Choosing the optimal annuitization time post retirement

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Abstract

In the context of decision making for retirees of a defined contribution pension scheme in the de-cumulation phase, we formulate and solve a problem of finding the optimal time of annuitization for a retiree having the possibility of choosing her own investment and consumption strategy. We formulate the problem as a combined stochastic control and optimal stopping problem. As criterion for the optimization we select a loss function that penalizes both the deviance of the running consumption rate from a desired consumption rate and the deviance of the final wealth at the time of annuitization from a desired target. We find closed form solutions for the problem and show the existence of three possible types of solutions depending on the free parameters of the problem. In numerical applications we find the optimal final wealth that triggers annuitization, compare it with the desired target and shed light on its dependence by both parameters of the financial market and parameters linked to the risk attitude of the retiree. Simulations of the behaviour of the risky asset seem to show that under typical situations optimal annuitization should occur a few years after retirement.

Keywords: defined contribution pension scheme, de-cumulation phase, stochastic optimal control, optimal annuitization time.

JEL classification: C61, D91, J26, G11, G23.

1 Introduction

In defined contribution pension schemes, the financial risk is borne by the member: contributions are fixed in advance and the benefits provided by the scheme depend on the investment performance experienced during the active membership and on the price of the annuity at retirement, in the case that the benefits are given in the form of an annuity. Therefore, the financial risk can be split into two parts: investment risk, during the accumulation phase, and annuity risk, focused at retirement. In order to limit the annuity risk — which is the risk that high annuity prices (driven by low bond yields) at retirement can lead to a lower than expected pension income — in many schemes the member has the possibility of deferring the annuitization of the accumulated fund. This possibility consists of leaving the fund invested in financial assets as in the accumulation phase, and allows for periodic withdrawals by the pensioner, until annuitization occurs (if ever). In UK this option is named "income drawdown option", in US the periodic withdrawals are called "phased withdrawals".

The current actuarial literature about the financial risk in defined contribution pension schemes is quite rich. Papers dealing with the financial risk in DC schemes in the accumulation phase are, for instance, Blake, Cairns and Dowd (2001), Booth and Yakoubov (2000), Boulier, Huang and Taillard (2001), Haberman and Vigna (2002), Khorasanee (1998) and Knox (1993). Arts and Vigna (2003) and Chiarolla, Longo and Stabile (2004) analyze both the accumulation and the distribution phase of a defined contribution pension scheme. The financial risk in the distribution phase of defined contribution pension schemes has been dealt with in many recent papers, including: Albrecht and Maurer (2002), Blake, Cairns and Dowd (2003), Gerrard, Haberman and Vigna (2004a), Gerrard, Haberman, Højgaard and Vigna (2004b), Gerrard, Haberman and Vigna (2006), Kapur and Orszag (1999), Khorasanee (1996), Milevsky (2001), Milevsky and Young (2002).

In this paper we assume that the retiree takes the income drawdown option: she defers the annuitization, meanwhile consumes some income withdrawn from the fund and invests the remainder of the fund. Such a pensioner has three principal degrees of freedom:

1 she can decide what investment strategy to adopt in investing the fund at her disposal;

2 she can decide how much of the fund to withdraw at any time between retirement and ultimate annuitization (if any);

3 she can decide when to annuitize (if ever).

The first two choices represent a classical inter-temporal decision making problem, which can be dealt with using optimal control techniques in the typical Merton (1971) framework (see Gerrard et al. (2006) for an example), whereas the third choice can be tackled by defining an optimal stopping time problem.

In this paper, we formulate a combined stochastic control and optimal stopping problem with the aim of outlining a decision tool that could help members of DC schemes in making their decisions regarding the three choices outlined above. The third choice, when to annuitize, has been analyzed with different approaches, for example, by Blake et al. (2003), Stabile (2006), Milevsky, Moore and Young (2006) and Milevsky and Young (2007). In this paper we find closed form solutions in terms of two constants z_0, z^* defined as solutions of given equations. We state and prove an algorithm for numerical solutions for z_0, z^* and apply this algorithm for numerical investigations of the optimization problem and its solution.

The remainder of the paper is organized as follows: section 2 outlines the general model, section 3 treats the model with quadratic utility functions, in which case a solution is constructed. In section 4 we verify that the constructed solution does solve the optimization problem. Section 5 concludes the paper with a numerical investigation of the problem.

2 The general model

2.1 Basics

A pensioner has a lump sum of size $x(0)$ at time 0, which can be invested either in a riskless asset paying interest at fixed rate r or in a risky asset, whose price evolves as a geometric Brownian motion with parameters λ and σ. The pensioner's force of mortality is supposed constant, equal to δ.

Up until the time of annuitization, the pensioner can choose what proportion of the fund to invest in the risky asset and can choose how much to withdraw from the fund. She is also able to select the time of eventual annuitization. The size of the annuity purchasable with sum x is kx, where $k > r$.

If the amount of money in the fund is ever exhausted, no further investment or withdrawal is permitted.

The pensioner derives utility $U_1(b)$ from a payment of size b before annuitization, $U_2(kx)$ from the same payment after annuitization. The introduction of two utility functions is to account for the fact that she might be wary of withdrawing money from the fund when this will increase the probability of ruin. Both U_1 and U_2 are assumed concave (but not necessarily strictly concave).

Notation:

- T_D is the pensioner's time of death, as measured from the time when the lump sum is received
- T is the time of annuitization
- T_0 is the time when the fund goes below 0
- $x(t)$ is the size of the fund at time t (where $t < \min(T, T_D, T_0)$)
- $y(t)$ is the proportion of the fund invested in the risky asset at time t
- $b(t)\,dt$ is the income withdrawn from the fund between time t and time $t + dt$.

We thus investigate the problem of choosing two continuous control variables, $y(t)$ and $b(t)$, and a stopping time, T, in such a way as to maximise the expectation of

$$\int_0^{T_D\wedge\tau} e^{-\rho t} U_1\big(b(t)\big)\,dt + \mathbf{1}_{\tau<T_D} \int_\tau^{T_D} e^{-\rho t} U_2\big(kx(\tau)\big)\,dt, \tag{2.1}$$

where $\tau = T \wedge T_0$, ρ is a subjective discount factor and the updating equation for x is

$$dx(t) = -b(t)\,dt + y(t)x(t)\big(\lambda\,dt + \sigma\,dB(t)\big) + r\big(1 - y(t)\big)x(t)\,dt,$$

where $B(\cdot)$ represents a standard Brownian motion.

Since mortality is assumed to operate independently of the evolution of the fund level, we can instead use the expecation of (2.1) with respect to the time of death as the objective function:

$$\int_0^{\tau} e^{-(\rho+\delta)t} U_1\big(b(t)\big)\, dt + \frac{e^{-(\rho+\delta)\tau}}{\rho+\delta} U_2\big(kx(\tau)\big). \tag{2.2}$$

The operation of such a scheme may be subject to local regulation:

- $b(t)$ may be restricted to lie in a given range $(b_{\min}, b_{\max})$, with both minimum and maximum values dependent on $x(0)$;
- there may be an upper limit on T, for example if pensioners are required to purchase an annuity by a given age;
- the investment strategy $y(t)$ may be constrained to be non-negative and/or to be no greater than unity, depending on rules regarding the possibility of borrowing to fund additional equity purchases or regarding the short selling of risky assets.

However, in this paper we treat only the situation of unconstrained controls.

Definition 1 (Admissible controls) *A control strategy* $(\{b(t) : t \geq 0\}, \{y(t) : t \geq 0\}, T)$ *is* admissible *if*

a) $\{b(t) : t \geq 0\}$, $\{y(t) : t \geq 0\}$ *and* T *are all adapted to the filtration generated by* $\{\big(x(t), B(t)\big) : t \geq 0\}$*;*

b) There is some constant $C_0 < \infty$ *such that, with probability 1,* $|y(t)x(t)| \leq C_0$ *for all* $t \leq T$.

Let $V_0(t, x)$ denote the supremal expected reward from time t onwards, given that the pensioner is still alive at that time and that $x(t) = x$. Then we have

$$V_0(t,x) = \max\left\{ \frac{e^{-(\rho+\delta)t}}{\rho+\delta} U_2(kx), V_0 + \sup_{b,y}\left[e^{-(\rho+\delta)t} U_1(b) + \frac{\partial V_0}{\partial t} + \mathcal{L}^{b,y} V_0(t,x) \right] \right\}, \tag{2.3}$$

where

$$\mathcal{L}^{b,y} V_0 = [-b + rx + (\lambda - r)xy] \frac{\partial V_0}{\partial x} + \tfrac{1}{2}\sigma^2 x^2 y^2 \frac{\partial^2 V_0}{\partial x^2}. \tag{2.4}$$

The compulsory termination of activity in the event of ruin implies that $V_0(t, 0-) = e^{-(\rho+\delta)t} U_2(0)/(\rho+\delta)$.

As the mechanism governing the evolution of the fund is time-homogeneous, we may deduce that V_0 takes the form

$$V_0(t,x) = e^{-(\rho+\delta)t} V(x), \tag{2.5}$$

so that, for each $x \geq 0$, either

$$V(x) \geq \frac{U_2(kx)}{\rho+\delta} \quad \text{and} \quad \sup_{b,y}\Big[U_1(b) - (\rho+\delta)V + \mathcal{L}^{b,y} V \Big] = 0 \tag{2.6}$$

or

$$V(x) = \frac{U_2(kx)}{\rho+\delta} \quad \text{and} \quad \sup_{b,y}\Big[U_1(b) - (\rho+\delta)V + \mathcal{L}^{b,y} V \Big] \leq 0 \tag{2.7}$$

A point x will be said to be in the continuation region if the first of these is the case, or in the stopping region if the second is true.

In Appendix A we prove the verification theorem which states conditions under which a function which satisfies (2.7) and (2.6) is the optimal value function.

2.2 Solution within the continuation region

If x is in the continuation region, then

$$\sup_{b,y}\big[U_1(b)-(\rho+\delta)V(x)+[-b+rx+(\lambda-r)xy]V'(x)+\tfrac{1}{2}\sigma^2x^2y^2V''(x)\big]=0.$$

We assume that there are no restrictions on y. The optimising value of y is therefore

$$y^*=y^*(x)=-\frac{(\lambda-r)V'(x)}{\sigma^2xV''(x)}, \tag{2.8}$$

as long as $V''(x)<0$. We shall be assuming that this holds for all x, i.e. that V is concave throughout the continuation region, otherwise there is no finite maximum for y..

The optimal value of b depends on the form of U_1, but we can write

$$b^*=b^*(x)=\arg\sup_b[U_1(b)-bV'(x)]. \tag{2.9}$$

Therefore

$$U_1\big(b^*(x)\big)-(\rho+\delta)V(x)-\big(b^*(x)-rx\big)V'(x)-\tfrac{1}{2}\beta^2\frac{V'(x)^2}{V''(x)}=0, \tag{2.10}$$

where β denotes the Sharpe ratio, $\beta=(\lambda-r)/\sigma$.

We make use of a method illustrated by Karatzas, Lehoczky, Sethi and Shreve (1986): define a function $X(z)$ to be the inverse of V', so that

$$V'\big(X(z)\big)=z \quad\text{and}\quad V''\big(X(z)\big)=1/X'(z).$$

The concavity of V implies that X is a decreasing function of z. We may then rewrite (2.6) as

$$U_1\big(b^*\big(X(z)\big)\big)-zb^*\big(X(z)\big)+rzX(z)-(\rho+\delta)V\big(X(z)\big)-\tfrac{1}{2}\beta^2z^2X'(z)=0. \tag{2.11}$$

The next step is to differentiate this equation with respect to z to obtain

$$\tfrac{1}{2}\beta^2z^2X''(z)+(\rho+\delta+\beta^2-r)zX'(z)-rX(z)=-b^*\big(X(z)\big). \tag{2.12}$$

The complementary function is of the form

$$X(z)=C_1z^{\alpha_1}+C_2z^{\alpha_2}, \tag{2.13}$$

where $\alpha_1>\alpha_2$ are the two roots of the quadratic

$$P(\alpha)=\tfrac{1}{2}\beta^2\alpha^2+(\rho+\delta+\tfrac{1}{2}\beta^2-r)\alpha-r.$$

Observe that the coefficient of α^2 in $P(\alpha)$ is positive and that $P(0)<0$, $P(-1)=\beta^2-r-\gamma=-(\rho+\delta)<0$. Therefore one root is positive, the other below -1. We assume that $\alpha_1>0>-1>\alpha_2$.

The particular solution is equal to $b_{\max}/r$ if $X(z)>x_2$, $b_{\min}/r$ if $X(z)<x_1$; in the case $x_1<X(z)<x_2$ let us denote the particular solution by $\xi(z)$. Thus the general solution takes the form

$$X(z) = \xi(z)+C_1z^{\alpha_1}+C_2z^{\alpha_2}, \tag{2.14}$$

$$V\big(X(z)\big) = \frac{1}{\rho+\delta}\left[\eta(z)+C_1\left(r-\tfrac{1}{2}\beta^2\alpha_1\right)z^{\alpha_1+1}+C_2\left(r-\tfrac{1}{2}\beta^2\alpha_2\right)z^{\alpha_2+1}\right], \tag{2.15}$$

where

$$\eta(z)=U_1\big(b^*\big(X(z)\big)\big)-zb^*\big(X(z)\big)+rz\xi(z)-\tfrac{1}{2}\beta^2z^2\xi'(z)$$

and we note that $\eta(z)=U_1(b_{\min})$ if $X(z)<x_1$ and $\eta(z)=U_1(b_{\max})$ if $X(z)>x_2$.

In the following sections we consider a special case, minimizing quadratic disutility functions, as treated in Gerrard et al. (2004b).

3 Quadratic model

3.1 Basics

In the formulation of the problem and the choice of the disutility function, we follow Gerrard et al. (2004b). We investigate the problem of choosing two continuous control variables, $y(t)$ and $b(t)$, and a stopping time, τ, in such a way as to minimise the expectation of

$$v \int_0^\tau e^{-(\rho+\delta)t} \left(b_0 - b(t)\right)^2 \, dt + \frac{w e^{-(\rho+\delta)\tau}}{\rho+\delta} \left(b_1 - kX(\tau)\right)^2 ,$$

where $\tau = \min(T, T_0)$, v and w are weights, k is the amount of annuity which can be purchased with one unit of money, and the updating equation for x is

$$dx(t) = -b(t)\, dt + y(t)x(t)(\lambda\, dt + \sigma\, dB(t)) + r(1 - y(t))x(t)\, dt.$$

This choice corresponds to $U_1(b) = v(b_0 - b)^2$ and $U_2(kx) = w(b_1 - kx)^2$.

The amount b_0, the income target until the annuity is purchased, will in many cases be equal to kx_0, the size of the annuity which could have been purchased if the retiree had annuitised immediately on retirement. This choice is reasonable, for UK regulations specify that the income drawn down from the fund before annuitisation cannot exceed kx_0.

The process evolves until either it is advantageous to annuitise or the fund falls to a negative value, in which case no further trading is permitted. The loss associated with annuitisation when the level of the fund is x, so that the annuity pays kx per unit time, is

$$K(x) = \frac{w}{\rho+\delta}(b_1 - kx)^2. \tag{3.1}$$

Remark 1

- The fact that annuitisation is compulsory when the fund level goes below zero implies that $V(0-) = K(0) = wb_1^2/(\rho+\delta)$.
- It is always possible to consume the interest received on the fund without investing in the risky asset. Therefore $V(x) \le v(b_0 - rx)^2/(\rho+\delta)$.
- It is always optimal to purchase an annuity if the fund level reaches b_1/k, since no further losses will be incurred in this case. If the fund level is above b_1/k, the investor can consume at rate b_0 then purchase an annuity if the fund level ever falls to b_1/k. Similarly, if the fund level is above b_0/r, she can consume b_0 without diminishing her fund. Therefore

$$V(x) = 0 \text{ for } x \ge \min\left(\frac{b_0}{r}, \frac{b_1}{k}\right).$$

- We assume that there is neither utility nor loss associated with the event of mortality before annuitisation.

Since the difference between b_0/r and b_1/k appears often, we define it:

$$D \stackrel{def}{=} \frac{b_0}{r} - \frac{b_1}{k}. \tag{3.2}$$

The formulation of the problem makes the possibility that $D < 0$ very atypical. In fact, the starting wealth is $x_0 = \frac{b_0}{k} < \frac{b_0}{r}$. In other words, the initial fund gives the possibility to buy a lifetime annuity of size b_0 which costs less than a perpetuity of size b_0. If $\frac{b_0}{r} < \frac{b_1}{k}$, the fund should cross $\frac{b_0}{r}$ before hitting the desired level $\frac{b_1}{k}$. If the fund reaches $\frac{b_0}{r}$, then, as already noted, it is optimal to invest the whole portfolio in the riskless asset and consume b_0, which gives to the pensioner the same outcome of immediate annuitization at retirement. Therefore, it would be impossible to reach the real goal which is being able to afford an annuity of size $b_1 > b_0$. Considering the fact that the utility from bequest in case of death before annuitization is here disregarded, immediate annuitization would then be preferable to the optimization program because it would avoid the ruin possibility. Thus the choice $D < 0$, although perfectly admissible from a mathematical point of view, is unreasonable in this context. For this reason, we will henceforth assume that $D > 0$.

3.2 The value function

The continuation region U is defined by

$$U := \{x \in \mathbf{R} : V(x) < K(x)\}$$

By application of (2.6), (2.7), and Theorem 12 in the Appendix, we will show that the value function of the problem satisfies the following variational inequality (HJB equation)

$$\begin{array}{llll} LV(x) = 0 & \text{and} & V(x) \leq K(x) & \text{for } x \in U \\ LV(x) \geq 0 & \text{and} & V(x) = K(x) & \text{for } x \in U^c \end{array} \tag{3.3}$$

where

$$LV(x) = \inf_{b,y}[v(b_0 - b)^2 - (\rho + \delta)V(x) + \mathcal{L}^{b,y}V(x)] \tag{3.4}$$

and where $\mathcal{L}^{b,y}$, as before, is the linear differential operator

$$\mathcal{L}^{b,y}V(x) = [-b + (\lambda - r)yx + rx]V' + \tfrac{1}{2}\sigma^2 y^2 x^2 V''.$$

In that part of the continuation region that lies between 0 and $\frac{b_1}{k}$ the optimal controls are given by

$$b^*(t) = b_0 + \frac{1}{2v}V_x\big(x(t)\big),$$

$$y^*(t) = -\frac{(\lambda - r)V_x\big(x(t)\big)}{\sigma^2 x(t)V_{xx}\big(x(t)\big)},$$

and the optimal stopping time τ^* is given by

$$\tau^* = \inf\{t \geq 0 : x(t) \notin U\}.$$

One of the difficult tasks consists in finding the continuation region U. However, exploiting the previous remark, we can prove the following:

Lemma 2 *The continuation region U contains the set $(\frac{b_1}{k}, +\infty)$, but $\frac{b_1}{k} \in U^c$.*

Therefore the only region where the problem is interesting is $\left[0, \frac{b_1}{k}\right)$.

Lemma 3 *If the set U_0 is defined by*

$$U_0 = \{x \in \mathbf{R} \ : \ LK(x) < 0\} \tag{3.5}$$

then $U_0 \subseteq U$.

Proof. If $x \in U^c$ then $V(x) = K(x)$ and $LV(x) \geq 0$, from which it follows that $LK(x) \geq 0$, i.e., $x \in U_0^c$.

Typically, one obtains information on the continuation region U by first analyzing the set U_0.

3.3 The analysis of the set U_0

The set U_0 under study is:

$$U_0 = \{x \ : LK(x) < 0\}$$

$$LK(x) = \inf_{b,y} \left\{v(b_0 - b)^2 - (\rho + \delta)K + [-b + (\lambda - r)yx + rx]K' + \tfrac{1}{2}\sigma^2 y^2 x^2 K''\right\} \tag{3.6}$$

Given the form (3.1) of $K(x)$, the minimising values of (3.6) are:

$$\hat{b}(x) = b_0 - \frac{kw}{v(\rho + \delta)}(b_1 - kx)$$

$$\hat{y}(x) = \beta \frac{b_1 - kx}{\sigma k x}.$$

By substitution, after some algebra, we obtain:

$$U_0 = \{x : w(b_1 - kx)\left[2krD - \phi(b_1 - kx)\right] < 0\}, \tag{3.7}$$

where D is given by (3.2), and

$$\phi = \rho + \delta + \beta^2 - 2r + k^2 \frac{w}{v(\rho + \delta)}. \tag{3.8}$$

Lemma 14, proved in Appendix B, allows us to deduce that the optimal behaviour when $\phi < 2krD/b_1$ is to purchase an annuity immediately, regardless of the value of x, so that $V(x) = K(x)$ for this range of values of ϕ. We therefore restrict attention to the case $\phi \geq 2krD/b_1$.

In this case,

$$U_0 = \left(-\infty, \frac{b_1}{k} - \frac{2rD}{\phi}\right) \cup \left(\frac{b_1}{k}, +\infty\right)$$

and therefore

$$U \supseteq \left[0, \frac{b_1}{k} - \frac{2rD}{\phi}\right) \cup \left(\frac{b_1}{k}, +\infty\right) \tag{3.9}$$

3.4 Solution within the continuation region

In the continuation region, the value function satisfies (see (3.3)):

$$\tfrac{1}{2}\beta^2\frac{(V')^2}{V''} + \frac{1}{4v}(V')^2 + (b_0 - rx)V' + (\rho+\delta)V = 0. \tag{3.10}$$

The optimal income to draw down and the optimal proportion of the fund to invest in the risky asset are, as usual,

$$b^*(t) = b_0 + \frac{1}{2v}V'\big(x(t)\big) \text{ and } y^*(t) = -\frac{\beta V'\big(x(t)\big)}{\sigma x V''\big(x(t)\big)}. \tag{3.11}$$

By application of the methodology illustrated in the general case, we define in this case X to be the negative of the inverse of V', so that

$$V'\big(X(z)\big) = -z.$$

The corresponding wealth function is:

$$X(z) = \frac{b_0}{r} - \frac{z}{2v(r-\gamma)} + C_1 z^{\alpha_1} + C_2 z^{\alpha_2}, \tag{3.12}$$

where γ is given by

$$\gamma = \rho + \delta + \beta^2 - r. \tag{3.13}$$

and C_1 and C_2 are constants to be determined by the boundary conditions. The corresponding value function is:

$$V\big(X(z)\big) = \frac{z^2}{4v(r-\gamma)} - \frac{1}{\rho+\delta}\left[A_1 C_1 z^{1+\alpha_1} + A_2 C_2 z^{1+\alpha_2}\right], \tag{3.14}$$

where

$$A_1 = r - \tfrac{1}{2}\beta^2\alpha_1, \qquad A_2 = r - \tfrac{1}{2}\beta^2\alpha_2. \tag{3.15}$$

Notice that the coefficients A_1 and A_2 are both positive. In fact, $P(2r/\beta^2) > 0$, so that $\alpha_i < 2r/\beta^2$ for $i = 1, 2$, thus $A_i = r - \frac{1}{2}\beta^2\alpha_i > 0$ for both i.

The optimal control functions can then be written as

$$y^*(X(z)) = -\frac{\beta}{\sigma}\frac{zX'(z)}{X(z)} \tag{3.16}$$

$$b^*(X(z)) = b_0 - \frac{z}{2v} \tag{3.17}$$

3.5 The boundary of the continuation region

According to (3.9), the form of the continuation region U is

$$U = [0, x^*) \cup (\tilde{x}, +\infty),$$

where $x^* \geq \frac{b_1}{k} - \frac{2rD}{\phi}$ and $\tilde{x} \leq \frac{b_1}{k}$. We begin by investigating $\tilde{x}$.

Lemma 4 $\tilde{x} = \frac{b_1}{k}$.

Proof. Since $V(x) \leq K(x)$, from $K(b_1/k) = K'(b_1/k) = 0$ it follows that $V(b_1/k) = V'(b_1/k) = 0$. Suppose that every interval of the form $(b_1/k - \epsilon, b_1/k)$ (for $\epsilon > 0$) contains an element of U. Then letting $\epsilon \to 0$ implies the existence of a z such that $X(z) = b_1/k$ satisfying $z = -V'(b_1/k) = 0$. However, if $z = 0$, then $X(z)$, which is given by (3.12), cannot be equal to b_1/k. This contradiction shows that the assumption was false. Therefore, for sufficiently small ϵ,

$$\left(\frac{b_1}{k} - \epsilon, \frac{b_1}{k}\right) \subset U^c,$$

and we conclude that $\tilde{x}$ cannot be less than b_1/k. □

Intuitively, this result can be explained by observing that if $\tilde{x}$ were strictly lower than $\frac{b_1}{k}$, then $\frac{b_1}{k}$ would stay in U, which is absurd, since it is clear that when reaching $\frac{b_1}{k}$ one should stop investing and annuitize to get zero loss.

It remains to determine x^*. One obvious characteristic is that

$$V(x^*) = K(x^*). \tag{3.18}$$

In addition, we may apply the "smooth fit principle" to obtain the further condition that

$$V'(x^*) = K'(x^*). \tag{3.19}$$

If we define z_* by $z_* = -V'(x^*)$, so that $X(z_*) = x^*$, then these two boundary conditions (3.18) and (3.19) can be written in the form

$$\begin{aligned} -z_* &= -\tfrac{2kw}{\rho+\delta}(b_1 - kx^*) \\ \tfrac{w}{\rho+\delta}(b_1 - kx^*)^2 &= \tfrac{z_*^2}{4v(r-\gamma)} - \tfrac{1}{\rho+\delta}\left[A_1C_1z_*^{1+\alpha_1} + A_2C_2z_*^{1+\alpha_2}\right] \\ x^* &= \tfrac{b_0}{r} - \tfrac{z_*}{2v(r-\gamma)} + C_1z_*^{\alpha_1} + C_2z_*^{\alpha_2} \end{aligned} \tag{3.20}$$

In addition, we require a boundary condition at $x = 0$. Since the pensioner is forced to purchase an annuity as soon as the fund becomes negative, one possible boundary condition is that $V(0) = K(0)$. A solution to the problem which satisfies this boundary condition will be called a Type 1 solution.

However, this is not the only possibility, since there exist strategies which ensure that the fund level never falls below 0. For example, the pensioner could stop investing in the risky asset as soon as x falls below ϵ, and instead consume only the interest on the fund. This leads to a penalty of $vb_0^2/(\rho+\delta)$ when $x = 0$, which may be strictly less than $K(0)$. Such a solution will be called a type 2 solution, and is characterized by the condition $\lim_{x\to 0} xy^*(x) = 0$, or, in other words, due to (3.16), there exists a value of z such that both $X(z) = 0$ and $X'(z) = 0$.

It is then clear that if

$$\frac{v}{w} < \left(\frac{b_1}{b_0}\right)^2 \tag{3.21}$$

then solution type 1 will not be feasible.

Although in general the solution $X(z)$ of (3.12) might not hit zero, any version of X which might be considered as a solution to the current problem must hit 0 at some point. We therefore define $z_0 = \inf\{z > 0 : X(z) = 0\}$, so that

$$\frac{b_0}{r} - \frac{z_0}{2v(r-\gamma)} + C_1z_0^{\alpha_1} + C_2z_0^{\alpha_2} = 0. \tag{3.22}$$

Then the boundary condition at z_0 corresponding to a Type 1 solution, $V(0) = K(0)$ is

$$\frac{z_0^2}{4v(r-\gamma)} - \frac{1}{\rho+\delta}\left[A_1 C_1 z_0^{1+\alpha_1} + A_2 C_2 z_0^{1+\alpha_2}\right] = \frac{wb_1^2}{\rho+\delta}, \tag{3.23}$$

while for a Type 2 solution the appropriate requirements, $X'(z_0) = 0$ and $V(0) \leq K(0)$, are

$$\begin{array}{rcl} \alpha_1 C_1 z_0^{\alpha_1 - 1} + \alpha_2 C_2 z_0^{\alpha_2 - 1} & = & \frac{1}{2v(r-\gamma)} \\ & \text{and} & \\ \frac{z_0^2}{4v(r-\gamma)} - \frac{1}{\rho+\delta}\left[A_1 C_1 z_0^{1+\alpha_1} + A_2 C_2 z_0^{1+\alpha_2}\right] & \leq & \frac{wb_1^2}{\rho+\delta} \end{array} \tag{3.24}$$

3.6 Construction of a solution

The method of construction is to start with a candidate value z_c for z_*, to derive appropriate values of C_1, C_2 and z_0 and to check whether this constitutes a solution to the problem.

Since $\frac{b_1}{k} > x^* \geq \frac{b_1}{k} - \frac{2rD}{\phi}$, we see from (3.20) that any solution z_* must satisfy

$$0 < z_* \leq \frac{4k^2 wrD}{\phi(\rho+\delta)} = z_U, \text{ say,}$$

and so we choose z_c in this range.

3.6.1 Signs of C_1 and C_2

From (3.20) it follows that the corresponding values of C_1 and C_2 must be

$$C_1(z_c) = \frac{2{z_c}^{-\alpha_1}}{\beta^2(\alpha_1-\alpha_2)}\left[-A_2 D - \phi\frac{\big(r-\gamma+\beta^2(1-\alpha_2)\big)(\rho+\delta)}{4k^2 w(\gamma-r)} z_c\right]. \tag{3.25}$$

$$C_2(z_c) = \frac{2{z_c}^{-\alpha_2}}{\beta^2(\alpha_1-\alpha_2)}\left[A_1 D + \phi\frac{\big(r-\gamma+\beta^2(1-\alpha_1)\big)(\rho+\delta)}{4k^2 w(\gamma-r)} z_c\right] \tag{3.26}$$

After some algebra, one can prove that $r-\gamma+\beta^2(1-\alpha_2) > 0$ and that $r-\gamma+\beta^2(1-\alpha_1) > 0 \iff r > \gamma$. From this it is possible to prove that $C_2(z_c) > 0$ if and only if

$$z_c < \frac{4k^2 wD}{\phi(\rho+\delta)}\left(r + \tfrac{1}{2}\beta^2\alpha_1\right) = z_U\left(1 + \frac{\beta^2}{2r}\alpha_1\right), \tag{3.27}$$

so this is always true for the range of values of z_c under consideration.

By means of a similar argument we find that $C_1(z_c) > 0$ if and only if

$$r > \gamma \quad \text{and} \quad z_c > z_U\left(1 + \frac{\beta^2}{2r}\alpha_2\right). \tag{3.28}$$

3.6.2 Behaviour of the function $X(z)$

Recall equation (3.12) giving the solution for $X(z)$. Since $X(z)$ depends on C_1 and C_2, we can regard it, too, as a function of z_c, denoted as $X(z; z_c)$. Notice that $\lim_{z\to 0} X(z; z_c) = +\infty$, as $\alpha_2 < 0$ and $C_2(z_c) > 0$.

Now observe that

$$\frac{\partial^2 X}{\partial z^2}(z; z_c) = \alpha_1(\alpha_1 - 1)C_1(z_c)z^{\alpha_1 - 2} + \alpha_2(\alpha_2 - 1)C_2(z_c)z^{\alpha_2 - 2}. \tag{3.29}$$

By investigating $P(1)$ we see that $\alpha_1 > 1 \iff r > \gamma$. Combining this result with (3.28), we notice that we have to consider three possible situations.

Situation 1: If $r < \gamma$ then $0 < \alpha_1 < 1$ and $C_1(z_c) < 0$, so the right hand side of (3.29) is positive, implying that X is convex, viewed as a function of z. In addition, $X(z; z_c) = \frac{z}{2v(\gamma - r)}(1 + o(1))$ as $z \to \infty$. Therefore X has a unique minimum value for each fixed z_c.

Situation 2: If $r > \gamma$ and $z_c > z_U(1 + \frac{1}{2}\beta^2\alpha_2/r)$ then $\alpha_1 > 1$ and $C_1(z_c) > 0$, again implying that X is convex. In this case $X(z; z_c) = C_1(z_c)z^{\alpha_1}(1 + o(1))$ as $z \to \infty$. Therefore X again has a unique minimum value for each z_c.

Situation 3: If $r > \gamma$ and $z_c < z_U(1 + \frac{1}{2}\beta^2\alpha_2/r)$. In this case $C_1(z_c) < 0$ and we conclude that $\frac{\partial X}{\partial z}(z; z_c) < 0$ for all z; indeed, as $z \to \infty$, $X(z; z_c) = C_1(z_c)z^{\alpha_1}\big(1 + o(1)\big) \to -\infty$.

Notice that in situation 3 one can have only type 1 solution, whereas situations 1 and 2 allow for both types of solution.

On differentiating (3.25) and (3.26), we find that

$$\frac{dC_1}{dz_c} = \frac{2(1 + \alpha_1)}{\beta^2(\alpha_1 - \alpha_2)} \cdot \frac{\phi(\rho + \delta){z_c}^{-(1+\alpha_1)}}{4k^2 w}(z_U - z_c) \tag{3.30}$$

$$\frac{dC_2}{dz_c} = -\frac{2(1 + \alpha_2)}{\beta^2(\alpha_1 - \alpha_2)} \cdot \frac{\phi(\rho + \delta){z_c}^{-(1+\alpha_2)}}{4k^2 w}(z_U - z_c). \tag{3.31}$$

For a fixed value of z we obtain

$$\frac{\partial}{\partial z_c}X(z; z_c) = \frac{2}{\beta^2(\alpha_1 - \alpha_2)} \cdot \frac{\phi(\rho + \delta){z_c}^{-1}}{4k^2 w}(z_U - z_c)\left\{(1 + \alpha_1)\left(\frac{z}{z_c}\right)^{\alpha_1} - (1 + \alpha_2)\left(\frac{z}{z_c}\right)^{\alpha_2}\right\}.$$

Every term is positive. So, as we decrease z_c, the value of $X(z; z_c)$ also decreases for each fixed z. We can conclude that $\inf_{z\geq 0} X(z; z_c)$ decreases as z_c decreases.

Proposition 5 *For z_c sufficiently small,* $\inf_{z>0} X(z; z_c) < 0$.

Proof. We can write

$$\begin{aligned}
X(z; z_c) &= \frac{b_0}{r} - \frac{z}{2v(r - \gamma)} + C_1(z_c)z^{\alpha_1} + C_2(z_c)z^{\alpha_2} \\
&= \frac{b_0}{r} - \frac{z}{2v(r - \gamma)} + \frac{2}{\beta^2(\alpha_1 - \alpha_2)}\left(\frac{z}{z_c}\right)^{\alpha_1}\left[-A_2 D - \phi\frac{\big(r - \gamma + \beta^2(1 - \alpha_2)\big)(\rho + \delta)}{4k^2 w(\gamma - r)}z_c\right] \\
&\quad + \frac{2}{\beta^2(\alpha_1 - \alpha_2)}\left(\frac{z}{z_c}\right)^{\alpha_2}\left[A_1 D + \phi\frac{\big(r - \gamma + \beta^2(1 - \alpha_1)\big)(\rho + \delta)}{4k^2 w(\gamma - r)}z_c\right] \\
&= \frac{b_0}{r} - \frac{\zeta z_c}{2v(r - \gamma)} + \frac{2}{\beta^2(\alpha_1 - \alpha_2)}\left\{\zeta^{\alpha_1}[-A_2 D + d_2 z_c] + \zeta^{\alpha_2}[A_1 D + d_1 z_c]\right\},
\end{aligned}$$

where $\zeta = z/z_c$, and d_1, d_2 are constants. Since $A_2 > 0$, we can choose ζ sufficiently large that

$$\frac{2D}{\beta^2(\alpha_1-\alpha_2)}[-A_2\zeta^{\alpha_1} + A_1\zeta^{\alpha_2}] < -4\frac{b_0}{r}.$$

Now choose z_c sufficiently small that

$$\max\left\{\frac{2|d_2|z_c}{\beta^2(\alpha_1-\alpha_2)}\zeta^{\alpha_1}, \frac{2|d_1|z_c}{\beta^2(\alpha_1-\alpha_2)}\zeta^{\alpha_2}\right\} < \frac{b_0}{r}.$$

If $r-\gamma > 0$ then it is easily seen that $X(z; z_c) < 0$. If, on the other hand, $r-\gamma < 0$ then choose z so small that $\frac{\zeta z_c}{2v(r-\gamma)} < \frac{b_0}{r}$. Then $X(\zeta z_c; z_c) < 0$, as required.□

To begin the construction process, we set $z_c = z_U$, so that $X(z; z_U)$ is a convex function of z and has a unique minimum. Depending on $r-\gamma$, we are then either in situation 1 or 2. What happens next depends on whether $\inf_{z\geq 0} X(z; z_U)$ is positive or negative.

Case 1: $\inf_{z\geq 0} X(z; z_U) \geq 0$

In this case we can progressively reduce z_c, which in turn reduces the minimum value of $X(z; z_c)$, until z_c is just large enough that $\inf_z X(z; z_c) = 0$, in other words, that $\frac{\partial}{\partial z}X(z; z_c) = 0$ at exactly the point when $X(z; z_c) = 0$: let z_M denote the value of z_c when this occurs. If, in this case, $V(0; z_M) \leq K(0)$, then the boundary conditions (3.24) are satisfied, so we have a Type 2 solution and the problem is solved: z_* is equal to z_M and z_0 is $\arg\min X(z; z_M)$.

If, however, $V(0; z_M) > K(0)$, then no Type 2 solution is possible, but we can still seek a Type 1 solution (notice that in this case (3.21) is violated). To this end, we continue to reduce z_c. For each z_c, define $z_0(z_c) = \inf\{z \geq 0 : X(z; z_c) \leq 0\}$. Then z_0 is a decreasing function of z_c. Consider $V(0; z_c) - K(0)$: by assumption this is positive when $z_c = z_M$.

z_0 is given by $0 = \frac{b_0}{r} - \frac{z_0}{2v(r-\gamma)} + C_1 z_0^{\alpha_1} + C_2 z_0^{\alpha_2}$. This implies that

$$\begin{aligned}
0 &= \left[-\frac{1}{2v(r-\gamma)} + \alpha_1 C_1 z_0^{\alpha_1-1} + \alpha_2 C_2 z_0^{\alpha_2-1}\right]\frac{\partial z_0}{\partial z_c} + z_0^{\alpha_1}\frac{\partial C_1}{\partial z_c} + z_0^{\alpha_2}\frac{\partial C_2}{\partial z_c} \\
&= \left[-\frac{1}{2v(r-\gamma)} + \alpha_1 C_1 z_0^{\alpha_1-1} + \alpha_2 C_2 z_0^{\alpha_2-1}\right]\frac{\partial z_0}{\partial z_c} \\
&\quad + \frac{2\phi(\rho+\delta)}{4k^2 w\beta^2(\alpha_1-\alpha_2)z_c}(z_U - z_c)\left\{(1+\alpha_1)\left(\frac{z_0}{z_c}\right)^{\alpha_1} - (1+\alpha_2)\left(\frac{z_0}{z_c}\right)^{\alpha_2}\right\}
\end{aligned} \tag{3.32}$$

In addition, $V(0; z_c) = \frac{z_0^2}{4v(r-\gamma)} - (\rho+\delta)^{-1}[A_1 C_1 z_0^{1+\alpha_1} + A_2 C_2 z_0^{1+\alpha_2}]$. This implies that

$$\begin{aligned}
\frac{\partial}{\partial z_c}V(0; z_c) &= \left\{\frac{z_0}{2v(r-\gamma)} - (\rho+\delta)^{-1}[(1+\alpha_1)A_1 C_1 z_0^{\alpha_1} + (1+\alpha_2)A_2 C_2 z_0^{\alpha_2}]\right\}\frac{\partial z_0}{\partial z_c} \\
&\quad - (\rho+\delta)^{-1}\left[A_1 z_0^{1+\alpha_1}\frac{\partial C_1}{\partial z_c} + A_2 z_0^{1+\alpha_2}\frac{\partial C_2}{\partial z_c}\right] \\
&= \left\{\frac{z_0}{2v(r-\gamma)} - \alpha_1 C_1 z_0^{\alpha_1} - \alpha_2 C_2 z_0^{\alpha_2}\right\}\frac{\partial z_0}{\partial z_c} \\
&\quad - \frac{2\phi(\rho+\delta)}{4k^2 w\beta^2(\alpha_1-\alpha_2)}(z_U - z_c)\left\{\alpha_1\left(\frac{z_0}{z_c}\right)^{1+\alpha_1} - \alpha_2\left(\frac{z_0}{z_c}\right)^{1+\alpha_2}\right\}
\end{aligned} \tag{3.33}$$

Putting these together gives

$$\frac{\partial}{\partial z_c}V(0; z_c) = \frac{2\phi(\rho+\delta)}{4k^2 w\beta^2(\alpha_1-\alpha_2)}(z_U - z_c)\left\{\left(\frac{z_0}{z_c}\right)^{1+\alpha_1} - \left(\frac{z_0}{z_c}\right)^{1+\alpha_2}\right\}$$

Every term on the RHS is positive, so $V(0; z_c)$ is an increasing function of z_c: as z_c decreases, $V(0; z_c)$ also decreases. There may come a value of z_c at which $V(0; z_c) = K(0)$. If so, the boundary condition (3.23) is satisfied and we have a Type 1 solution to the problem, since by construction $X'(z) < 0$ for all $z < z_0$.

We should check that $V(0; z_c)$ really does reach $K(0)$ eventually. Let us consider what happens when z_c is close to 0. In this case

$$C_2(z_c) = \frac{2A_1 D {z_c}^{-\alpha_2}}{\beta^2(\alpha_1 - \alpha_2)}(1 + O(z_c)), \quad C_1(z_c) = -\frac{2A_2 D {z_c}^{-\alpha_1}}{\beta^2(\alpha_1 - \alpha_2)}(1 + O(z_c))$$

and therefore

$$X(z; z_c) = \frac{b_0}{r} - \frac{\zeta z_c}{2v(r-\gamma)} + \frac{2D}{\beta^2(\alpha_1 - \alpha_2)}\left[A_1 \zeta^{\alpha_2} - A_2 \zeta^{\alpha_1}\right] + O(z_c),$$

where $\zeta = z/z_c$. This implies that $z_0(z_c) = \zeta_0 z_c(1 + O(z_c))$, where ζ_0 is the solution to

$$\frac{2D}{\beta^2(\alpha_1 - \alpha_2)}\left[A_2 \zeta^{\alpha_1} - A_1 \zeta^{\alpha_2}\right] = \frac{b_0}{r}.$$

(This definitively does have a solution $\zeta_0 > 1$ because putting $\zeta = 1$ on the LHS gives D, which is less than b_0/r, whereas when $\zeta \to \infty$ the LHS diverges to $+\infty$.)

Now

$$V(0; z_c) = V \circ X(z_0(z_c); z_c) = \frac{\zeta_0^2 {z_c}^2}{4v(r-\gamma)} + \frac{2A_1 A_2 D \zeta_0 z_c}{\beta^2(\alpha_1 - \alpha_2)(\rho + \delta)}\left[\zeta_0^{\alpha_1} - \zeta_0^{\alpha_2}\right] + O(z_c)$$

Therefore $\lim_{z_c \to 0} V(0; z_c) = 0 < K(0)$, as required.

Case 2: $\inf_{z \geq 0} X(z; z_U) < 0$

In this case no Type 2 solution is possible. We define $z_0(z_c)$ as above. If $V \circ X(z_0(z_U); z_U) < K(0)$, then no type 1 solution is possible either, since reducing the value of z_c below z_U will only have the effect of decreasing $V(0; z_c)$, and there will be no value of z_c which gives $V(0; z_c) = K(0)$. If, however, $V \circ X(z_0(z_U); z_U) \geq K(0)$, then progressively reducing z_c will eventually result in a value such that $V \circ X(z_0(z_c); z_c) = K(0)$, which corresponds to a type 1 solution.

4 Application of the verification theorem

We are now in a position to state and prove a theorem showing that the constructed solution satisfies the verification theorem (Theorem (12)).

Theorem 6 *Assume that $D > 0$ and that $\phi \geq 2krD/b_1$. Suppose that there exist constants C_1, C_2, z_0 and z_* with $0 < z_* < z_0 < \infty$, such that*

a) the function $X(z)$ given by (3.12) satisfies the boundary conditions (3.20), (3.22) and either (3.23) or (3.24);

b) $-\infty < X'(z) < 0$ for all $z_* \leq z < z_0$.

Then

(i) For each $z \in (z_*, z_0)$ there is a corresponding $x \in (0, x^*)$ such that $X(z) = x$;

(ii) The function V given by

$$\begin{array}{ll} V(x) = 0 & \text{for } x \geq \frac{b_1}{k} \\ V(x) = K(x) & \text{for } x^* \leq x \leq \frac{b_1}{k} \\ V\big(X(z)\big) \text{ is given by (3.14)} & \text{for } z_* \leq z \leq z_0 \end{array} \tag{4.1}$$

is the optimal value function;

(iii) The optimal time to annuitise is $\tau^* = \inf\{t : x(t) \in U^c\}$, where the continuation set U is given by

$$U = [0, x^*) \cup \left(\frac{b_1}{k}, \infty\right);$$

(iv) For values of x belonging to the continuation region U, the optimal controls are given by

$$y^*(t) = -\frac{\lambda - r}{\sigma^2} \cdot \frac{V'\big(x(t)\big)}{x(t)\, V''\big(x(t)\big)}, \qquad b^*(t) = b_0 + \frac{1}{2v} V'\big(x(t)\big).$$

In order to prove the theorem we need to prove the following proposition.

Proposition 7 *Suppose $(C_1, C_2, z_0, z_*, x^*)$ constitutes either a type 1 solution or a type 2 solution constructed as above. Then*

$$V(x) - K(x) \leq 0 \qquad \text{for} \qquad 0 \leq x \leq x^*.$$

Proof. Note to begin with that an essential feature of the method of construction is that $X'(z) < 0$ for $0 < z < z_0$.

The proof consists of a series of lemmas.

Lemma 8 *Suppose there exists $\tilde{x} \in (0, x^*)$ such that*

$$\begin{array}{ll} V'(x) \leq K'(x) & \text{for } 0 < x < \tilde{x} \\ V'(x) \geq K'(x) & \text{for } \tilde{x} < x < x^*. \end{array} \tag{4.2}$$

Then

$$V(x) - K(x) \leq 0 \qquad \text{for} \qquad 0 \leq x \leq x^*.$$

Proof. We know that $V(0) - K(0) \leq 0$ and $V(x^*) = K(x^*)$. For any $x \in (0, \tilde{x}]$,

$$V(x) - K(x) = V(0) - K(0) + \int_0^x (V'(s) - K'(s))\, ds \leq 0;$$

similarly, for any $x \in (\tilde{x}, x^*)$,

$$V(x) - K(x) = -\int_x^{x^*} (V'(s) - K'(s))\, ds \leq 0$$

□

Lemma 9 *Define $F(z) = V'\big(X(z)\big) - K'\big(X(z)\big)$ for $z \in (z_*, z_0)$.*
(a) If $F(z) > 0$ for $z_* < z < z_0$, then $V(x) \leq K(x)$ for $0 < x < x^*$.
(b) If F is concave on (z_*, z_0), then either there exists $\tilde{x} \in (0, x^*)$ such that the condition (4.2) is satisfied or F is strictly positive on (z_*, z_0).

Proof. (a) $V(X(z)) - K(X(z)) = \int_{z_*}^{z}[V'(X(\zeta)) - K'(X(\zeta))]X'(\zeta)\,d\zeta = \int_{z_*}^{z} F(\zeta)X'(\zeta)\,d\zeta \leq 0$.
(b) Suppose F is concave for $z \in (z_*, z_0)$. Recall that $F(z_*) = V'(X(z_*)) - K'(X(z_*)) = 0$ and $\int_{z_*}^{z_0} F(z)X'(z)\,dz = V(0) - K(0) \leq 0$. F cannot be strictly negative throughout (z_*, z_0), since this would violate the integral condition. Therefore either F is strictly positive or there exists some $\tilde{z} \in (z_*, z_0)$ such that $F(z)$ is positive for $z_* < z < \tilde{z}$ and negative for $\tilde{z} < z < z_0$. □

Lemma 10 *$V'(X(z)) - K'(X(z))$ is either concave for $z \in (z_*, z_0)$ or strictly positive for $z \in (z_*, z_0)$.*

Proof.

$$F(z) = V'(X(z)) - K'(X(z)) = -z + \frac{2k^2 w}{\rho + \delta}\left(\frac{b_1}{k} - X(z)\right)$$

If either (a) $r < \gamma$ or (b) $r > \gamma$ and $z_U > z_* > z_U(1 + \frac{1}{2}\beta^2\alpha_2/r)$, then

$$F''(z) = -\frac{2k^2 w}{\rho + \delta}X''(z) < 0,$$

proving that F is concave.

If, on the other hand, $r > \gamma$ and $z_* < z_U(1 + \frac{1}{2}\beta^2\alpha_2/r)$, then $C_1 < 0$ and

$$F'(z) = -1 - \frac{2k^2 w}{\rho + \delta}\left[-\frac{1}{2v(r - \gamma)} + \alpha_1 C_1 z^{\alpha_1 - 1} + \alpha_2 C_2 z^{\alpha_2 - 1}\right] = \frac{\phi}{r - \gamma} - \frac{2k^2 w}{\rho + \delta}\left[\alpha_1 C_1 z^{\alpha_1 - 1} + \alpha_2 C_2 z^{\alpha_2 - 1}\right].$$

Every term on the right hand side is positive, so F is strictly increasing on the range (z_*, z_0). As $F(z_*) = 0$, it follows that $F(z) > 0$ for $z_* < z < z_0$. □

The proof of the proposition is now straightforward by application of the previous lemmas. □

4.1 Proof of Theorem 6

(i) is clear, since the function $X(z)$ given by (3.12) is continuous and strictly decreasing, hence invertible over the range.

(ii) In order to show that the function V defined in the Theorem is the optimal value function, we need to show, first, that it satisfies (3.3). Second, that the controls specified in the Theorem are admissible.

The first requirement is that

$$\inf_{b,y}\left\{\mathcal{L}^{b,y}K - (\rho + \delta)K + \frac{v}{\rho + \delta}(b_0 - b)^2\right\} \geq 0 \text{ for all } x \in U^c.$$

This is guaranteed by the fact that $U^c \subset U_0^c$ (see 3.5). Furthermore, $V(x) = K(x)$ by definition.

Next we need to show that

$$\inf_{b,y}\left\{\mathcal{L}^{b,y}V - (\rho + \delta)V + \frac{v}{\rho + \delta}(b_0 - b)^2\right\} = 0 \text{ for all } x \in [0, x^*).$$

By construction, the function V does satisfy this condition as long as $V''(x) > 0$ for all $x \in (0, x^*)$, i.e., as long as $V''\big(X(z)\big) > 0$ for all $z \in (z_*, z_0)$. But $V''\big(X(z)\big) = -1/X'(z)$, so condition (b) of the Theorem is sufficient to demonstrate that this is true.
Furthermore, we have $V(x) \leq K(x)$ for $x \in (0, x^*)$ (see Proposition 7).

Next we turn to the proof of admissibility. By construction, $b^*(t)$ and $y^*(t)$ are functions of $x(t)$, and τ^* is adapted to the filtration generated by $x(t)$. It therefore remains only to prove that $|y^*(t)x(t)|$ has a finite bound with probability 1. Under the stated policy, given that $x(t) = x$, we have

$$xy^*(x) = -\frac{\lambda - r}{\sigma^2} \cdot \frac{V'(x)}{V''(x)} = -\frac{\lambda - r}{\sigma^2} \, zX'(z),$$

Now $|X'|$ is a continuous function on a compact interval, so has a finite maximum, C_0, say, over the interval. Thus $|y^*(t)x(t)| \leq \frac{\lambda - r}{\sigma^2} C_0 z_0$ for all $t \leq \tau^*$.

(iii) follows from Theorem 12. Showing that U takes this shape is rather technical and follows from the analysis contained in section 3.3.

(iv) follows from Theorem 12, by observing that b^* and y^* are the minimizers of $LV(x)$.

This ends the proof. □

5 Numerical application

In this section we show two numerical applications of the model presented.

Firstly, with the help of a Perl program that finds the solution with the methodology described in section 3.6 above, we have found the triplet solution (z_0, z^*,x^*) with a number of different scenarios for market conditions and demographic assumptions. Recalling the form of the continuation region $U = [0, x^*)$, where $x^* \leq b_1/k$, it seems of crucial interest to study the dependence of the width of the continuation region on the parameters of the problem. This is done by analyzing the ratio $x^*/(b_1/k)(\leq 1.)$ Results are reported in section 5.1.

Secondly, we have chosen a typical scenario for all the parameters and have simulated the behaviour of the risky asset, by means of Monte Carlo simulations. We have then analyzed the optimal investment/consumption strategies and time of optimal annuitization as well as size of the annuity upon annuitization. Results are reported in section 5.2.

5.1 Dependence of the solution on the scenario

Recall that in a realistic setting some of the parameters are chosen by the retiree and some are given.
Parameters that can be chosen are the weights given to penalty for running consumption, v, and to penalty for final annuitization, w. We remark that the relevant quantity is the ratio of these weights, w/v. Another parameter chosen by the retiree is the targeted level of annuity, b_1, while it is reasonable to assume that the level of interim consumption b_0 is given and depends on the size of the fund at retirement. A typical choice for b_0 is the size of annuity purchasable at retirement with the initial fund x_0. Thus, typically b_1 is a multiple of b_0, and the relevant quantity is b_1/b_0. It is easy to see this ratio as a measure of the risk aversion of the retiree: the higher b_1/b_0, the lower the risk aversion and vice versa.
The parameters given are r, λ, σ (financial market), δ (demographic assumptions) and k (financial

and demographic assumptions).
A parameter that is somehow arbitrary and somehow given is ρ, the intertemporal discount factor: although subjective by its own nature, in typical situations cannot differ too much from the riskfree rate of return r. However, what is relevant in the problem is the sum $\rho + \delta$, which measures the patience of the retiree for future events.

By varying the values of $r \in (0.03, 0.05)$, $\lambda \in (0.07, 0.12)$, $\sigma \in (0.1, 0.25)$, $\rho \in (0.03, 0.05)$, $\delta \in (0.005, 0.02)$, $k \in (0.07, 0.1)$, $b_1/b_0 \in (1.2, 2)$, $w/v \in (0.275, 1.25)$, and combining them in many possible ways, we have observed the following results:

1. with typical values of the market parameters, situation 1 ($r < \gamma$) is the most likely to occur
2. the case of no solution seems to occur only with situation 2 ($r > \gamma$)
3. with typical values, solution type 2 is the most frequent one
4. everything else being equal, solution type 2 becomes solution type 1 when
   (a) decreasing β; furthermore, if β is decreased too much solution type 1 becomes "no solution"
   (b) decreasing $\frac{w}{v}$
   (c) decreasing $\frac{b_1}{b_0}$ (provided that the values of ρ and $\frac{w}{v}$ are respectively high and low enough to permit solution type 1)
   (d) increasing $\rho + \delta$
5. everything else being equal, the ratio $\frac{x^*}{(b_1/k)}$
   (a) increases by increasing β, in both solutions type 1 and 2
   (b) increases by increasing $\frac{w}{v}$, in both solutions type 1 and 2
   (c) increases by increasing $\frac{b_1}{b_0}$, in both solutions type 1 and 2
   (d) generally slightly decreases by increasing $\rho + \delta$ when the problem has solution type 2, slightly increases by increasing $\rho + \delta$ when the problem has solution type 1

The results 5a, 5b and 5c for solution type 2 are illustrated in figures 1, 2 and 3 below (similar figures can be obtained for solution type 1). For instance, figure 1 reports β on the x-axis and the ratio $\frac{x^*}{b_1/k}$ on the y-axis, the legenda reports the values of all the other relevant parameters (left constant in order to isolate the effect of β on the width of the continuation region). All the figures show two different lines to report some of the variety of combinations of parameters tested. Similarly, figures 2 reports w/v on the x-axis, and figure 3 reports b_1/b_0 on the x-axis.

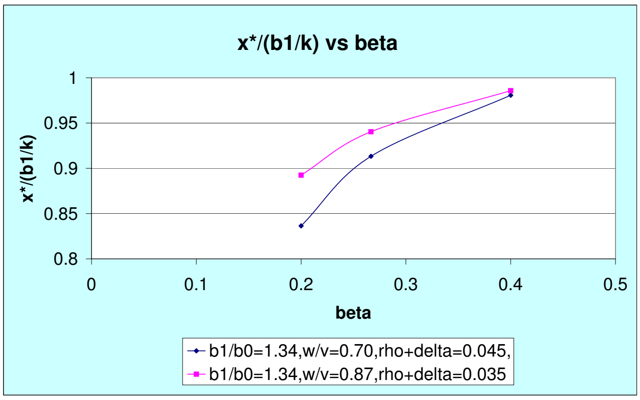

Figure 1.

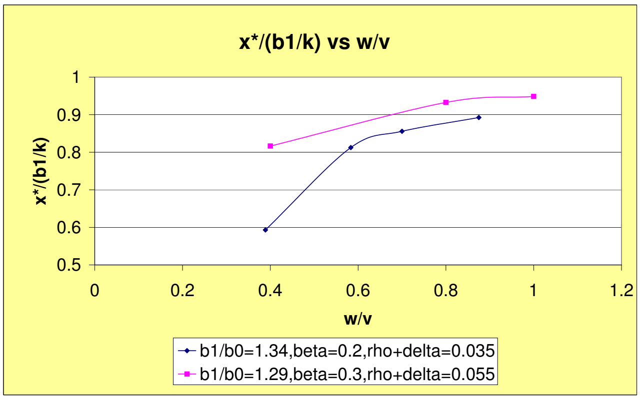

Figure 2.

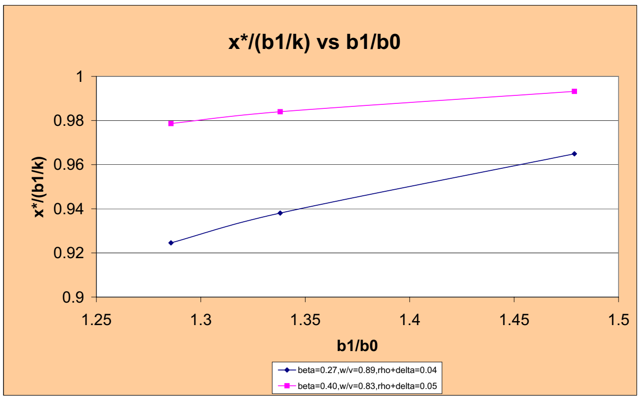

Figure 3.

The observed results can be explained. Due to 4a, 4b, 4c and 5a, 5b, 5c, it is clear that, everything else being equal, by increasing either β or w/v or b_1/b_0 one passes from solution type 1 to solution type 2 and the ratio $\frac{x^*}{(b_1/k)}$, that determines the width of the continuation region, increases as well.

This shows that, in general, the continuation region is larger with solution type 2 than with solution type 1.

The intuition behind this is that it is optimal for longer time to trade your own wealth if you choose and/or are given a set of parameters that lead you to solution type 2 than if you are in a solution type 1 case. On the other hand, if you choose and/or are given a set of parameters that lead your problem to a solution type 1, then you are likely to annuitize earlier than if you are in a solution type 2 case. This is consistent also with the fact that annuitization occurs in solution type 1 also in the case of ruin, whereas it does not with a solution type 2.

The dependence of the type of solution from the parameters is now easy to understand and explain. In fact, if β is high, the risky asset is a good one compared to the riskless one, and in this situation it is reasonable to delay annuitization as much as possible (this result was also found in Gerrard et al. (2004a)). If w/v is high, the penalty to be paid in case of annuitization before reaching b_1/k is high compared to the choice investment/consumption, which is then preferable. If b_1/b_0 is high, the retiree has a low risk aversion, thus will be likely to take chances in the financial market instead of locking his position in an annuity. Furthermore, higher values of $\rho + \delta$ are associated to old retirees, who have higher force of mortality and higher subjective discount factor (as they are less patient for future events), and it is reasonable to expect them to be more willing to annuitize rather then continuing investing in the market.

5.2 Simulations

In this application we consider the position of a male retiree aged 60, who retires with initial fund $x_0 = 1000$. We have selected the following values of the parameters:

$$r = 0.04,\ \lambda = 0.08,\ \sigma = 0.1,\ \rho + \delta = 0.045,\ w = v = 0.04,\ b_0 = 69.95,\ b_1 = 120,\ k = 0.095$$

This implies

$$\beta = 0.4 \qquad \frac{w}{v} = 1, \qquad \frac{b_1}{b_0} = 1.72, \qquad \frac{b_1}{k} = 1263.16,$$

In turn, the solution (Type 2) is

$$x^* = 1257.14, \qquad \frac{x^*}{b_1/k} = 0.995$$

We have simulated the behaviour of the risky asset with Monte Carlo simulations in 1000 scenarios, and in each scenario we have adopted the optimal investment and consumption strategies until the minimum between time of annuitization and 15 years. The choice of a terminal time of the optimization program is consistent with current regulation in UK, whereby annuitization becomes compulsory at age 75.

An interesting result is that the probability of annuitization within 15 years from retirement is 88.60% and on average optimal annuitization occurs after 4.66 years after retirement. The mean size of annuity is 90.39 and in 43.90% of the cases the annuity value lies between 90 and 100. More detailed information about optimal annuitization time and size of annuity can be gathered from the histograms reported in figures 4 and 5. In figure 5, the presence of a number of annuity values of size lower than 60 is motivated by the cases in which optimal annuitization does not occur within the time frame (114 cases out of 1000).

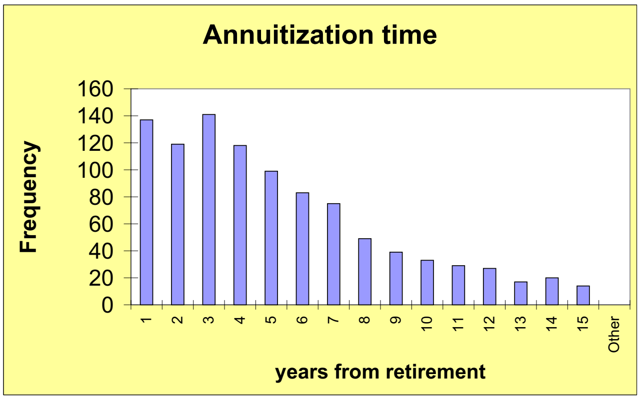

Figure 4.

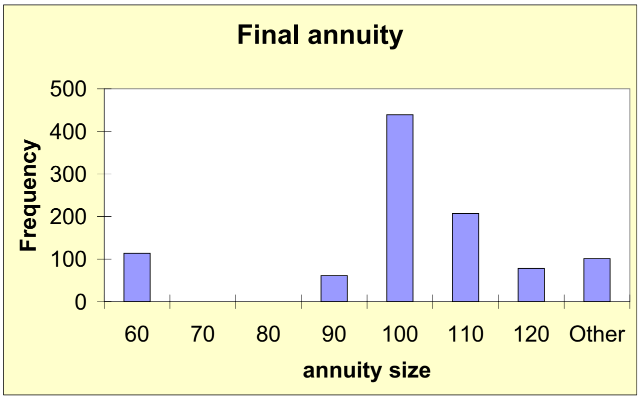

Figure 5.

Figure 6 reports some statistics of the optimal consumption in the 15 years after retirement (obviously, in the 886 scenarios in which optimal annuitization occurs before age 75, the consumption reported after annuitization time is the annuity value). The interim target consumption, b_0 is reported for comparison. We notice that on average optimal consumption is higher than the annuity purchasable at retirement, as well as the annuity obtained at optimal annuitization. This highlights the financial convenience for the retiree of deferment of annuitization until a more propitious time, in this particular scenario.

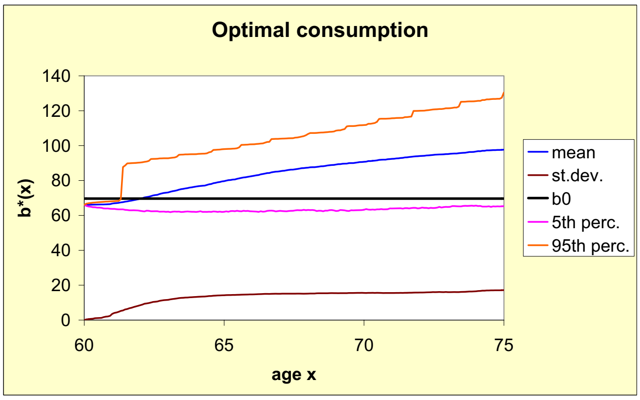

Figure 6.

In future research it would be useful to investigate and compare simulation results with different scenarios for market conditions. A comparison of the proposed model with a model with compulsory annuitization at terminal date and without possibility of earlier annuitization, though not easy, would also be interesting.

Appendix

A For the general case

Lemma 11 *Assume there exists a C^2 function W that satisfies (2.6) and (2.7) and that for all admissible controls*

$$\mathbb{E}\int_0^t y(s)x(s)W'\big(x(s)\big)e^{-(\rho+\delta)s}\,B(ds) = 0. \tag{1.1}$$

for all t. Then $W(x) \geq V(x)$ for all x.

Proof. By Dynkin's Formula and (1.1) we have for any control and stopping time T, that

$$\mathbb{E}[e^{-(\rho+\delta)T}W\big(x(T)\big) - W(x)] = \mathbb{E}\int_0^T e^{-(\rho+\delta)s}\left[\mathcal{L}^{b(s),y(s)}W\big(x(s)\big) - (\rho+\delta)W\big(x(s)\big)\right] ds. \tag{1.2}$$

From (2.6) and (2.7) the integrand on the right hand side is smaller than $-e^{-(\rho+\delta)s}U_1\big(b(s)\big)$. Hence we obtain

$$W(x) \geq \mathbb{E}\int_0^T e^{-(\rho+\delta)s}U_1(b(s))ds + e^{-(\rho+\delta)T}W(X(T)).$$

From (2.6) and (2.7)

$$W(X(T)) \geq \frac{U_2(X(T))}{\rho+\delta}$$

and it follows that $W(x) \geq V(x)$.

Theorem 12 (Verification theorem) *Let W be as in Lemma 11. Assume that $[0,\infty)$ can be split into two regions A and B such that (2.6) is satisfied in A and (2.7) in region B. Let $B^*(x), Y^*(x)$ be the maximizers of the second term in (2.7)and define the controls $y^*(t) = Y^*(x^*(t))$ and $b^*(t) = B^*(x^*(t))$, where $x^*(t)$ is the solution of (2.1b) with $y(t), b(t)$ replaced by $y^*(t), b^*(t)$. Define $T^* = \inf\{t > 0 | x^*(t) \in A\}$. Assume that*

$$\mathbb{E}[e^{-(\rho+\delta)t} W(X^*(t)) 1_{T^*=\infty}] \to 0 \tag{1.3}$$

when $t \to \infty$. Then $y^*(t), b^*(t)$ are optimal controls and T^* the optimal stopping time. Furthermore, the function $W(x) = V(x)$.

Proof. Consider the controls $y^*(t), b^*(t)$. Since on $t < T^*$, $X^*(t) \in B$ and we get

$$\begin{aligned}
&\mathbb{E}[e^{-(\rho+\delta)(T^*\wedge t)} W(x^*(T^* \wedge t)) - W(x)] \\
&= \mathbb{E} \int_0^{T^*\wedge t} e^{-(\rho+\delta)s} (\mathcal{L}^{b^*(s), y^*(s)} W(X^*(s)) - (\rho+\delta) W(x^*(s))) ds \\
&= -\mathbb{E} \int_0^{T^*\wedge t} e^{-(\rho+\delta)s} U_1(b^*(s)) ds
\end{aligned}$$

Letting $t \to \infty$, we get by (1.3) that

$$\begin{aligned}
W(x) &= \mathbb{E} \int_0^{T^*\wedge t} e^{-(\rho+\delta)s} U_1(b^*(s)) ds + e^{-(\rho+\delta)T^*} W(x^*(T^*)) 1_{T^*<\infty} \\
&= \mathbb{E} \int_0^{T^*\wedge t} e^{-(\rho+\delta)s} U_1(b^*(s)) ds + e^{-(\rho+\delta)T^*} \frac{U_2(x^*(T))}{\rho+\delta} 1_{T^*<\infty}.
\end{aligned}$$

Now the result follows by applying Lemma 11.

Corollary 13 *Assume that $(\rho+\delta)^{-1} U_2(x)$ satisfies (1.1) and*

$$\sup_{b,y} [U_1(b) - U_2(x) + \mathcal{L}^{b,y} (\rho+\delta)^{-1} U_2(x)] \le 0 \tag{1.4}$$

for all x. Then $T^* = 0$ and $V(x) = (\rho+\delta)^{-1} U_2(x)$.

Proof. The proof follows easily from Lemma 11 and Lemma **??**. That (1.3) is satisfied in this case is obvious.

B For the special case

Lemma 14 *Assume that $\phi \le 2krD/b_1$. Then, for any $x(0) \in [0, b_1/k]$, the optimal behaviour is to annuitise immediately, implying that $V(x) = K(x)$.*

Proof. The proof follows from Corollary 13. By (5.2) we have that (1.4) is fulfilled. Notice we still need to show that (1.1) is satisfied. It is simple if we have bounds on y, otherwise it is not trivial, so that is still an open problem. The general technique is to show that we can reduced the set of admissible controls to those satisfying (1.1), e.g. by showing that if (1.1) is not satisfied, the return function will be infinite.

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