right] = A_0^2 \cdot \left(e^{2T\left(\mu+\sigma^2\right)} - e^{T\left(2\mu+\sigma^2\right)}\right) = A_0^2 \cdot r^{2T} \cdot \left(e^{T\sigma^2} - 1\right)$$

For $1 \le t \le T$ one has using assumption (M1) that

$$\begin{aligned}
&Var\left[P_t \cdot \exp\left(Z_{t,T}\right) - X_t \cdot \exp\left(Z_{t+1,T}\right)\right] \\
&= Var\left[E\left[P_t \cdot \exp\left(Z_{t,T}\right) - X_t \cdot \exp\left(Z_{t+1,T}\right) \middle| Z_{t,T}, Z_{t+1,T}\right]\right] \\
&+ E\left[Var\left[P_t \cdot \exp\left(Z_{t,T}\right) - X_t \cdot \exp\left(Z_{t+1,T}\right) \middle| Z_{t,T}, Z_{t+1,T}\right]\right] \\
&= Var\left[\mu_{P_t} \cdot \exp\left(Z_{t,T}\right) - \mu_{X_t} \cdot \exp\left(Z_{t+1,T}\right)\right] \\
&+ E\left[\sigma_{P_t}^2 \cdot \exp\left(2Z_{t,T}\right) + \sigma_{X_t}^2 \cdot \exp\left(2Z_{t,T}\right) - 2Cov\left[P_t, X_t\right] \cdot \exp\left(Z_{t,T} + Z_{t+1,T}\right)\right] \\
&= \mu_{P_t}^2 \cdot Var\left[\exp\left(Z_{t,T}\right)\right] + \mu_{X_t}^2 \cdot Var\left[\exp\left(Z_{t+1,T}\right)\right] - 2\mu_{P_t}\mu_{X_t} \cdot Cov\left[\exp\left(Z_{t,T}\right), \exp\left(Z_{t+1,T}\right)\right] \\
&+ \sigma_{P_t}^2 \cdot E\left[\exp\left(2Z_{t,T}\right)\right] + \sigma_{X_t}^2 \cdot E\left[\exp\left(2Z_{t+1,T}\right)\right] \\
&- 2Cov\left[P_t, X_t\right] \cdot \left(E\left[\exp\left(Z_{t,T}\right)\right] \cdot E\left[\exp\left(Z_{t+1,T}\right)\right] + Cov\left[\exp\left(Z_{t,T}\right), \exp\left(Z_{t+1,T}\right)\right]\right)
\end{aligned}$$

By assumption (M2) Z_t is independent from $Z_{t+1,T}$ and one has

$$\begin{aligned}
&Cov\left[\exp\left(Z_{t,T}\right), \exp\left(Z_{t+1,T}\right)\right] = Cov\left[\exp\left(Z_t\right) \cdot \exp\left(Z_{t+1,T}\right), \exp\left(Z_{t+1,T}\right)\right] \\
&= E\left[\exp\left(Z_t\right)\right] \cdot Var\left[\exp\left(Z_{t+1,T}\right)\right] = r^{2(T-t)+1} \cdot \left(e^{(T-t)\sigma^2} - 1\right).
\end{aligned}$$

Inserted in the preceding expression one gets

$$\begin{aligned}
&Var\left[P_t \cdot \exp\left(Z_{t,T}\right) - X_t \cdot \exp\left(Z_{t+1,T}\right)\right] \\
&= \mu_{P_t}^2 \cdot r^{2(T-t+1)}\left(e^{(T-t+1)\sigma^2} - 1\right) + \mu_{X_t}^2 \cdot r^{2(T-t)}\left(e^{(T-t)\sigma^2} - 1\right) - 2\mu_{P_t}\mu_{X_t} \cdot r^{2(T-t)+1}\left(e^{(T-t)\sigma^2} - 1\right) \\
&+ \sigma_{P_t}^2 \cdot r^{2(T-t+1)} e^{(T-t+1)\sigma^2} + \sigma_{X_t}^2 \cdot r^{2(T-t)} e^{(T-t)\sigma^2} - 2Cov\left[P_t, X_t\right] \cdot r^{2(T-t)+1} e^{(T-t)\sigma^2}.
\end{aligned}$$

On the other side, for $1 \le s < t \le T$ one has

$$\begin{aligned}
&Cov\left[P_s \exp\left(Z_{s,T}\right) - X_s \exp\left(Z_{s+1,T}\right), P_t \exp\left(Z_{t,T}\right) - X_t \exp\left(Z_{t+1,T}\right)\right] \\
&= E\left[P_s P_t\right] \cdot Cov\left[\exp\left(Z_{s,T}\right), \exp\left(Z_{t,T}\right)\right] + Cov\left[P_s, P_t\right] \cdot E\left[\exp\left(Z_{s,T}\right)\right] \cdot E\left[\exp\left(Z_{t,T}\right)\right] \\
&- E\left[X_s P_t\right] \cdot Cov\left[\exp\left(Z_{s+1,T}\right), \exp\left(Z_{t,T}\right)\right] + Cov\left[X_s, P_t\right] \cdot E\left[\exp\left(Z_{s+1,T}\right)\right] \cdot E\left[\exp\left(Z_{t,T}\right)\right] \\
&- E\left[P_s X_t\right] \cdot Cov\left[\exp\left(Z_{s,T}\right), \exp\left(Z_{t+1,T}\right)\right] + Cov\left[P_s, X_t\right] \cdot E\left[\exp\left(Z_{s,T}\right)\right] \cdot E\left[\exp\left(Z_{t+1,T}\right)\right] \\
&+ E\left[X_s X_t\right] \cdot Cov\left[\exp\left(Z_{s+1,T}\right), \exp\left(Z_{t+1,T}\right)\right] + Cov\left[X_s, X_t\right] \cdot E\left[\exp\left(Z_{s+1,T}\right)\right] \cdot E\left[\exp\left(Z_{t+1,T}\right)\right] \\
&= r^{2T-(s+t)+2} \cdot \left(\mu_{P_s}\mu_{P_t} \cdot \left(e^{(T-t+1)\sigma^2} - 1\right) + Cov\left[P_s, P_t\right] \cdot e^{(T-t+1)\sigma^2}\right) \\
&- r^{2T-(s+t)+1} \cdot \left(\mu_{X_s}\mu_{P_t} \cdot \left(e^{(T-t+1)\sigma^2} - 1\right) + Cov\left[X_s, P_t\right] \cdot e^{(T-t+1)\sigma^2}\right) \\
&- r^{2T-(s+t)+1} \cdot \left(\mu_{P_s}\mu_{X_t} \cdot \left(e^{(T-t)\sigma^2} - 1\right) + Cov\left[P_s, X_t\right] \cdot e^{(T-t)\sigma^2}\right) \\
&+ r^{2T-(s+t)} \cdot \left(\mu_{X_s}\mu_{X_t} \cdot \left(e^{(T-t)\sigma^2} - 1\right) + Cov\left[X_s, X_t\right] \cdot e^{(T-t)\sigma^2}\right),
\end{aligned}$$

where the covariance terms are obtained as follows. Since $Z_{s,T} = Z_{s,t-1} + Z_{t,T}$ and $Z_{s,t-1}$ is independent from $Z_{t,T}$, one obtains

$$\begin{aligned} &Cov[\exp(Z_{s,T}), \exp(Z_{t,T})] = Cov[\exp(Z_{s,t-1}) \cdot \exp(Z_{t,T}), \exp(Z_{t,T})] \\ &= E[\exp(Z_{s,t-1})] \cdot Var[\exp(Z_{t,T})] = r^{2T-(s+t)+2} \cdot \left(e^{(T-t+1)\sigma^2} - 1\right). \end{aligned}$$

In a similar way, for $1 \le t \le T$ one has

$$\begin{aligned} &Cov[A_0 \exp(Z_{1,T}), P_t \exp(Z_{t,T}) - X_t \exp(Z_{t+1,T})] \\ &= A_0 \mu_{P_t} \cdot Cov[\exp(Z_{1,T}), \exp(Z_{t,T})] - A_0 \mu_{P_t} \cdot Cov[\exp(Z_{1,T}), \exp(Z_{t+1,T})] \\ &= A_0 r^{2T-t} \cdot \left(\mu_{P_t} r \cdot \left(e^{(T-t+1)\sigma^2} - 1\right) - \mu_{X_t} \cdot \left(e^{(T-t)\sigma^2} - 1\right)\right). \end{aligned}$$

Gathering all terms together appropriately one obtains finally (3.8). ◊

4. Technical values of a life insurance contract

We consider a life insurance contract with level premium payments subject to the single mortality cause of decrement over the time horizon $[0, n]$ in a discrete time setting. At the time points $t \in \{1,2,...,n\}$ the contract offers to a policyholder the following benefits:

D_t : death benefit paid end of period in case the policyholder dies in time period $(t-1, t]$
E_n : survival benefit paid in case the policyholder survives the whole period at time n

To perform in the last Section concrete numerical calculations, we require the following classical technical values as presented in Hürlimann(1988):

π_t^R : risk premium for the time period $(t-1, t]$ due at time $t-1$
π_t^S : saving premium for the time period $(t-1, t]$ due at time $t-1$
$\pi^N = \pi_t^R + \pi_t^S$: net premium for the time period $(t-1, t]$ due at time $t-1$
$\pi_t^{E,R}$: expense risk premium for the time period $(t-1, t]$ due at time $t-1$
$\pi_t^{E,S}$: expense saving premium for the time period $(t-1, t]$ due at time $t-1$
$\pi^E = \pi_t^{E,R} + \pi_t^{E,S}$: expense premium for the time period $(t-1, t]$ due at time $t-1$
$\pi = \pi^N + \pi^E$: gross premium for the time period $(t-1, t]$ due at time $t-1$
${}_tV^N$: the net actuarial reserve required at time t such that ${}_nV^N = E_n$
${}_tV^E$: the expense reserve required at time t such that ${}_nV^E = 0$
${}_tV = {}_tV^N + {}_tV^E$: the gross actuarial reserve required at time t
c_t : the operating cost charge for the time period $(t-1, t]$ due at time t

The idea of including the expense reserve into the gross actuarial reserve is due to Zillmer(1831-1893) (e.g. Gerber(1986), p. 103). In this traditional setting, pricing is based on a single decrement mortality table with entries

q_x : probability a life aged x will die within one year
$p_x = 1 - q_x$: probability a life aged x will survive to age $x+1$

and some fixed technical interest rate i. The technical discount factor is denoted by $v = (1+i)^{-1}$. The technical values satisfy the following relationships:

$$\pi_t^R = v \cdot q_{x+t-1} \cdot \left(D_t - {}_tV^N\right), \quad t = 1, ..., n-1, \quad \pi_n^R = 0 \tag{4.1}$$

$$_tV^N = \left({}_{t-1}V^N + \pi_t^S\right) \cdot (1+i) \tag{4.2}$$

$$\pi_t^{E,R} = -v \cdot q_{x+t-1} \cdot {}_tV^E, \quad t = 1, ..., n-1, \quad \pi_n^{E,R} = 0 \tag{4.3}$$

$$c_t = \left({}_{t-1}V^E + \pi_t^{E,S}\right) \cdot (1+i) - {}_tV^E, \quad t = 1, ..., n \tag{4.4}$$

These formulas are given in Hürlimann(1988), p. 184, where (4.4) for the operating cost charge is not considered in the classical texts on life insurance.

<u>Example 4.1</u>.

For the classical endowment contract such that $D_1 = D_2 = ... = D_n = E_n = 1$ and for an acquisition cost rate α, one obtains the closed form formula

$$c_t = \left(\pi^E - \alpha \cdot \pi^N\right) \cdot (1+i) - \alpha \cdot i, \quad t = 1, ..., n. \tag{4.5}$$

Indeed, inserting the expression $_tV^E = -\alpha \cdot \left(1 - {}_tV^N\right)$ (e.g. Gerber(1986), p. 102) for the expense reserve into the formula (4.4) for the operating cost charge, one gets with the expressions (4.1) and (4.3) for the expense risk premiums and the risk premiums:

$$\begin{aligned}
c_t &= \left(-\alpha + \alpha \cdot {}_{t-1}V^N + \pi^E - \pi_t^{E,R}\right)(1+i) + \alpha - \alpha \cdot {}_tV^N \\
&= -\alpha \cdot i + \alpha \cdot \left({}_{t-1}V^N \cdot (1+i) - {}_tV^N\right) + \pi^E \cdot (1+i) - \alpha \cdot q_{x+t-1} \cdot \left(1 - {}_tV^N\right) \\
&= \pi^E \cdot (1+i) - \alpha \cdot i + \alpha \cdot \left({}_{t-1}V^N \cdot (1+i) - {}_tV^N - \pi_k^R \cdot (1+i)\right) \\
&= \pi^E \cdot (1+i) - \alpha \cdot i - \alpha \cdot \left(\pi_t^S \cdot (1+i) + \pi_t^R \cdot (1+i)\right) = \left(\pi^E - \alpha \cdot \pi^N\right) \cdot (1+i) - \alpha \cdot i
\end{aligned}$$

Due to regulatory laws, insurance companies are usually allowed to only guarantee relatively low technical interest rates to their policyholders. To compensate for this, policyholders are typically entitled to participate in the gross surplus of an insurance company. Additional variable bonus payments are periodically credited to the policyholder's account or bonus fund. Specification of the bonus fund depends upon the bonus policy of a life insurance company. Usually, the bonus highly depends upon the realized investment return I_t in period $(t-1, t]$. For simplicity and illustration we will assume a bonus rate of the type $I_t^b = \max(I_t - \delta, i)$, where δ is some interest spread. In general, the bonus rate is at least equal to the guaranteed technical interest rate. A possible specification of the size of the bonus fund of a n-year life insurance, which is viewed as a deterministic saving account and denoted by B_t, may be the following one. The bonus fund at the end of a period consists of the of the bonus fund at the beginning of a period accumulated with the bonus interest on the bonus fund and the excess bonus interest above the technical interest on the net actuarial reserves and net premiums. Expressed in a formula one has

$$B_t = B_{t-1} \cdot \left(1 + I_t^b\right) + \left({}_{t-1}V^N + \pi^N\right) \cdot \left(I_t^b - i\right). \tag{4.6}$$

The bonus fund of a policyholder is paid out by death or by survival at expiration date. We are interest in the expected value of the bonus fund at time t. Consider the bonus relevant capital in time period $(t-1, t]$, which is denoted and given by

$$c_t^b = \left({}_{t-1}V^N + \pi^N\right), \; t \in \{1, ..., n\}. \tag{4.7}$$

Rewrite (4.6) into the form

$$B_t = \left(B_{t-1} + {}_{t-1}V^N + \pi^N\right) \cdot \left(1 + I_t^b\right) - \left({}_{t-1}V^N + \pi^N\right) \cdot (1+i) \tag{4.8}$$

Then, similarly to (3.2) one has

$$B_t = \sum_{j=1}^{t} \left\{c_j^b \cdot \left(1 + I_j^b\right) - c_j^b \cdot (1+i)\right\} \cdot \prod_{k=j+1}^{t} \left(1 + I_k^b\right) \tag{4.9}$$

In general, the valuation of guaranteed stochastic funds of the type (4.9) is rather complex because it involves implicit or embedded options. Two simple models to handle this problem have been proposed in Hürlimann(2006). As a simple approximation, we assume here that the stochastic bonus rate exceeds the sum of the technical interest and the interest spread with almost certainty, that is we assume that $I_t^b \geq \delta + i$ with probability one. Then the formula (3.7) of Theorem 3.1 can be applied with I_t replaced by $I_t - \delta$ to get the expected bonus fund

$$b_t = E[B_t] = \sum_{j=1}^{t} \left\{c_j^b \cdot (r - \delta) - c_j^b \cdot (1+i)\right\} \cdot (r - \delta)^{t-j} \tag{4.10}$$

5. <u>The SST risk measure of a life insurance contract</u>

Consider the stochastic process $E_t = A_t - L_t, \quad t \in \{0, 1, ..., T\}$, of the equity at time t of a life insurance contract, where the assets A_t satisfy a recursive relationship of the type (3.1) and the liabilities L_t include the net actuarial reserves, the expense reserves and the bonus fund.

For simplicity, we assume that the insurance company is risk neutral with respect to mortality. This means that mortality risk is assumed to be diversifiable. In fact, if the portfolio of an insurance company is not too small, the law of large numbers for mortality risks can be applied and random cash flows can be replaced by expected cash flows. The evaluation is illustrated for an endowment contract with technical values as considered in Section 4.

To determine the mean and variance of the assets according to the assumptions of Section 3, one needs therefore the expected cash-in flow of premiums given by

$$p_t = E[P_t] = {}_{t-1}p_x \cdot \pi , \tag{5.1}$$

and the expected cash-out flow of insurance costs, which includes besides the expected insurance benefits the expected cost charges and the expected bonus payments, given by

$$x_t = E[X_t] = {}_{t-1}p_x \cdot q_{x+t-1} \cdot (1+b_t) + \delta_t^n \cdot {}_n p_x \cdot (1+b_n) + {}_t p_x \cdot c_t , \tag{5.2}$$

where $\delta_t^n = 1$ if $t = n$ and $\delta_t^n = 0$ otherwise. From Theorem 3.1 with time horizon $[0,t]$ and deterministic cash flows, one obtains the following formulas

$$E[A_t] = r^t \cdot \left\{ A_0 + \sum_{j=1}^{t} r^{-j} \cdot \left(p_j \cdot r - x_j\right) \right\} \tag{5.3}$$

$$Var[A_t] = r^{2t} \cdot \left\{ \begin{array}{l} A_0^2 \cdot \left(e^{t\sigma^2} - 1\right) + A_0 \cdot \sum_{j=1}^{t} r^{-j} \cdot \left\{ \left(p_j r - x_j\right)\left(e^{(t-j)\sigma^2} - 1\right) + p_j r e^{(t-j)\sigma^2}\left(e^{\sigma^2} - 1\right) \right\} \\ + \sum_{j=1}^{t} r^{-j} \cdot \left\{ \left(p_j r - x_j\right)^2 \left(e^{(t-j)\sigma^2} - 1\right) + \left(p_j r\right)^2 e^{(t-j)\sigma^2}\left(e^{\sigma^2} - 1\right) \right\} \\ + 2 \cdot \sum_{1 \le i < j \le t} r^{-(i+j)} \cdot \left(p_i r - x_i\right) \cdot \left\{ \left(p_j r - x_j\right)\left(e^{(t-j)\sigma^2} - 1\right) + p_j r e^{(t-j)\sigma^2}\left(e^{\sigma^2} - 1\right) \right\} \end{array} \right\} \tag{5.4}$$

By the risk-neutral assumption on the mortality risk, the liabilities are viewed as deterministic quantity with expected value

$$E[L_t] = (1 - \delta_t^n) \cdot {}_t p_x \cdot \left({}_t V^N + {}_t V^E + b_t\right). \tag{5.5}$$

To evaluate the SST target capital according to Section 2, we consider the stochastic process of the *shortfall risk-bearing capital* at time t, which is defined as market consistent discounted value at time $t = 0$ of the negative equity at time t, that is

$$SC_t = D_t \cdot \left(L_t - A_t\right), \tag{5.6}$$

where D_t is the market consistent discount factor, that is the actual price at time $t = 0$ of a unit zero coupon bond, which pays one unit at time t. To evaluate the SST risk measure, it is necessary to specify the distribution of (5.6). In general, it is possible to consider Gamma or elliptical type distributions (e.g. Hürlimann(2001), Landsman and Valdez(2003), Valdez(2005), Furman and Landsman(2005/07) and references). However, by the made simplifying assumptions, the standard normal distribution assumption will suffice for our purpose. Therefore, we assume that $SC_t,\ t \in \{1,...,T\}$, is normally distributed with mean and variance

$$E[SC_t] = D_t \cdot (E[L_t] - E[A_t]), \quad Var[SC_t] = D_t^2 \cdot Var[A_t]. \tag{5.7}$$

Similarly, the difference in shortfall risk-bearing capital $\Delta SC_t = SC_t - SC_{t-1}$ (required in the formula (2.5) for the SST risk measure) is also normally distributed with mean $E[\Delta SC_t] = E[SC_t] - E[SC_{t-1}]$ and variance determined by

$$Var[\Delta SC_t] = D_t^2 \cdot Var[A_t] - D_{t-1} \cdot (2rD_t - D_{t-1}) \cdot Var[A_{t-1}], \tag{5.8}$$

where use has been made of the following covariance expression

$$\begin{aligned} &Cov[A_{t-1}, A_t] = Cov[A_{t-1}, A_{t-1} \cdot \exp(Z_t) + p_t \cdot \exp(Z_t) - x_t] \\ &= E[\exp(Z_t)] \cdot Var[A_{t-1}] = r \cdot Var[A_{t-1}]. \end{aligned} \tag{5.9}$$

Section 4 and 5 contain now all specifications required to evaluate the SST target capital of an endowment contract using both the SST risk measure (2.5) and the SST coherent risk measure (2.6). The next Section provides concrete numerical calculations.

6. <u>A numerical example</u>

We illustrate the numerical evaluation of the SST target capital with a 10-year endowment contract for a life aged $x = 40$ with 1000 units as level insured sum. The technical interest rate is chosen at $i = 2.5\%$, the acquisition cost rate at $\alpha = 4\%$ and the operating cost rate (in percent of the gross premium) at $\beta = 5\%$. The probabilities of death and survival in Table 6.1 are taken from the illustrative life table in Bowers et al.(1986), p. 560. Table 6.2 provides the technical values of the endowment contract and Table 6.3 lists expected values needed in the formulas of Section 5.

<u>Table 6.1</u>: life table

year	probability of death q_{x+t-1}	probability of survival ${}_{t-1}p_x$
1	0.0027812	1.00000
2	0.0029818	0.99722
3	0.0032017	0.99425
4	0.0034427	0.99106
5	0.0037070	0.98765
6	0.0039966	0.98399
7	0.0043141	0.98006
8	0.0046621	0.97583
9	0.0050436	0.97128
10	0.0054617	0.96638

Table 6.2: Technical values of an endowment contract

net premium			expense premium			gross premium		
88.719			8.406			97.125		
year	${}_tV^N$	${}_tV^E$	π_t^S	π_t^R	$\pi_t^{E,S}$	$\pi_t^{E,R}$	c_t	b_t
1	88.402	-36.464	86.246	2.473	8.307	0.099	4.978	1.996
2	179.101	-32.836	86.331	2.388	8.310	0.096	4.978	6.076
3	272.185	-29.113	86.446	2.273	8.315	0.091	4.978	12.391
4	367.750	-25.290	86.595	2.124	8.321	0.085	4.978	21.100
5	465.901	-21.364	86.787	1.932	8.329	0.077	4.978	32.372
6	566.754	-17.330	87.030	1.689	8.338	0.068	4.978	46.389
7	670.438	-13.182	87.332	1.387	8.351	0.055	4.978	63.341
8	777.097	-8.916	87.705	1.014	8.365	0.041	4.978	83.430
9	886.891	-4.524	88.162	0.557	8.384	0.022	4.978	106.874
10	1000	0	88.719	0	8.406	0	4.978	133.902

Table 6.3: expected values of various quantities

	expected values							
year	cash-in or gross premium	benefits	cost charge	bonus benefits	cash-out without bonus	cash-out with bonus	gross actuarial reserve	bonus fund
1	97.125	2.781	4.964	0.006	7.745	7.751	51.793	1.991
2	96.855	2.974	4.949	0.018	7.923	7.941	145.423	6.041
3	96.566	3.183	4.933	0.039	8.116	8.156	240.900	12.280
4	96.257	3.412	4.916	0.072	8.328	8.400	338.231	20.839
5	95.926	3.661	4.898	0.119	8.559	8.678	437.419	31.854
6	95.570	3.933	4.878	0.182	8.811	8.993	538.467	45.464
7	95.188	4.228	4.857	0.268	9.085	9.353	641.369	61.810
8	94.777	4.549	4.835	0.380	9.384	9.764	746.118	81.034
9	94.335	4.899	4.810	0.524	9.709	10.233	852.701	103.281
10	93.860	966.380	4.784	129.400	971.164	1100.564	0	0

The confidence level in the SST risk measure is set at $\alpha = 99\%$, the cost of capital rate at $i_{CoC} = 6\%$ and the market discount factor corresponds for simplicity to a risk-free rate of 3%. The bonus spread is taken constant and equal to $\delta = 0.25\%$. We observe that in all of our examples, which depend on the choice of the expected rate of return and the volatility of return, the expected shortfall of the first year shortfall risk-bearing capital is negative, that is $ES_\alpha[SC_1] < 0$. We assume that at the entry date of the endowment contract there is no liability, that is $L_0 = 0$, hence $E_0 = A_0$. This implies that the implicit equation $TC = A_0$ obtained by setting the SST target capital to its initial capital has a unique solution A_0. By (2.4)-(2.6) the obtained unique solutions correspond either to a vanishing SST risk measure

$R_{\alpha}^{SST}[SC]=0$ or to a vanishing SST coherent risk measure $R_{\alpha}^{SST,c}[SC]=0$. A unique solution is either *called optimal SST initial capital* or *optimal SST coherent initial capital*. An immediate reinterpretation yields the following rule of thumb. Under the made assumptions, the risk margin (RM) or market value margin (MvM) of an endowment contract coincides in absolute value with the expected shortfall of the first year shortfall risk-bearing capital.

Table 6.4 summarizes various values of the optimal SST (not coherent and coherent) initial capital without or with bonus. We note a dramatic decrease of the optimal SST coherent initial capital compared to the original optimal SST initial capital. The obtained values are very sensitive to the volatility parameter. The dependence upon the expected rate of return is less sensitive but opposite in behaviour for an endowment without or with bonus participation.

Table 6.4: optimal SST initial capitals

return	optimal SST initial capital					
	without bonus			with bonus		
	volatilities			volatilities		
	8%	10%	12%	8%	10%	12%
4%	45.331	77.600	118.712	50.637	83.584	125.635
5%	43.417	76.167	118.075	53.076	87.084	130.751
6%	41.477	74.724	117.463	55.588	90.713	136.101
return	optimal SST coherent initial capital					
	without bonus			with bonus		
	volatilities			volatilities		
	8%	10%	12%	8%	10%	12%
4%	-2.345	9.915	23.740	2.309	14.864	29.074
5%	-4.580	7.877	21.931	3.868	16.871	31.639
6%	-6.824	5.844	20.143	5.486	18.964	34.325

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