A Pension Fund Model with Surplus: an Infinite Dimensional Stochastic Control Approach

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Abstract

This paper deals with the management of a pension fund with surplus. The mathematical problem is a stochastic control problem with delay, which is approached by the tool of the dynamic programming in infinite dimension. We show the equivalence between the one-dimansional delay problem and the associated infinite dimensional problem without delay. Then we prove that the value function is continuous and that it is a constrained viscosity solution of the HJB equation associated.

Keywords: Pension Funds, Stochastic Optimal Control with Delay, Dynamic Programming in Infinite Dimension, Hamilton-Jacobi-Bellman Equations, Viscosity Solutions.

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1 Introduction

This paper deals with the management of a defined contribution pension fund with benefits composed by a minimum guarantee term and a surplus term. In particular we want to find the optimal portfolio strategy assuming that the manager can invest in two assets (a risky and a riskless one, in a standard Black-Scholes market) and maximizes an intertemporal utility function depending on the current wealth over an infinite horizon. Our problem is similar to optimal portfolio selection problems but it has some special features due to the nature and the social target of the pension funds: in particular the presence of contributions and benefits, the presence of constraints on the investment strategies, the presence of solvency constraints for the wealth.

Stochastic optimization approaches to defined contribution plans with the constraint that the wealth must not be inferior to a minimum guarantee at a terminal date (socalled European guarantee) have been introduced in [1] and [12]. In these models they assume that the terminal date corresponds to the retirement of a representative worker and they apply the traditional Merton approach maximizing the total expected discounted utility from final wealth exceeding the promised guarantee. More recently in [15] the authors solve an optimal allocation problem for an investor which maximizes utility from final wealth but is constrained to stay above the guarantee at every intermediate date (so-called American guarantee). However these models are concerning with the management of contributions and benefits of a single representative participant to the pension fund (see, e.g., [1], [2], [12], [21]) and do not take into account the dynamical evolution of contributions and benefits which are related with new workers who adhere to the pension fund and those who have accrued the right of pension.

In the paper [13] the authors propose and study a continuous time stochastic model of optimal allocation for a defined contribution pension fund with minimum guarantee. Their target is to maximize the expected utility from current wealth over an infinite horizon, whereas, as described, usually portfolio selection models for pension funds maximize the expected utility from final wealth over a finite horizon. In this model the dynamics of wealth takes directly into account the flows of contributions and benefits; moreover the level of wealth is constrained to stay above a solvency level. The fund manager can invest in a riskless asset and in a risky asset but borrowing and short selling are prohibited. The model is formulated as an optimal stochastic control problem with state and control constraints and is treated by the dynamic programming approach, showing that the value function of the problem is a regular solution of the associated Hamilton-Jacobi-Bellman equation. Then they apply verification techniques to get the optimal allocation strategy in feedback form and to study its properties, giving finally a special example with explicit solution.

However usually pension funds provide for their members also a surplus term as benefit besides a minimum guarantee term. Such a term could depend on a contract *a priori* defined by the fund and it is reasonable to think that this contract is a function depending on the past wealth of the fund in the last period: the higher is the performance of the fund in the last period, the higher is the surplus paid to the fund members. From a mathematical point of view this unavoidably leads to a delay term in the state equation, making the problem considerably more difficult to treat. In our paper we add to the model of [13] a surplus term, preserving its remaining setting. Stochastic control problems with delay can be treated with the methodology described in [17], [34], [35], but this approach applies only in very specific cases and in particular it leaves out our problem. A more powerful tool is represented by the passage to infinite dimension, which applies in more generality (for this kind of technique see e.g. [10], [22], [25]). In particular this approach reduces the non-markovian delay problem to an infinite-dimensional markovian one, allowing to apply the dynamic programming in the infinite-dimensional context. We will follow this last approach.

In Section 2 we give a brief survey on the model described in [13] and then we focus our attention on the modelling of the surplus contract which may be introduced in the model. In particular, by technical reasons, we choose for our model a surplus contract which guarantees the concavity of the value function of the optimization problem. Actually this contract is, from a financial point of view, less meaningful than other ones. Nevertheless it represents a first starting point for simplifying a problem which is hard to treat in itsef. We hope to study other type of surplus contracts in future works.

In Section 3, after having given a formal setting of the stochastic optimization problem which we want to investigate, we pass to investigate some qualitative results on the value function: we prove results about finiteness, time-dependence, monotonicity and concavity.

Section 4 is devoted to rewrite the problem in an infinite-dimensional setting: following the approach contained, for example, in [43] for the deterministic case and in [6], [10], [25] for the stochastic case, we use the tool of the infinite-dimensional approach to study the problem, which essentially consists in defining a new state variable carrying the present and the past (in the last period) of the old state variable and a new state equation. However, unlike the cited references, here we have to use a more specific approach, because in our case the delay term is non-linear and this forces us to work in a suitable subspace of the Hilbert space H where we impose the infinite-dimensional problem living. This new state equation, which is an infinite-dimensional SDE, is the formal counter-part of the one-dimensional SDDE of the original problem. The next step is to prove a result of existence and uniqueness of mild solutions (and associated results on continuous dependence on initial data) for a class of infinite-dimensional SDE, in which our equation falls within. This class of equations is very specific and it is not covered by [9], so that we have to treat it in a specific way. After that, to give sense to our approach, we prove a result which actually shows the equivalence between the one-dimensional delay problem and the infinite-dimensional problem without delay.

In Section 5 we investigate the continuity properties of the value function. We are able to prove that the domain of the value function has not empty interior part and, thanks to the concavity, that it is continuous in the interior part of its domain with respect to the norm of the Hilbert space where it is defined. This fact will allow to extend the value function to an open set of H. Moreover we will show that, when the boundary is absorbing, the value function is continuous up to the boundary.

Finally Section 6 is devoted to study the Hamilton-Jacobi-Bellman equation associated with our control problem. This is a fully nonlinear infinite dimensional PDE and it cannot be treated with the techniques of [7] or [8], where existence and uniqueness of regular solution is discussed in the semi-linear case. Therefore we follow a viscosity approach; the nearest paper seems to be [32], where a second-order fully non-linear equation is studied and an existence-uniqueness result is proved. Nevertheless our problem is more difficult with respect to the one of [32], due to its particular features: the equation is defined in a subspace of the Hilbert space, it is a boundary problem due to the state constraint and the hamiltonian is not continuous on the state variable with respect to the norm of the Hilbert space. The viscosity approach in infinite dimension with state constraint is studied in [5] and [33] in the deterministic case. Here we give a definition of constrained viscosity solution and prove that the value function of our control problem solves the Hamilton-Jacobi-Bellman equation in this sense. In particular we give a definition of viscosity solution using test functions in the spirit of the common literature for viscosity solutions in infinite dimension, see in particular [32], so that a uniqueness result, at least if we define the analogous problem without state constraint, seems to be not too far.

2 The model

In this section, for convenience of the reader, we give a brief survey on the model described in [13] and we proceed to model the surplus term, which is main feature of our paper.

Over an infinite continuous-time model we consider a financial market which is:

- competitive, i.e. we assume that the investor's behavior is optimizing: she/he optimizes her/his utility function on the whole time horizon;
- frictionless, i.e. all assets are perfectly divisible and there are no transaction costs or taxes;
- arbitrage free, i.e. there is no opportunity to gain without assuming risk with not null probability;
- default free, i.e. financial institutions issuing assets cannot default;
- continuously open, i.e. the investor can continuously trade in the market.

Moreover we assume that:

- the investor is price taker, i.e. she/he cannot affect the probability distribution of the available assets: this hypothesis is usual in literature regarding financial management models of pension funds and it is realistic if the single agent does not invest a big amount of money; as a matter of fact, the volume of assets exchanged by pension funds is such that they could affect the price of assets (i.e. investor may be price maker) but we do not deal with this fact here.
- the investor faces the following trading constraints: borrowing and short positions are not allowed;
- the investor maximizes the expected utility from the current fund wealth over an infinite horizon.

Finally we impose that the pension fund wealth must be above a suitable positive function which we call *solvency level*.

2.1 The wealth dynamics

To set up the mathematical model, we consider a complete probability space $(\Omega, \mathcal{F}, P)$ on which a standard one-dimensional Brownian motion $(B(t))_{t\geq 0}$ is defined. The filtration $(\mathcal{F}_t)_{t\geq 0}$, representing the information available while time goes on, will be the completion of the natural filtration induced by the Brownian motion; this is a continuous filtration with respect to which B is still a Brownian motion (see [30] for a detailed discussion). The financial market is the classical Black-Scholes market, composed of two kinds of assets: a riskless asset and a risky asset.

So the price of the riskless asset, denoted by $S^0(t)$, $t \geq 0$, evolves according to the deterministic equation

$$\frac{dS^0(t)}{S^0(t)} = rdt, \qquad S^0(0) = 1,$$

where $r \geq 0$ is the instantaneous spot rate of return.

The price of risky asset $S^1(t)$, $t \geq 0$, follows an Itô process and satisfies the stochastic equation

$$\frac{dS^1(t)}{S^1(t)} = \mu dt + \sigma dB(t),$$

where $\mu \geq r$ is the instantaneous rate of expected return and $\sigma > 0$ is the instantaneous rate of volatility.

The drift μ can be expressed by the relation $\mu = r + \sigma\lambda$, where $\lambda \geq 0$ is the instantaneous risk premium of the market, i.e. the price per unit of volatility that the market assigns to the randomness expressed by the standard Brownian motion B.

We want to find an optimal portfolio strategy of a defined contribution pension fund with a minimum guarantee between the two assets just described. We suppose that the minimum guarantee is fixed exogenously; for an optimal design see [12].

The optimal allocation policy is determined applying the expected utility criterium: the fund manager invests the pension fund wealth between the two alternative investments maximizing her/his utility. Then the decision variable is represented by the proportion of wealth that the manager can invest respectively in the two assets offered by the market.

We suppose that the pension fund starts its activity at the date $t = 0$ and that at this time it owns a starting amount of wealth $x_0 \geq 0$. Let us denote by $x(t)$, $t \geq 0$, the $(\mathcal{F}_t)_{t\geq 0}$- progressively measurable process (state variable) which describes the amount of the pension fund wealth at every time. As we said in the introduction, we assume that a *solvency constraint* must be respected: the process $x(\cdot)$ describing the fund wealth is subject to the following constraint

$$x(t) \geq l(t) \quad P\text{-a.s.}, \tag{1}$$

for each $t \geq 0$, where the positive deterministic process $l(t)$, $t \geq 0$, is a given datum which represents the solvency level. This hypothesis is important since it prevents from improper behavior of the fund manager. If this assumption is not held, she/he could keep the fund wealth at a too low level for long periods.

We denote by $\theta(t)$, $t \geq 0$, the $(\mathcal{F}_t)_{t\geq 0}$- progressively measurable process (control variable) which represents the proportion of fund wealth to invest in the risky asset. Therefore $\theta(t) \in [0,1]$ for each $t \geq 0$, due to the borrowing and short selling constraints and to the assumption $x(t) \geq l(t) \geq 0$ for every $t \geq 0$. Then the dynamics of wealth is expressed by the following state equation

$$\begin{cases} dx(t) = \frac{\theta(t)x(t)}{S^1(t)} dS^1(t) + \frac{[1-\theta(t)]x(t)}{S^0(t)}\ dS^0(t) + [c(t) - b(t)]dt, & t \geq 0, \\ x(0) = x_0, \end{cases} \tag{2}$$

where $\dfrac{\theta(t)x(t)}{S^1(t)}$ and $\dfrac{[1-\theta(t)]\,x(t)}{S^0(t)}$ are respectively the quantities of portfolio invested in risky and riskless asset, whereas the non-negative progressively measurable processes $c(t)$ and $b(t)$ indicate respectively the contribution flow and the benefit flow at time $t \geq 0$.

The state equation (2) can be rewritten in the following way:

$$\begin{cases} dx(t) = [\theta(t)\sigma\lambda + r]\,x(t)dt + c(t) - b(t)dt + \sigma\theta(t)x(t)dB(t) + [c(t) - b(t)]dt, & t \geq 0, \\ x(0) = x_0. \end{cases} \tag{3}$$

2.2 Demography of the fund, contributions, benefits

We focus our analysis on the financial features of the problem, so that we do not deal with demographic risk, which would introduce an extra source of randomness, making the market incomplete. Thus we semplify the demographic model assuming demographic stationarity hypotheses: we assume that the flow of people who enters in the fund is constant on time and equal to $\bar{c} > 0$ starting from time $t = 0$ and that there is an exogenous constant $T > 0$ which represents the time during which the members adhere to the pension fund. Thus the exit flow of population from the fund begins at time $t = T$ and it is also constant and equal to $\bar{c}$ after that date, exactly balancing the entrance flow. Thus the population within the fund at time $t \in [0,T]$ is equal to $\bar{c}t$, whereas after time T the population within the fund is equal to $\bar{c}T$. Due to the demographic assumptions, we assume that the aggragate contribution flow increases linearly on time in the interval $[0,T]$ and is equal to a constant $c > 0$ after time T; for instance, as done in [13], we can imagine that each contributor who is adhering to the fund pays to the fund a contribution rate equal to αw, where $\alpha \in (0,1)$ and $w > 0$ is the (constant) wage rate (which has the dimension euros/time) earned by each contributor. In this case we can write the aggregate contributions flow as:

$$c(t) := \begin{cases} \alpha w \cdot \bar{c}t, & 0 \leq t < T, \\ \alpha w \cdot \bar{c}T, & t \geq T; \end{cases} \tag{4}$$

therefore, in this case, the aggregate contributions flow after time T is the constant $c = \alpha w \cdot \bar{c}T$.

About the benefit flow, we suppose that it starts at time T, when the first retirements occur, and that, after that date, it is composed by two terms, a minimum guarantee flow and a surplus flow. Again, by the demographic stationarity assumptions, we suppose that the minimum guarantee flow is a constant $g \geq c$, which is, in some way, the capitalization of the contributions paid to the fund by the members in retirement. For instance, we can imagine, as done in [13], that the fund pays to the generic member in retirement as (lumpsum) minimum guarantee the capitalization at a minimum guarantee rate $\delta \in [0,r]$ of the contributions paid by her/him in the time interval during which she/he was adhering to the fund. In this case, coherently with (4), we can write the aggregate minimum guarantee flow, for $t \geq T$, as:

$$g = \bar{c} \int_{t-T}^{t} (\alpha w)\, e^{\delta(t-u)} du, \tag{5}$$

i.e.

$$g = \begin{cases} c, & \text{if } \delta = 0, \\ \bar{c} \cdot (\alpha w)\, \frac{e^{\delta T}-1}{\delta} & \text{if } \delta > 0; \end{cases} \tag{6}$$

in particular we have $g \geq c$. The previous inequality means in particular that, unless $\delta = 0$, the current contributions do not permit to pay even only the current minimum guarantee. Nevertheless we will show that, in our setup for the benefits, under suitable assumptions on the surplus contract and on the solvency level, the fund manager can always pay the whole current benefits, mantaining the wealth level of the fund above the solvency level.

The inequality $\delta \leq r$ could be justified thinking to the fact that often the participants to the pension fund do not have same time to enter to the financial market as the fund manager. Moreover we recall that in the actual market, but it is not the case of our framework which has neither transactional or informational costs, the fund manager can usually get higher interest rate than the fund members.

About the surplus flow, we suppose that it depends on the past of the fund wealth in the last period $[t-T,t]$: the idea behind that is that the fund pays something more than the minimum guarantee to its member in retirement, if the fund growth was good in the period during which they were adhering to the fund. We describe our model for the surplus term in the next subsection.

2.3 Surplus

Many pension funds provide for their members a surplus premium over the minimum guarantee. It is natural supposing a surplus contract related to a performance index of the fund growth on the last period, i.e. in general it is natural think about a contract mathematically represented by a path dependent function $S\left(t, x(\cdot)|_{[t-T,T]}\right)$. We choose as expression for the surplus term the function

$$S\left(t, x(\cdot)|_{[t-T,t]}\right) = f_0\big(x(t) - \kappa x(t-T)\big), \tag{7}$$

where $f_0 : \mathbb{R} \to [0, +\infty)$ is increasing, convex and Lipschitz continuous with Lipschitz constant K_0 and where $\kappa > 0$. Referrinng to the minimum guarantee rate δ in (5), a typical choice for κ could be $\kappa = e^{\delta T}$, while a possible choice for $f_0(\cdot)$ could be $f_0(x) = \xi x^+$, where $\xi > 0$.

With the form (7) for the surplus, the equation (3) for the wealth process $x(\cdot)$ becomes a stochastic delay differential equation. Notice that the delay term is nonlinear and this represents a further complication. As said in the introduction, following the approach of e.g. [10], [22], [25], we treat the problem by passing to an infinite-dimensional equivalent problem.

Remark 2.1. *Of course other expressions for the surplus contract are possible; in particular it is meaningful to consider a surplus term which is a function of the ratio* $\frac{x(t)}{x(t-T)}$ *rather than of the difference* $x(t) - \kappa\, x(t-T)$*, i.e.*

$$S(t, x(\cdot)|_{[t-T,t]}) = f_0\left(\frac{x(t)}{x(t-T)}\right).$$

For example, referring to (4) for the contribution flow model, the cumulative benefit expression could be

$$b(t) = g + S\left(t, x(\cdot)|_{[t-T,t]}\right) = \bar{c} \cdot \alpha w \cdot \frac{e^{\left[\delta + \xi\left(\frac{1}{T}\log\frac{x(t)}{x(t-T)} - \delta\right)^+\right]T} - 1}{\delta + \xi\left(\frac{1}{T}\log\frac{x(t)}{x(t-T)} - \delta\right)^+},$$

where $\xi \in (0,1)$ *is the retrocession rate, i.e. the share of the fund return exceeding* δ *awarded to the fund members. In this case the fund corresponds as return rate to its members in retirement a minimum guarantee rate* δ *plus a fraction* $\xi \in (0,1)$ *of the fund return rate (in the period during which these members were adhering to the fund) exceeding* δ*.*

This kind of contract leads to technical complications, due to existence problems in the state equation and, mainly, to the unavoidable loss of concavity of the value function, but it seems more meaningful from a financial point of view; we hope to investigate this contract in future works. #

2.4 The solvency level

Usually a solvency level is imposed by law to guarantee that the company is able to pay at least part of the due benefits at each time $t \geq 0$. Without imposing this constraint the fund manager is allowed to use strategies that may bring him to be not able to pay the current benefits. We assume that the solvency level has the following structure:

- at the beginning the fund must hold a given minimum startup level $l_0 := l(0) \geq 0$;
- in the first period, i.e. for $t \in [0, T]$, the solvency level is a continuous and strictly increasing deterministic function;
- after time T, when the fund enters in the stationary phase, the solvency level is identically equal to $l := l(T)$.

Remark 2.2. *A possible form for the solvency level in the accumulation phase* $[0, T]$ *could be that proposed in [19], i.e.*

$$l(t) = l_0 \, e^{\beta t} + \int_0^t (\alpha w \cdot \bar{c}s) \, e^{\beta(t-s)} ds, \tag{8}$$

where the capitalization rate $\beta \leq r$ *can be chosen with regard to the other parameters (it could be, e.g.,* $\beta = r$*). In this case* l_0 *is a minimum startup level for the fund, which has to be capitalized at rate* $\beta \leq r$ *toghether with the contributions collected until to time* t*.* #

2.5 The utility function and the objective functional

Also to semplify, we will focus our attention on the control problem starting from the initial time T, when the delay term appears in the state equation making the mathematical problem more interesting (the study of the optimization problem in the period $[0, T]$ is the object of [19]). The objective functional (expressing the total expected discounted utility coming from the wealth) which we want to maximize is given by

$$\mathbb{E}\left[\int_T^{+\infty} e^{-\rho(t-T)} U\left(x\left(t\right)\right) dt\right], \tag{9}$$

where ρ is the discount individual rate and U is a utility function.

The main motivation behind the choice of the previous optimization criterion is quite intuitive: since the surplus term depends on the performance of the fund, a good management of the fund with regard to the functional above will have a good correspective in the surplus payments. Therefore here we adopt directly the point of view of the contributors: the fund's manager is delegated by the contributors to invest in the risky market in order to perform their benefits.

We set up the following assumptions about the utility function:

Hypothesis 2.3. *(i)* $U : [l, +\infty) \longrightarrow [-\infty, +\infty)$ *is the fund manager's utility function, which is assumed to be finite in* $(l, +\infty)$*, strictly increasing, strictly concave and belonging to the class* $C^2\big((l, +\infty); \mathbb{R}\big)$*.*

(ii) There exist constants $C > 0$*,* $\beta \in [0, 1)$ *for which we have*

$$\rho > \beta r + \frac{\lambda^2}{2} \cdot \frac{\beta}{1-\beta} \tag{10}$$

and

$$U\left(x\right) \leq C(1 + x^{\beta}). \tag{11}$$

Let us give some comment on the above Hypothesis 2.3. First of all recall that $l(t) \equiv l$ for $t \geq T$, so the utility function is defined where the wealth process $x(\cdot)$ must live. If we consider the problem for $t < T$ then we should also allow U to be defined for some $x < l$, but this is not the case we are studying. Concerning Hypothesis 2.3-(i) observe that all utility functions of the form $U\left(x\right) = \frac{(x-l)^{1-\gamma}-1}{1-\gamma}$ for $\gamma > 0$ (which for $\gamma = 1$ has to be intended $U\left(x\right) = \log\left(x - l\right)$) always satisfy it. About the condition (10), it guarantees that the functional in (9) is well-defined as proved in Proposition 3.10.

3 The stochastic control problem

Now we come to the precise formulation of the problem. First of all notice that the initial time $t = 0$ has been chosen as the first time of operations of the fund, but we have chosen T as initial time for the optimzation problem. However it also makes sense, in order to apply a dynamic programming approach, to look to a pension fund that is already running after a given amount of time $s \geq T$ so to establish an optimal decision policy from s on.

On the probability space of the previous section let $(\mathcal{F}_t^s)_{t\geq s}$ the completion of the filtration generated by the process $(B^s(t))_{t\geq s} := (B(t) - B(s))_{t\geq s}$; the control process $(\theta(t))_{t\geq s}$ is an $(\mathcal{F}_t^s)$-progressively measurable process with values in $[0, 1]$.

Consider the convex sets

$$\mathcal{C} := \left\{\eta = (\eta_0, \eta_1(\cdot)) \in [l, +\infty) \times C([-T, 0); \mathbb{R}) \;\Big|\; \lim_{\zeta\to 0} \eta_1(\zeta) = \eta_0\right\} \tag{12}$$

and

$$\mathcal{D} := \left\{\eta \in \mathcal{C} \mid \eta_1(\cdot) \geq l(T + \cdot)\right\}, \tag{13}$$

where $l(\cdot)$ (the solvency level) is given according with the assumptions of section 2.4. We have $\mathcal{D} \subset \mathcal{C} \subset E$, where

$$E := \left\{(x_0, x_1(\cdot)) \in \mathbb{R} \times C([-T, 0); \mathbb{R}) \;\Big|\; \lim_{\zeta\to 0} x_1(\zeta) = x_0\right\} \tag{14}$$

the space E is a Banach space when endowed with the norm

$$\|(x_0, x_1(\cdot))\|_E = |x_0| + \sup_{\zeta\in[-T,0)} |x_1(\zeta)|.$$

We allow the past-present η belonging to the space $\mathcal{C}$ (see Remark 3.1 below). Now set $\eta \in \mathcal{C}$ and consider the following stochastic delay differential equation for the dynamics of the wealth

$$\begin{cases} dx(t) = & [(\sigma\lambda\theta(t) + r)\, x(t) + c - g]\, dt \\ & -f_0\, (x(t) - \kappa x(t - T))\, dt + \sigma\theta(t)x(t)dB(t), \\ x(s) = \eta_0,\;\; x(s + \zeta) = \eta_1(\zeta), & \zeta \in [-T, 0). \end{cases} \tag{15}$$

Referring to the equation (3) for the general model, the term $c(\cdot)$ is a constant c, as discussed in section 2.2, wheras the term $b(\cdot)$ is given by the minimum guarantee constant flow g, again as discussed in section 2.2, and by the surplus term given according with (7). We set from now on $q := g - c \geq 0$.

Remark 3.1. *Of course, from a financial point of view, the natural space of initial data for initial time $s \geq T$ would be*

$$\mathcal{D}_s := \{(\eta_0, \eta_1(\cdot)) \in [l, +\infty) \times C([-T, 0); \mathbb{R}) \mid \lim_{\zeta\to 0} \eta_1(\zeta) = \eta_0,\;\; \eta_1(\cdot) \geq l(s + \cdot)\} \subset \mathcal{C},$$

which is, in particular, time-dependent. Nevertheless it makes sense (and it is convenient from a mathematical point of view) to enlarge the set of initial data for initial time $s \geq T$ to the wider class $\mathcal{C}$ in order to consider a set of initial data not time-dependent. #

For $s \geq T$, let $C_{\mathcal{P}}([s, +\infty); L^2(\Omega))$ be the space of the mean-square continuous $(\mathcal{F}_t^s)_{t\geq s}$-progressively measurable process.

Theorem 3.2. *For any given $(\mathcal{F}_t^s)_{t\geq s}$-progressively measurable and $[0, 1]$-valued control process $(\theta(t))_{t\geq s}$, the state equation (15) admits a unique strong solution in the class $C_{\mathcal{P}}([s, +\infty); L^2(\Omega))$.*

Proof. This follows by Theorem 2.1, Chapter II, in [39], or by Theorem 6.16, Chapter 1, in [44]. #

We denote the unique strong solution of equation (15) by $x\,(t; s, \eta, \theta(\cdot))$.

3.1 The set of the admissible strategies

In the framework above we define the set of the admissible control strategies for initial time $s \geq T$ and initial past-present $\eta(\cdot) \in \mathcal{C}$ by

$$\Theta_{ad}(s,\eta) := \{\theta(\cdot) \text{ prog. meas. w.r.t. } (\mathcal{F}_t^s)_{t\geq s} \mid x\,(t;s,\eta,\theta(\cdot)) \geq l \text{ for } t \geq s\}\,. \tag{16}$$

Now let us recall the Girsanov's Theorem in the version that we need for the following work:

Theorem 3.3 (Girsanov). *Let $(\Omega,\mathcal{F},P)$ be a probability space on which there is defined a Brownian motion $(B(t))_{t\geq 0}$; let $\mathcal{F}^B = (\mathcal{F}_t^B)_{t\geq 0}$ be the filtration generated by the Brownian motion. For an $\mathcal{F}^B$-progressively measurable process γ consider the processes*

$$\tilde{B}(t) = B(t) + \int_0^t \gamma(s)ds,$$

$$L(t) = \exp\left(-\int_0^t \gamma(s)dB(s) - \frac{1}{2}\int_0^t \gamma^2(s)ds\right);$$

assume that γ satisfies the Novikov's condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \gamma^2(s)ds\right)\right] < \infty, \qquad\qquad t \in [0,+\infty).$$

Then $(\tilde{B}(s))_{0\leq s\leq t}$ is a Brownian motion under the probability $\tilde{P}_t << P|_{\mathcal{F}_t^B}$ on $\mathcal{F}_t^B$ defined by the Radon-Nikodym derivative $L(t)$. #

Lemma 3.4. *Let $\eta \in \mathcal{C}$; then $\Theta_{ad}(T,\eta) \neq \emptyset$ if and only if the null strategy $\theta(\cdot) \equiv 0$ (corresponding to the riskless investment of the whole wealth at every time) is admissible.*

Proof. Of course if $\theta(\cdot) \equiv 0$ belongs to $\Theta_{ad}(s,x)$, then $\Theta_{ad}(s,x) \neq \emptyset$. The proof of the converse implication is an application of the previous Girsanov's Theorem. Indeed let $\theta(\cdot)$ be an admissible control for initial time T and initial past-present $\eta \in \mathcal{C}$; by Girsanov's Theorem, for $u > T$ we can write the dynamics of $x(t;T,\eta,\theta(\cdot))$ in the interval $[T,u]$, under the probability $\tilde{P}_u$ given by the Girsanov's transformation, as

$$\begin{cases} dx(t) = [rx(t) - q]\,dt - f_0(x(t) - \kappa x(t-T))dt + \sigma\theta(t)x(t)d\tilde{B}(t), \\ x(T) = \eta_0,\; x(T+\zeta) = \eta_1(\zeta), \qquad \zeta \in [-T,0), \end{cases}$$

where $\tilde{B}(t) := B(t) + \lambda t$ is a Brownian motion under $\tilde{P}_u$ in the interval $[0,u]$. Passing to the expectations and taking into account the convexity of the map f_0 and the Jensen's inequality, we get

$$\begin{cases} d\tilde{\mathbb{E}}_u[x(t)] \;=\; \left[r\tilde{\mathbb{E}}_u[x(t)] - q\right]dt - \tilde{\mathbb{E}}_u\left[f_0(x(t) - \kappa x(t-T))\right]dt, \\ \qquad\qquad\quad \leq\; \left[r\tilde{\mathbb{E}}_u[x(t)] - q\right]dt - f_0\left(\tilde{\mathbb{E}}_u[x(t)] - \kappa\tilde{\mathbb{E}}_u[x(t-T)]\right)dt, \\ \tilde{\mathbb{E}}_u[x(T)] = \eta_0,\; \tilde{\mathbb{E}}_u[x(T+\zeta)] = \eta_1(\zeta), \qquad \zeta \in [-T,0). \end{cases} \tag{17}$$

By assumption $x(t;T,\eta,\theta(\cdot)) \geq l$ for $t \geq T$, so that also $\tilde{\mathbb{E}}_u[x(t;T,\eta,\theta(\cdot))] \geq l$ for $t \in [T,u]$. But the dynamics of $x(t;T,\eta,0)$ is given by

$$\begin{cases} dx(t) = [rx(t) - q]\,dt - f_0\,(x(t) - \kappa x(t-T))\,dt, \\ x(T) = \eta_0,\; x(T+\zeta) = \eta_1(\zeta), \qquad \zeta \in [-T,0). \end{cases} \tag{18}$$

Therefore, by the classic comparison criterion for ordinary differential equations applied to (17) and (18), we get $x(t;T,\eta,0) \geq \tilde{\mathbb{E}}_u[x(t;T,\eta,\theta(\cdot))] \geq l$ for $t \in [T,u]$. By the arbitrariness of u the conclusion follows. #

Set $k := l - \kappa\, l_0$; because of Lemma below, we are led to introduce the following:

Hypothesis 3.5.

$$f_0(k) \leq rl - q.$$

Lemma 3.6. *If Hypothesis 3.5 holds true, then the null strategy $\theta(\cdot) \equiv 0$ belongs to $\Theta_{ad}(T,\eta)$ for all $\eta \in \mathcal{D}$. In particular $\Theta_{ad}(T,\eta)$ is not empty for each $\eta \in \mathcal{D}$.*

Proof. Suppose that hypotesis 3.5 holds true. Let us consider the state trajectory $x(\cdot)$ corresponding to the null strategy; at time t, supposing the constraint satified in the past and so in particular $x(t-T) \geq l_0$, we have, taking into account that f_0 is increasing,

$$\begin{aligned} dx(t) &= (rx(t)-q)dt - f_0(x(t)-\kappa x(t-T))dt \\ &\geq (rx(t)-q)dt - f_0(x(t)-\kappa\, l_0)dt. \end{aligned}$$

Thus, whenever $x(t) = l$ (if there is the case), we have

$$dx(t) \geq (rl-q)dt - f_0(l-\kappa\, l_0)dt = (rl-q)dt - f_0(k)dt \geq 0,$$

i.e. $x(t)$ is increasing at t and so the claim is proved. #

By Lemma 3.6 we will assume from now on that Hypothesis (3.5) holds true. Let us see the behaviour of the solution in some special cases: we give a lemma which will be useful afterwards.

Lemma 3.7. *Let $\eta \in \mathcal{D}$. Then:*

1. *If $rl = q$ and $\eta_0 = l$, then $\Theta_{ad}(T,\eta) = \{0\}$ and $x(\cdot;T,\eta,0) \equiv l$.*
2. *If $\eta_0 > l$, then $x(\cdot;T,\eta,0) \geq l + \beta$ for some $\beta > 0$.*

Proof. 1. Let $\eta_0 = l = q/r$. By Lemma 3.6 we know that $0 \in \Theta_{ad}(T,\eta)$. With the same argument of Lemma 3.4 (taking into account that in this case $rl - q = f_0(k) = 0$), one can show that, if $\theta(\cdot)$ is an admissible strategy, then it has to be $x(t;T,\eta,\theta(\cdot)) \geq l$ and $\tilde{\mathbb{E}}_u[x(t;T,\eta,\theta(\cdot))] \equiv l$ for $t \in [T,u]$ (for arbitrary $u \geq T$), so $x(\cdot;T,\eta,\theta(\cdot)) \equiv l$ and we can say that $\theta(\cdot) \equiv 0$ is the unique admissible strategy.

2. Let us consider the state trajectory $x(\cdot)$ corresponding to the null strategy $\theta(\cdot) \equiv 0$; let us define $t_0 := \inf\left\{t \geq T \mid x(t) = l + \frac{\eta_0 - l}{2}\right\} > T$. If $t_0 = +\infty$, we have concluded. Otherwise, set $\varepsilon := l(t_0 - T) - l_0$; by the assumptions done in section 2.4 on the solvency level we have $\varepsilon > 0$. Let us suppose that $\frac{\eta_0 - l}{2} \leq \kappa\varepsilon$; for $t \geq t_0$, $x(t-T) \geq l(t_0 - T)$; therefore

$$\begin{aligned} f_0(x(t)-\kappa x(t-T)) &\leq f_0(x(t)-\kappa\, l(t_0-T)) \\ &= f_0(x(t)-\kappa\,(l_0+\varepsilon)). \end{aligned}$$

Thus, whenever $x(t) = l + \frac{\eta_0 - l}{2}$ (e.g. in t_0),

$$\begin{aligned} dx(t) &= [rx(t)-q]dt - f_0(x(t)-\kappa\, x(t-T))dt \\ &\geq [rl-q]dt - f_0\left(l + \frac{\eta_0 - l}{2} - \kappa\,(l_0+\varepsilon)\right) dt \\ &\geq [rl-q]dt - f_0(k)\, dt \geq 0. \end{aligned}$$

Then $x(t) \geq l + \frac{\eta_0 - l}{2}$, for $t \geq t_0$, and the claim follows.
In the case that $\frac{\eta_0 - l}{2} > \kappa\varepsilon$, let $t_1 := \inf\{t \geq t_0 \mid x(t) = l + \kappa\varepsilon\}$; arguing as above we can show that $x(t) \geq l + \kappa\varepsilon$, for $t \geq t_1$, and the claim follows also in this case. #

Remark 3.8. *The results of Lemma 3.6 and Remark 3.4 hold also for initial time $s > T$; this follows by Theorem 2.10, Chapter 1, of [44] and by the fact that our system is time-independent. Indeed, by the cited result, there is a one-to-one correspondence between the strategies starting from T and the strategies starting from $s > T$; to make more clear this point, consider the measurable space*

$(C([T,+\infty);\mathbb{R}),\mathcal{B}(C([T,+\infty);\mathbb{R})))$, *with the filtration* $(\mathcal{B}_t(C([T,+\infty);\mathbb{R})))_{t\geq T}$ *defined in the following way:* $\mathcal{B}_t(C([T,+\infty);\mathbb{R}))$ *is the* σ*-algebra on* $C([T,+\infty);\mathbb{R})$ *induced by the projection*

$$\begin{array}{rccc}\pi: & C([T,+\infty);\mathbb{R}) & \longrightarrow & (C([T,t];\mathbb{R}),\mathcal{B}(C([T,t];\mathbb{R}))) \\ & \zeta(\cdot) & \longmapsto & \zeta(\cdot)|_{[T,t]},\end{array}$$

i.e. the smallest σ*-algebra which makes* π *measurable; intuitively a measurable application with respect to* $\mathcal{B}_t(C([T,+\infty);\mathbb{R}))$ *is an application which does not distinguish between two functions of* $C[T,+\infty)$ *which coincide on* $[T,t]$*. If* $(\theta_T(t))_{t\geq T}$ *is a strategy starting from* T*, there exists an adapted process* ψ *on* $[T,+\infty)\times C([T,+\infty);\mathbb{R})$ *such that*

$$\theta_T(t)(\cdot)=\psi(t,B^T(\cdot)),\quad t\geq T;$$

then the shifted strategy

$$\theta_s(t)(\cdot)=\psi(t-s+T,B^s(\cdot)),\quad t\geq s,$$

starts from s *and, by the recalled time-homogeneity of our equation,* $\theta_T\in\Theta_{ad}(T,\eta)$ *if and only if* $\theta_s\in\Theta_{ad}(s,\eta)$*.* #

We can summarize Lemma 3.4, Lemma 3.6 and Remark 3.8 in the following

Proposition 3.9. *Let* $s\geq T$ *and* $\eta\in\mathcal{C}$*; then* $\Theta_{ad}(s,\eta)\neq\emptyset$ *if and only if* $\theta(\cdot)\equiv 0$ *is admissible. Moreover, under Hypothesis 3.5, the null strategy is admissible for each initial time* $s\geq T$ *and for each initial past-present* $\eta\in\mathcal{D}$*; in particular* $\Theta_{ad}(s,\eta)\neq\emptyset$ *for each initial time* $s\geq T$ *and for each initial past-present* $\eta\in\mathcal{D}$*.* #

3.2 The value function and its properties

As we said in Section 2, the objective functional which we want to maximize over the set of admissible strategies $\theta(\cdot)\in\Theta_{ad}(s,\eta)$ is

$$J\left(s,\eta;\theta\left(\cdot\right)\right):=\mathbb{E}\left[\int_s^{+\infty}e^{-\rho t}U\left(x\left(t;s,\eta,\theta(\cdot)\right)\right)dt\right],\quad s\geq T,$$

where U satisfies Hypothesis 2.3.

Proposition 3.10. *Let us suppose that (10) and (11) hold true and let* $s\geq T$*,* $\eta\in\mathcal{C}$*,* $\theta(\cdot)\in\Theta_{ad}(s,\eta)$*; then*

$$\mathbb{E}\left[\int_s^{+\infty}e^{-\rho(t-T)}[U^+(x(t))]dt\right]<+\infty.$$

Proof. Let C be the constant appearing in (11). We have, taking into account that $x(t)\geq l>0$,

$$\begin{aligned}\mathbb{E}\left[\int_s^{+\infty}e^{-\rho(t-T)}[U^+(x(t))]dt\right] &\leq \mathbb{E}\left[\int_s^{+\infty}e^{-\rho(t-T)}C(1+x(t)^\beta)dt\right]\\ &\leq C\int_s^{+\infty}e^{-\rho(t-T)}\left(1+\mathbb{E}\left[x(t)^\beta\right]\right)dt. \qquad (19)\end{aligned}$$

For fixed $u\geq s$, let us consider, on the interval $[s,u]$, under the probability $\tilde{P}_u$ and the $\tilde{P}_u$-Brownian motion $\tilde{B}(t)=B(t)+\lambda t$, $s\leq t\leq u$, given by Girsanov's Theorem, the problem

$$\begin{cases}dx(t)=\left[rx(t)-q\right]dt-f_0(x(t)-\kappa x(t-T))dt+\sigma\theta(t)x(t)d\tilde{B}(t),\\ x(s)=\eta_0,\ x(s+\zeta)=\eta_1(\zeta),\qquad \zeta\in[-T,0);\end{cases}$$

and compare that with the following one:

$$\begin{cases}dy(t)=ry(t)dt+\sigma\theta(t)y(t)d\tilde{B}(t),\\ y(s)=\eta_0;\end{cases}$$

of course, by comparison criterion, if $\theta(\cdot)$ is an admissible control for the problem, then $l \leq x(t;s,\eta,\theta(\cdot)) \leq y(t;s,\eta,\theta(\cdot))$ for $s \leq t \leq u$. Now

$$\mathbb{E}\left[x(t)^{\beta}\right] \leq \mathbb{E}\left[y(t)^{\beta}\right] = \tilde{\mathbb{E}}\left[y(t)^{\beta}\exp\left(\lambda\tilde{B}(t) - \frac{1}{2}\lambda^2 t\right)\right];$$

by Holder inequality

$$\begin{aligned}\tilde{\mathbb{E}}\left[y(t)^{\beta}\exp\left(\lambda\tilde{B}(t) - \frac{1}{2}\lambda^2 t\right)\right] &\leq \tilde{\mathbb{E}}\left[y(t)\right]^{\beta}\tilde{\mathbb{E}}\left[\exp\left[\left(\lambda\tilde{B}(t) - \frac{1}{2}\lambda^2 t\right)\frac{1}{1-\beta}\right]\right]^{1-\beta} \\ &\leq \eta_0^{\beta}e^{\beta r(t-s)}e^{-\frac{1}{2}\lambda^2 t}e^{\frac{\lambda^2 t}{2(1-\beta)}} = \eta_0^{\beta}e^{-\beta rs}e^{\left(\beta r + \frac{\beta}{2(1-\beta)}\lambda^2\right)t}.\end{aligned}$$

Thus, by (10), (19) and by the arbitrariness of u, the claim follows. #

We define the *value function*

$$V(s,\eta) := \sup_{\theta(\cdot)\in\Theta_{ad}(s,\eta)} \mathbb{E}\left[\int_s^{+\infty} e^{-\rho(t-T)}U\left(x\left(t;s,\eta,\theta\right)\right)dt\right], \quad s \geq T,\ \eta \in \mathcal{C}, \tag{20}$$

with the convention $\sup\emptyset = -\infty$.

Definition 3.11. *(i) Let $s \geq T$, $\eta \in \mathcal{C}$. An optimal strategy for initial data (s,η) is a strategy $\theta^*(\cdot) \in \Theta_{ad}(s,\eta)$ such that for the corresponding trajectory $x^*(t) := x(t;s,\eta,\theta^*(\cdot))$ we have*

$$V(s,\eta) = J(s,\eta;\theta^*(\cdot)) = \mathbb{E}\left[\int_s^{+\infty} e^{-\rho(t-T)}U(x^*(t))dt\right].$$

The couple $(\theta^*(\cdot), x^*(\cdot))$ is called an optimal pair.

(ii) Let $s \geq T$, $\eta \in \mathcal{C}$, $\varepsilon > 0$. An ε-optimal strategy for initial data (s,η) is a strategy $\theta^{\varepsilon}(\cdot) \in \Theta_{ad}(s,\eta)$ such that for the corresponding trajectory $x^{\varepsilon}(t) := x(t;s,\eta,\theta^{\varepsilon}(\cdot))$ we have

$$V(s,\eta) - \varepsilon \leq J(s,\eta;\theta^{\varepsilon}(\cdot)) = \mathbb{E}\left[\int_s^{+\infty} e^{-\rho(t-T)}U(x^{\varepsilon}(t))dt\right].$$

The couple $(\theta^{\varepsilon}(\cdot), x^{\varepsilon}(\cdot))$ is called an ε-optimal pair. #

We have the following Proposition, giving the dependence of the value function on the time variable:

Proposition 3.12. *Let $s \geq T$ and $\eta \in \mathcal{C}$; then*

$$V(s,\eta) = e^{-\rho(s-T)}V(T,\eta).$$

Proof. The proof is standard and follows by Remark 3.8, by the time-independence of the state equation, by the fact that the utility function is not time-dependent and by the fact that the optimization problem is over an infinite horizon. #

By the previous Proposition we are reduced to study the function $V(T,\eta)$; with a slight abuse of notation we set $J(\eta;\theta(\cdot)) := J(T,\eta;\theta(\cdot))$, $V(\eta) := V(T,\eta)$ and $\Theta_{ad}(\eta) := \Theta_{ad}(T,\eta)$. Let us see some other propreties of this function; firstly we have, by the proof of Proposition 3.10, the following result about the upper finiteness of the value function:

Corollary 3.13. *Let $\eta \in \mathcal{C}$; there exists $K > 0$ such that $V(\eta) \leq K(1+\eta_0^{\beta})$, where β is given by the Hypothesis 2.3-(ii).*

Proof. Let C be the constant appearing in (11) and let $\theta(\cdot) \in \Theta_{ad}(\eta)$. We have, setting $x(t) := x(t; T, \eta, \theta(\cdot))$ and repeating the estimate in the proof of Proposition 3.10,

$$J(\eta; \theta(\cdot)) \le C \int_T^{+\infty} e^{-\rho(t-T)} \left(1 + \eta_0^\beta e^{-\beta r T} e^{\left(\beta r + \frac{\beta}{2(1-\beta)}\lambda^2\right)t}\right) dt \le K(1 + \eta_0^\beta),$$

for some $K > 0$ not depending on $\theta(\cdot)$, so that the claim follows. #

Proposition 3.14. *Let $\eta, \eta' \in \mathcal{C}$ such that $\eta_0 \le \eta_0'$, $\eta_1(\cdot) \le \eta_1'(\cdot)$; then $V(\eta) \le V(\eta')$.*

Proof. Let us consider a control $\theta(\cdot) \in \Theta_{ad}(\eta)$ and call $x(\cdot), x'(\cdot)$ the state trajectories corresponding respectively to η, η' with control $\theta(\cdot)$, i.e. $x(t) := x(t; T, \eta, \theta(\cdot))$, $x'(t) := x(t; T, \eta', \theta(\cdot))$. Ir results for the dynamics of $x'(\cdot)$ in $[T, 2T]$, by monotonicity of f_0,

$$dx'(t) \ge [r + \sigma\lambda\theta(t)]x'(t)dt - q\,dt - f_0(x'(t) - \kappa x(t-T))dt + \sigma\theta(t)x'(t)dB(t),$$

whereas we have for the dynamics of $x(\cdot)$ in $[T, 2T]$,

$$dx(t) = [r + \sigma\lambda\theta(t)]x(t)dt - q\,dt - f_0(x(t) - \kappa x(t-T))dt + \sigma\theta(t)x(t)dB(t);$$

by comparison criterion (see e.g. [30], Proposition 2.18) it results $x'(t) \ge x(t)$ on $[T, 2T]$. Thus we can iterate the argument and conclude that $\theta(\cdot) \in \Theta_{ad}(\eta')$, i.e $\Theta_{ad}(\eta) \subset \Theta_{ad}(\eta')$, and $x'(t) \ge x(t)$, so that we can conclude by monotonicity of U. #

Proposition 3.15. *Let $\eta \in \mathcal{D}$; we have the following statements about the lower finiteness of the value function:*

1. *If $U(l) > -\infty$, then $V(\eta) > -\infty$.*
2. *If $U(l) = -\infty$, $rl - q = 0$ and $\eta_0 = l$, then $V(\eta) = -\infty$.*
3. *If $U(l) = -\infty$ and $\eta_0 > l$, then $V(\eta) > -\infty$.*
4. *If $U(l) = -\infty$, $rl - q > f_0(k)$ and $\eta_0 = l$, we have to distinguish two cases:*
   - *If U is integrable in l^+, then $V(\eta) > -\infty$.*
   - *If U is not integrable in l^+, then $V(\eta) = -\infty$.*

Proof.
1. Of course $V(\eta) \ge J(\eta; 0) \ge \int_T^{+\infty} e^{-\rho(t-T)} U(l)dt \ge U(l)/\rho$ and this statement is proved.
2. By Lemma 3.7-(1) $\Theta_{ad}(\eta) = \{0\}$ and $x(\cdot; T, \eta, 0) \equiv l$, so that $J(\eta; 0) = -\infty$ and $V(\eta) = -\infty$.
3. By Lemma 3.7-(2) we know that $x(\cdot; T, \eta, 0) \ge l + \beta$, for some $\beta > 0$. Therefore $V(\eta) \ge J(\eta, 0) \ge \int_T^{+\infty} e^{-\rho(t-T)} U(l + \beta)dt > -\infty$.
4. See the Appendix. #

Remark 3.16. *Let us define*

$$D(V) = \{\eta \in \mathcal{C} \mid V(\eta) > -\infty\}$$

and

$$\mathcal{D}_0 := \{\eta \in \mathcal{D} \mid \eta_0 > l\};$$

by the previous Proposition we get $\mathcal{D}_0 \subset D(V)$; moreover, again by the previous Proposition, if $U(l) \in \mathbb{R}$, then $\mathcal{D} \subset D(V)$. #

Proposition 3.17. *The function $\eta \mapsto V(\eta)$ is concave on $\mathcal{C}$.*

Proof. See the Appendix. #

4 The equivalent infinite dimensional formulation

In this section we will formulate an infinite-dimensional stochastic control problem equivalent to the one of the previous section. We refer to [43] for this kind of approach in the deterministic case and to [6], [10], [25] in the stochastic case.

Let us set $L^2[-T,0] := L^2([-T,0];\mathbb{R})$, $W^{1,2}[-T,0] := W^{1,2}([-T,0];\mathbb{R})$ and consider the Hilbert space

$$H = \mathbb{R} \times L^2[-T,0] \tag{21}$$

with inner product

$$\langle x, y\rangle = x_0 y_0 + \int_{-T}^{0} x_1(\xi) y_1(\xi) d\xi$$

and norm

$$\|x\| = \left(|x_0|^2 + \int_{-T}^{0} |x_1(\xi)|^2 d\xi \right)^{1/2},$$

where $x_0, x_1(\cdot)$ denote respectively the $\mathbb{R}$-valued and the $L^2[-T,0]$-valued components of the generic element $x \in H$; let also $(B(t))_{t\geq 0}$ be the same Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$ of the previous sections. Let us consider the space E defined in (14); for $x \in E$ we have the estimate $\|x\|_H \leq (1+T)^{1/2}\|x\|_E$, so that we have the continuous and dense embedding

$$\begin{array}{rccc} \iota : & (E, \|\cdot\|_E) & \longrightarrow & (H, \|\cdot\|_H), \\ & x & \longmapsto & x. \end{array}$$

Given an $\mathcal{F}^T$-progressively measurable and $[0,1]$-valued process $\theta(\cdot)$ and $x \in E$, we can consider the infinite dimensional H-valued stochastic evolution equation starting at time T

$$\begin{cases} dX(t) = AX(t)dt + \sigma\lambda\theta(t)\Phi X(t)dt - F(X(t))dt + \sigma\theta(t)\Phi X(t)dB(t), \\ X(T) = x \in E; \end{cases} \tag{22}$$

where

- $A : D(A) \subset H \to H$ is the unbounded linear operator defined by $(x_0, x_1(\cdot)) \mapsto (rx_0, x_1'(\cdot))$, with

  $$D(A) = \{(x_0, x_1(\cdot)) \in H \mid x_1(\cdot) \in W^{1,2}[-T,0],\ x_0 = x_1(0)\};$$

  above by $x_1(0)$ we mean the evaluation at $\zeta = 0$ of the unique (absolutely) continuous representative of $x_1 \in W^{1,2}[-T,0]$.

- $F : E \to H$ is the nonlinear map

  $$\begin{pmatrix} x_0 \\ x_1(\cdot) \end{pmatrix} \mapsto \begin{pmatrix} f(x_0, x_1(\cdot)) \\ 0 \end{pmatrix},$$

  where $f : E \to \mathbb{R}$, $(x_0, x_1(\cdot)) \mapsto f_0(x_0 - \kappa\, x_1(-T)) + q$.

- $\Phi : H \to H$ is the linear operator defined by $\Phi x := (x_0, 0)$.

It is well known that A is a closed densely defined operator and that it is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on H; more precisely it is defined by

$$S(t)(x_0, x_1(\cdot)) = \left(x_0 e^{rt}, I_{[-T,0]}(t+\cdot)\ x_1(t+\cdot) + I_{[0,+\infty)}(t+\cdot)\ x_0 e^{r(t+\cdot)} \right);$$

in particular $S(t)$ maps E in itself and

$$S(t)(x_0, 0) = \left(x_0 e^{rt}, I_{[0,+\infty)}(t+\cdot)\ x_0 e^{r(t+\cdot)} \right).$$

Concerning the norm of the semigroup, we have the estimate

$$
\begin{aligned}
\|S(t)x\|_H^2 &\leq \left|x_0 e^{rt}\right|^2 + 2\int_{-T}^{0} \left|I_{[-T,0]}(t+\zeta)\ x_1(t+\zeta)\right|^2 d\zeta + 2\int_{-T}^{0} \left|I_{[0,+\infty)}(t+\zeta)\ x_0 e^{r(t+\zeta)}\right|^2 d\zeta \\
&\leq (3+2T)e^{2rt}\|x\|_H^2; \qquad (23)
\end{aligned}
$$

so $\|S(t)\|_{\mathcal{L}(H)} \leq Me^{\omega t}$, with $M = (3+2T)^{1/2}$ and $\omega = r$.

4.1 Mild solutions: existence and uniqueness

In this section we give the definition of *mild solution* of (22) and investigate the existence and uniqueness of such a solution.

Definition 4.1. *We define mild solution of (22) an E-valued process $(X(t))_{t\geq T}$ which satisfies the integral equation*

$$
\begin{aligned}
X(t) &= S(t-T)x + \sigma\lambda \int_T^t \theta(\tau) S(t-T)\left[\Phi X(\tau)\right] d\tau - \int_T^t S(t-\tau) F(X(\tau)) d\tau \\
&\quad + \sigma \int_T^t \theta(\tau) S(t-\tau)\left[\Phi X(\tau)\right] dB(\tau). \qquad (24)
\end{aligned}
$$

Remark 4.2. *In this remark we want to explain the reasons for the choice of the space E as subspace in which looking for a solution of (22). This space has three important properties:*

- *it gives sense to the term $F(x)$, $x \in E$;*
- *it is invariant for the semigroup $S(\cdot)$ and moreover $S(\cdot)$ is a C_0-semigroup also on $(E, \|\cdot\|_E)$;*
- *the value function (20) is defined on the points of this space.*

The first two properties are essential for working with the state equation (22). The third one allows to link the one-dimensional optimal control problem with delay defined in the previous section with the infinite-dimensional one which we will define in this section. #

Notice that in the equation (24) the noise is one dimensional. In order to be able to manipulate this equation and link the $L^2[-T,0]$-valued integration (stochastic or deterministic) with the canonical $\mathbb{R}$-valued integration, we need some technical considerations, which we recall for the reader's convenience. Let $a, b \in \mathbb{R}$, $a < b$ and let G be a Banach space; we can define the space $L^2_{\mathcal{P}}([a,b]; L^2(\Omega; G))$ of the square-integrable G-valued progressively measurable process and the space $C_{\mathcal{P}}([a,b]; L^2(\Omega; G))$ of the progressively measurable G-valued processes and mean-square continuous. Of course the stochastic or deterministic integral of a generic $X \in L^2_{\mathcal{P}}([a,b]; L^2(\Omega; G))$ belongs to $C_{\mathcal{P}}([a,b]; L^2(\Omega; G))$.

Now let us consider the case when $G = L^2[-T,0]$; for given $U \in L^2(\Omega; L^2[-T,0])$ we can define for $\zeta \in [-T,0]$, the $\mathbb{R}$-valued random variable

$$
Z(\zeta)(\omega) := \overline{\tilde{U}(\omega)}(\zeta),
$$

where we denoted with the symbols $\bar{f}$, $\tilde{U}$ the pointwise well-defined representatives of generic $f \in L^2[-T,0]$ and $U \in L^2(\Omega; L^2[-T,0])$. With a slight abuse of notation, we will write $U(\zeta)$ for denoting the equivalence class in $L^2(\Omega; \mathbb{R})$ of the $\mathbb{R}$-valued random variable $Z(\zeta)$ defined above. Note that, choosing other representatives, i.e. defining

$$
Z'(\zeta)(\omega) := \overline{\overline{\tilde{\tilde{U}}(\omega)}}(\zeta)
$$

and denoting by $U'(\zeta)$ the equivalence class in $L^2(\Omega; \mathbb{R})$ of $Z'(\zeta)$, we would have $U(\zeta) = U'(\zeta)$ (in $L^2(\Omega; \mathbb{R})$), for a.e. $\zeta \in [-T,0]$. The following Lemmata are easy to prove

Lemma 4.3. *Let $U \in L^2(\Omega; L^2[-T,0])$; then we have $U(\zeta) \in L^2(\Omega;\mathbb{R})$ for a.e. $\zeta \in [-T,0]$.* #

Lemma 4.4. *Let $X, Y \in L^2(\Omega; L^2[-T,0]))$; suppose that, for a.e. $\zeta \in [-T,0]$, $X(\zeta) = Y(\zeta)$ in $L^2(\Omega;\mathbb{R})$. Then $X = Y$ in $L^2(\Omega; L^2[-T,0])$.* #

The desired link between the $L^2[-T,0]$-valued integration and the $\mathbb{R}$-valued integration is given by the following:

Lemma 4.5. *Let $X \in L^2_{\mathcal{P}}([a,b]; L^2(\Omega; L^2[-T,0]))$. Then we have, for a.e. $\zeta \in [-T,0]$, the following equalities in $L^2(\Omega;\mathbb{R})$:*

- $\left(\int_a^b X(t)dt\right)(\zeta) = \int_a^b X(t)(\zeta)dt;$
- $\left(\int_a^b X(t)dB(t)\right)(\zeta) = \int_a^b X(t)(\zeta)dB(t).$ #

Now let E be defined as in (14) and H as in (21). As we said, if $x \in E$, we have $\|x\|_H \leq (1+T)^{1/2}\|x\|_E$; therefore we have the continuous and dense embedding

$$\begin{array}{rccc} \iota: & L^2_{\mathcal{P}}\big([a,b]; L^2(\Omega; (E, \|\cdot\|_E))\big) & \longrightarrow & L^2_{\mathcal{P}}\big([a,b]; L^2(\Omega; (H, \|\cdot\|_H))\big), \\ & X & \longmapsto & X, \end{array}$$

and the continuous and dense embedding

$$\begin{array}{rccc} \tilde{\iota}: & C_{\mathcal{P}}\big([a,b]; L^2(\Omega; (E, \|\cdot\|_E))\big) & \longrightarrow & C_{\mathcal{P}}\big([a,b]; L^2(\Omega; (H, \|\cdot\|_H))\big), \\ & X & \longmapsto & X. \end{array}$$

In the rest of this section, unless differently specified, we will mean the spaces E and H respectively endowed with the norm $\|\cdot\|_E$ and $\|\cdot\|_H$, which make them Banach spaces.

Lemma 4.6. *Consider, for $a, b \in \mathbb{R}$, $T \leq a < b$, the linear map*

$$\begin{array}{rccc} \chi: & L^2\big((\Omega, \mathcal{F}^T_a); E\big) & \longrightarrow & L^2_{\mathcal{P}}\big([a,b]; L^2(\Omega; H)\big), \\ & \psi & \longmapsto & S(\cdot - a)\psi. \end{array}$$

Then, for any $\psi \in L^2((\Omega, \mathcal{F}^T_a); E)$, we have $\chi(\psi) \in C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big)$. Moreover there exists a constant C_{b-a} such that

$$\|\chi(\psi)\|_{C_{\mathcal{P}}([a,b];L^2(\Omega;E))} \leq C_{b-a}\|\psi\|_{L^2(\Omega;E)}.$$

Proof. Let $\psi \in L^2((\Omega, \mathcal{F}^T_a); E)$; for $a \leq t \leq b$, we can choose $S(t-a)\psi$ taking values in E, because $S(t-a)$ maps E in itself. Moreover $S(\cdot)$ is a strongly continuous semigroup also on the space E, so that (see e.g. [18]) there exists a constant C_{b-a} such that $\|S(t-a)\|_{\mathcal{L}(E)} \leq C_{b-a}$, $t \in [a,b]$. Therefore, for $t \in [a,b]$, $\|S(t-a)\psi\|_E \leq C_{b-a}\|\psi\|_E$, so that $S(t-a)\psi \in L^2(\Omega; E)$.
The fact that $\chi(\psi) \in C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big)$ follows by dominated convergence, since, by the property of strong continuity of the semigroup, if $t_0 \in [a,b]$ and $[a,b] \ni t \to t_0$, then $S(t-a)\psi \longrightarrow S(t_0-a)\psi$ in E (pointwise on Ω after having chosen a representative for the random variable ψ) and moreover $\|S(t-a)\psi\|_E \leq C_{b-a}\|\psi\|_E$.
Finally the last statement follows again because $\|S(t-a)\psi\|_E \leq C_{b-a}\|\psi\|_E$. #

Lemma 4.7. *For $a, b \in \mathbb{R}$, $T \leq a < b$ and for a given $\mathcal{F}^T$-progressively measurable and $[0,1]$-valued process $\theta(\cdot)$, consider the linear map*

$$\begin{array}{rccc} \gamma_\theta: & C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big) & \longrightarrow & C_{\mathcal{P}}\big([a,b]; L^2(\Omega; H)\big) \\ & X(\cdot) & \longmapsto & \int_a^{\cdot} \theta(\tau) S(\cdot - \tau)\left[\Phi X(\tau)\right] d\tau. \end{array}$$

Then, for any $X \in C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big)$, we have $\gamma_\theta(X) \in C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big)$. Moreover we have the estimate

$$\|\gamma_\theta(X)\|^2_{C_{\mathcal{P}}([a,b];L^2(\Omega;E))} \leq 4e^{2r(b-a)}(b-a)^2\|X\|^2_{C_{\mathcal{P}}([a,b];L^2(\Omega;E))}.$$

Proof. See the Appendix. #

Lemma 4.8. *Let $g : E \to \mathbb{R}$ be a Lipschitz continuous map, with Lipschitz constant C_g, and consider the map associated with it*

$$\begin{array}{rccc} G: & E & \longrightarrow & H, \\ & x & \longmapsto & \begin{pmatrix} g(x) \\ 0 \end{pmatrix}; \end{array} \tag{25}$$

(of course, if $X(\cdot) \in C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big)$, then $g(X(\cdot)) \in C_{\mathcal{P}}([a,b]; L^2(\Omega; \mathbb{R}))$).
For $T \leq a < b$, consider the nonlinear map

$$\begin{array}{rccc} \gamma: & C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big) & \longrightarrow & C_{\mathcal{P}}\big([a,b]; L^2(\Omega; H)\big) \\ & X(\cdot) & \longmapsto & \int_a^{\cdot} S(\cdot - \tau) G(X(\tau)) d\tau. \end{array}$$

Then, for any $X \in C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big)$, we have $\gamma(X) \in C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big)$. Moreover, for $X, Y \in C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big)$, we have the estimate

$$\|\gamma(X) - \gamma(Y)\|^2_{C_{\mathcal{P}}([a,b];L^2(\Omega;E))} \leq 4C_g^2 e^{2r(b-a)} (b-a)^2 \|X - Y\|^2_{C_{\mathcal{P}}([a,b];L^2(\Omega;E))}.$$

Proof. See the Appendix. #

Lemma 4.9. *For $a, b \in \mathbb{R}$, $T \leq a < b$ and for a fixed $\mathcal{F}^T$-progressively measurable and $[0,1]$-valued process $\theta(\cdot)$, consider the linear map*

$$\begin{array}{rccc} \tilde{\gamma}_\theta: & C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big) & \longrightarrow & C_{\mathcal{P}}\big([a,b]; L^2(\Omega; H)\big) \\ & X(\cdot) & \longmapsto & \int_a^{\cdot} \theta(\tau) S(\cdot - \tau) \left[\Phi X(\tau)\right] dB(\tau). \end{array}$$

Then, for any $X \in C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big)$, we have $\tilde{\gamma}_\theta(X) \in C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big)$. Moreover we have the estimate

$$\|\tilde{\gamma}_\theta(X)\|^2_{C_{\mathcal{P}}([a,b];L^2(\Omega;E))} \leq 10 e^{2r(b-a)} (b-a) \|X\|^2_{C_{\mathcal{P}}([a,b];L^2(\Omega;E))}.$$

Proof. See the Appendix. #

Now we are ready to prove the desired existence and uniqueness result:

Theorem 4.10. *Let $G : E \to H$ be a map defined as in (25). Then, for a fixed $\mathcal{F}^T$-progressively measurable and $[0,1]$-valued process $\theta(\cdot)$ and for a fixed $x \in E$, the integral equation*

$$\begin{aligned} X(t) \quad = \quad & S(t-T)x + \sigma\lambda \int_T^t \theta(\tau) S(t-\tau) \left[\Phi X(\tau)\right] d\tau - \int_T^t S(t-\tau) G(X(\tau)) d\tau \\ & + \sigma \int_T^t \theta(\tau) S(t-\tau) \left[\Phi X(\tau)\right] dB(\tau). \end{aligned} \tag{26}$$

admits a unique solution in the space $C_{\mathcal{P}}\big([T, +\infty); L^2(\Omega; E)\big)$.

Proof. Let $a \geq T$ and $\psi \in L^2((\Omega, \mathcal{F}_a^T); E)$; for $b > a$ such that

$$e^{r(b-a)} \left[(2C_g + \sigma\lambda)(b-a) + 10^{1/2} \sigma (b-a)^{1/2} \right] < 1, \tag{27}$$

consider the map

$$\begin{array}{rccc} \Gamma_{\psi,\theta}: & C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big) & \longrightarrow & C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big), \\ & X & \longmapsto & \chi(\psi) + \sigma\lambda\gamma_\theta(X) - \gamma(X) + \sigma\tilde{\gamma}_\theta(X); \end{array}$$

by the previous Lemmata and by (27) this map is a contraction on $C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big)$ and so it admits a unique fixed point in this space. This fixed point is the solution of the equation (26) in the interval

$[a,b]$ for initial time a and initial condition ψ. So starting from T we can find a unique $X \in C_{\mathcal{P}}\big([T, T+(b-a)]; L^2(\Omega; E)\big)$ such that, for $t \in [T, T+(b-a)]$,

$$\begin{aligned} X(t) &= S(t-T)x + \sigma\lambda \int_T^t \theta(\tau) S(t-\tau) \left[\Phi X(\tau)\right] d\tau - \int_T^t S(t-\tau) G(X(\tau)) d\tau \\ &\quad + \sigma \int_T^t \theta(\tau) S(t-\tau) \left[\Phi X(\tau)\right] dB(\tau). \end{aligned}$$

Then we can iterate the argument by $(b-a)$-steps (notice that the achievement of (27) depends only on the difference $b-a$) and get the solution in the whole interval $[T, +\infty)$ by the semigroup property of $S(\cdot)$. #

Remark 4.11. *In the proof of Theorem 4.10 we could obtain the solution directly on the whole interval $[T, +\infty)$ by using an* exponential *norm on the space $C_{\mathcal{P}}([T, +\infty); L^2(\Omega; E))$; indeed, if we endow this space with the* exponential norm

$$\|X\|_{C_{\mathcal{P}}\left([T,+\infty); L^2(\Omega;E)\right)} = \sup_{t\in[a,b]} e^{-\lambda t} \left(\mathbb{E}\left[\|X(t)\|_E^2\right]\right)^{1/2},$$

then we could find $\lambda > 0$ such that $\Gamma_{\psi,\theta}$ becomes a contraction on $C_{\mathcal{P}}\big([T, +\infty); L^2(\Omega; E)\big)$ and get the fixed point directly in this space. #

Remark 4.12. *Notice that our specific map F satisfies the condition of Theorem 4.10, since by assuumption the surplus map f_0 is Lipschitz continuous on $(E, \|\cdot\|)$ (whereas it is not even continuous in $(E, \|\cdot\|_H)$); thus we can apply the previous result to get existence and uniqueness of mild solution for the equation (22). Notice also that we cannot use the theory of [9]: the nearest result in that book is the Theorem 7.19, but here we do not require the dissipativity of G; in particular the map F, which is in our interest, is such that $F - \omega I$ is not dissipative for any $\omega \geq 0$. Indeed, for $\omega \geq 0$, we have*

$$\begin{aligned} \langle (F-\omega I)x - (F-\omega I)y, x-y\rangle &= (x_0 - y_0)\big[f_0(x_0 - \kappa x_1(-T)) - f_0(y_0 - \kappa y_1(-T))\big] \\ &\quad -\omega \left[|x_0 - y_0|^2 + \int_{-T}^0 (x_1(\zeta) - y_1(\zeta))^2 d\zeta\right]. \end{aligned}$$

We want to show that, for each $\omega \geq 0$, there exist $x, y \in E$ such that the right hand-side in the previous expression is strictly positive. Fix $\omega \geq 0$ and let $y = 0$, $x_0 > 0$; by convexity of f_0 we can do the expression $f_0(x_0 - \kappa x_1(-T)) - f_0(0)$ arbitrarely large by moving $x_1(-T)$ and, at the same time, to take fix x_0 and $|x_0|^2 + \int_{-T}^0 x_1(\zeta)^2 d\zeta$; this shows what claimed. #

4.2 The link between the stochastic delay problem and the infinite dimensional problem

At the beginning of this section we have defined the infinite dimensional equation (22) and we have proved an existence and uniqueness result for such a problem. Now, to give sense to our approach, we want to link (22) with (15). The link is given by the following result:

Proposition 4.13. *For a fixed $\mathcal{F}^T$-progressively measurable and $[0,1]$-valued process $\theta(\cdot)$ and for a fixed $x \in E$, let $x(t)$ be the unique solution of the stochastic delay differential equation*

$$\begin{cases} dx(t) = [r + \sigma\lambda\theta(t)]\, x(t)\, dt - [f_0(x(t) - \kappa x(t-T)) + q]\, dt + \sigma\theta(t)x(t)dB(t), \\ x(T) = x_0, \ \ x(T+\zeta) = x_1(\zeta), \qquad \zeta \in [-T, 0). \end{cases} \tag{28}$$

Then $X(t) := (x(t), x(t+\zeta)|_{\zeta\in[-T,0]})$ is the unique mild solution of (22) in $C_{\mathcal{P}}([T, +\infty); L^2(\Omega; E))$.

Proof. See the Appendix. #

Thanks to the previous equivalence result, we can rewrite our optimization problem in the infinite-dimensional setting: consider the class of equations, for $\theta(\cdot) \in \Theta_{ad}(x)$, $x \in \mathcal{C}$,

$$\begin{cases} dX(t) = AX(t)dt + \sigma\lambda\theta(t)\Phi X(t)dt - F(X(t))dt + \sigma\theta(t)\Phi X(t)dB(t), \\ X(T) = x. \end{cases}$$

and denote the mild solution to the previous equation by $X(t;T,x,\theta(\cdot))$. Thanks to Proposition 4.13 the objective functional defined in Section 3.2 can be rewritten as

$$J\left(x;\theta\left(\cdot\right)\right) = \mathbb{E}\left[\int_T^{+\infty} e^{-\rho(t-T)} U\left(X_0\left(t;T,x,\theta(\cdot)\right)\right)dt\right].$$

and the value function as

$$V\left(x\right) = \sup_{\theta(\cdot)\in\Theta_{ad}(x)} J(x;\theta(\cdot)).$$

4.3 Continuous dependence on initial data

In this subsection we investigate the continuous dependence on the initial data for the mild solution of equation (22) with respect to the $\|\cdot\|_E$-norm and with respect to the $\|\cdot\|_H$-norm.

Proposition 4.14. *In the hypotheses of Theorem 4.10, let $x, y \in E$ be two initial data for the equation and denote by $X(x), X(y)$ the solution associated respectively to x, y. Then, for $u \geq T$, there exists a constant $K_u > 0$ such that*

$$\|X(x) - X(y)\|_{C_{\mathcal{P}}([T,u];L^2(\Omega;E))} \leq K_u \|x - y\|_E.$$

Proof. Let us consider the map

$$\begin{array}{rccc} \Gamma_\theta: & L^2((\Omega,\mathcal{F}_a^T);E) \times C_{\mathcal{P}}\big([a,b];L^2(\Omega;E)\big) & \longrightarrow & C_{\mathcal{P}}\big([a,b];L^2(\Omega;E)\big), \\ & (\psi, X) & \longmapsto & \chi(\psi) + \sigma\lambda\gamma_\theta(X) - \gamma(X) + \sigma\tilde{\gamma}_\theta(X); \end{array}$$

we have already proved in Theorem 4.10 that, for $(b-a)$ small enough, there exists $C < 1$ such that

$$\|\Gamma_\theta(\psi, X) - \Gamma_\theta(\psi, Y)\|_{C_{\mathcal{P}}([a,b];L^2(\Omega;E))} \leq C\,\|X - Y\|_{C_{\mathcal{P}}([a,b];L^2(\Omega;E))}.$$

Moreover, by Lemma 4.6, we know that

$$\begin{array}{rcl} \|\Gamma_\theta(\psi, X) - \Gamma_\theta(\psi', X)\|_{C_{\mathcal{P}}([a,b];L^2(\Omega;E))} & = & \|\chi(\psi - \psi')\|_{C_{\mathcal{P}}([a,b];L^2(\Omega;E))} \\ & \leq & C_{b-a}\|\psi - \psi'\|_{L^2(\Omega;E)}. \end{array}$$

Thus, denoting by $X(\psi), X(\psi')$ the solution in $[a,b]$ to our equation, starting from ψ, ψ' respectively, we have

$$\begin{array}{rcl} \|X(\psi) - X(\psi')\|_{C_{\mathcal{P}}([a,b];L^2(\Omega;E))} & = & \|\Gamma_\theta(\psi, X(\psi)) - \Gamma_\theta(\psi', X(\psi'))\|_{C_{\mathcal{P}}([a,b];L^2(\Omega;E))} \\ & \leq & C_{b-a}\|\psi - \psi'\|_{L^2(\Omega;E)} \\ & & +C\|X(\psi) - X(\psi')\|_{C_{\mathcal{P}}([a,b];L^2(\Omega;E))}, \end{array}$$

therefore

$$\|X(\psi) - X(\psi')\|_{C_{\mathcal{P}}([a,b];L^2(\Omega;E))} \leq (1-C)^{-1} C_{b-a}\|\psi - \psi'\|_{L^2(\Omega;E)}.$$

Thus, starting from T with initial conditions x, y and iterating by $(b-a)$-steps until reaching u, we get the claim. #

Thanks to Proposition 4.13, we can prove a continuous dependence on initial data result with respect to the H-norm under the null control:

Proposition 4.15. *Let $x, y \in E$ be two initial data for the equation (22) and denote by $X(x), X(y)$ the mild solutions associated respectively to x, y, both under the null control $\theta(\cdot) \equiv 0$. Then, for each $u \geq T$, there exists a constant $K_u > 0$ such that*

$$\|X(x) - X(y)\|_{C([T,u];(E,\|\cdot\|_H))} \leq K_u \|x - y\|_H.$$

Proof. See the Appendix. #

By the previous result we get the following very useful result:

Corollary 4.16. *Let $x, y \in E$ two initial data for the equation (22) with null control $\theta(\cdot) \equiv 0$ and denote by $X(x), X(y)$ the mild solution associated respectively to x, y. Then, for $u \geq T$, there exists a constant $K_u > 0$ such that*

$$\sup_{t\in[T,u]} |X_0(x)(t) - X_0(y)(t)| \leq K_u \|x - y\|_H.$$

#

5 Continuity of the value function

In this section we prove the continuity of the value function with respect to the $\|\cdot\|_H$-norm in the interior part of its domain and extend it to a $\|\cdot\|_H$-continuous function defined on an open set of $(H, \|\cdot\|_H)$. Moreover, in the special case of absorbing boundary, i.e. when $rl = q$ and $U(l) > -\infty$ (see Proposition 5.6-(6)), we prove that the value function is $\|\cdot\|_H$-continuous up to the boundary. Actually, for brevity, we will prove the lower semicontinuity of the value function at the boundary and only the idea of the proof of the upper semicontinuity. We will mean the topological notions in the $\|\cdot\|_H$ norm. Nevertheless we will often use subscripsts to stress in which topological space we are working.

Let us recall that the domain of V is

$$D(V) := \{x \in \mathcal{C} \mid V(x) > -\infty\}$$

and that, by Proposition 3.15, we have $\mathcal{D}_0 \subset D(V)$, where $\mathcal{D}_0$ was defined in Remark 3.16.

Lemma 5.1. *Let $x \in \mathcal{D}_0$; then there exists $\varepsilon > 0$ such that V is bounded from below on $B_{(E,\|\cdot\|_H)}(x, \varepsilon)$.*

Proof. Consider the null strategy $\theta(\cdot) \equiv 0$ and set $X(t) := X(t; T, x, 0)$; by Lemma 3.7-(2) there exists $\beta > 0$ such that $X_0(t) \geq l + \beta$. Set, for $y \in B_{(E,\|\cdot\|_H)}(x, \varepsilon)$, $Y(t) := Y(t; T, y, 0)$. Take $\varepsilon < \beta/2K_{2T}$, where $K_{2T} > 0$ is the constant given in the estimate of Corollary 4.16; by the same Corollary, if $\|x - y\|_H \leq \varepsilon$, then

$$\sup_{t\in[T,2T]} |X_0(t) - Y_0(t)| \leq \beta/2,$$

so that we have $Y_0(t) \geq l + \beta/2$ on $[T, 2T]$. Thus, for $y \in B_{(E,\|\cdot\|_H)}(x, \varepsilon)$,

$$\mathbb{E}\left[\int_T^{2T} e^{-\rho(t-T)} U(Y_0(t))dt\right] \geq \frac{1 - e^{-\rho T}}{\rho}\, U(l + \beta/2).$$

Let $z = (z_0, z_1(\cdot))$, where $z_0 = l + \beta/2$ and $z_1 : [-T, 0) \to \mathbb{R}$ is the constant function $\zeta \mapsto l + \beta/2$. By Proposition 4.13 we have $Y(2T) \geq z$, in the sense of Proposition 3.14. By the semigroup property of the mild solution $Y(\cdot)$, we have, for $t \geq 2T$, $Y(t) = X(t; 2T, Y(2T), 0)$, so that $Y_0(t) \geq X_0(t; 2T, z, 0)$, for $t \geq 2T$. Since $z \in \mathcal{D}_0$, by Lemma 3.7-(2) and Proposition 4.13 we have $X_0(t; 2T; z, 0) \geq l + \beta_z$, where $\beta_z > 0$ does not depend on $y \in B_{(E,\|\cdot\|_H)}(x, \varepsilon)$. Thus we can write

$$\begin{aligned} V(y) &\geq \mathbb{E}\left[\int_T^{2T} e^{-\rho(t-T)} U(Y_0(t))dt\right] + \mathbb{E}\left[\int_{2T}^{+\infty} e^{-\rho(t-T)} U(Y_0(t))dt\right] \\ &= \frac{1 - e^{-\rho T}}{\rho}\, U(l + \beta/2) + \frac{1 - e^{-\rho T}}{\rho}\, U(l + \beta_z/2). \end{aligned}$$

The estimate holds uniformly without regard to the point $y \in B_{(E,\|\cdot\|_H)}(x,\varepsilon)$ chosen, so that the claim is proved. #

With the previous result we have proved in particular that $Int_{(E,\|\cdot\|_H)}(D(V))$ is not empty and that $\mathcal{D}_0 \subset Int_{(E,\|\cdot\|_H)}(D(V))$. The proof of the following Lemma can be found e.g. in [16], Chapter 1, Corollary 2.4.

Lemma 5.2. *Let Z be a topological vector space and $f : Z \to \bar{\mathbb{R}}$ concave and proper. Let us define*

$$D(f) := \{z \in Z \mid f(z) > -\infty\};$$

if f is bounded from below on a neighborhood of some $z_0 \in D(f)$, then $Int(D(f))$ is not empty and) f is continuous on $Int(D(f))$.

Thus from Lemma 5.1 and Lemma 5.2 above we can get the following result:

Corollary 5.3. *The value function V is continuous on $Int_{(E,\|\cdot\|_H)}(D(V))$.* #

For semplicity of notation we set

$$\mathcal{V} := Int_{(E,\|\cdot\|_H)}(D(V)). \tag{29}$$

We want to extend V to a continuous function defined on an open set of $(H, \|\cdot\|_H)$ containing $\mathcal{V}$. Notice that, if $\mathcal{A}$ is an open set of $(H, \|\cdot\|_H)$, then $\mathcal{A} \cap E$ is still $\|\cdot\|_H$-dense in $\mathcal{A}$.

Proposition 5.4. *There exist an open set $\mathcal{O}$ of $(H, \|\cdot\|_H)$ and a continuous function $\bar{V} : \mathcal{O} \to \mathbb{R}$ such that:*

1. *$\mathcal{O} \supset \mathcal{V}$ and $\bar{V}|_{\mathcal{V}} = V$.*
2. *$\mathcal{V} = \mathcal{O} \cap E$ and $\mathcal{O} = Int_{(H,\|\cdot\|_H)}\left(Clos_{(H,\|\cdot\|_H)}(\mathcal{V})\right)$.*
3. *$\mathcal{O}$ is convex and $\bar{V}$ is concave on $\mathcal{O}$.*

Proof. See the Appendix. #

Hereafter, with a slight abuse of notation, we still indicate the extended value function $\bar{V}$ on $\mathcal{O}$ by V.

Now we want to study the continuity properties of value function on the boundary. We start with a topological lemma which makes clear the link between the boundary $Fr_{(H,\|\cdot\|_H)}(\mathcal{O})$ of $\mathcal{O}$ in $(H, \|\cdot\|_H)$ and the boundary $Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$ of $\mathcal{V}$ in $(E, \|\cdot\|_H)$. Notice that $Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$ is not empty: indeed, by Proposition 3.15, we have $\{x \in \mathcal{D} \mid x_0 = l\} \subset Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$.

Lemma 5.5. *We have the following statements:*

1. *$E \setminus \mathcal{V} = (H \setminus \mathcal{O}) \cap E$.*
2. *$Clos_{(E,\|\cdot\|_H)}(\mathcal{V}) = Clos_{(H,\|\cdot\|_H)}(\mathcal{O}) \cap E$.*
3. *$Fr_{(E,\|\cdot\|_H)}(\mathcal{V}) = Fr_{(H,\|\cdot\|_H)}(\mathcal{O}) \cap E$.*

Proof. See the Appendix. #

Refining the assumptions on the parameters of the model the boundary becomes absorbing:

Proposition 5.6. *Let $rl = q$, $U(l) > -\infty$ and let $\mathcal{C} \subset E$ be the convex set defined in (12). Then the following statements hold:*

1. *We have $D(V) = \{x \in \mathcal{C} \mid 0 \in \Theta_{ad}(x)\} = Clos_{(E,\|\cdot\|_H)}(\mathcal{V})$.*

2. *Let $x \in D(V)$; then either there exists $\beta > 0$ such that $X_0(t;T,x,0) \geq l+\beta$ for all $t \geq T$ (first case) or there exists $s \in [T,2T)$ such that $X_0(s;T,x,0) = l$ (second case). In the second case $X_0(t;T,x,0) = l$, for every $t \geq s$.*

3. *We have $x \in \mathcal{V}$ if and only if $x \in D(V)$ and there exists $\beta > 0$ such that $X(t;T,x,0) \geq l+\beta$, for all $t \geq T$.*

4. *We have $x \in Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$ if and only if $x \in D(V)$ and there exists $s \in [T,2T)$ such that $X(s;T,x,0) = l$. In this case $X_0(t;T,x,0) = l$, for $t \geq s$.*

5. *We have $Fr_{(E,\|\cdot\|_H)}(\mathcal{V}) = \{x \in \mathcal{C} \mid \Theta_{ad}(x) = \{0\}\}$.*

6. *The boundary $Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$ is absorbing for the problem, in the sense that, if $x \in Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$, then we have $X(t;T,x,0) \in Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$ for all $t \geq T$.*

Proof. See the Appendix. #

Proposition 5.7. *Let $U(l) > -\infty$, $rl = q$. Then the (extended) value function $V : \mathcal{O} \cup Fr_{(E,\|\cdot\|_H)}(\mathcal{V}) \to \mathbb{R}$ is continuous at the boundary $Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$.*

Proof. **(i)** First of all notice that, since V is continuous on $\mathcal{O}$ and $\mathcal{V}$ is dense in $\mathcal{O}$, without loss of generality we can prove that

$$V(x) \leq \liminf_{\substack{y \to x \\ y \in Clos_{(E,\|\cdot\|_H)}(\mathcal{V})}} V(y).$$

Let $x \in Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$ and set $X(t) := X(t,T,x,0)$; by Proposition 5.6-(5) we know that the only admissible strategy is the null one and, by Proposition 5.6-(4), there exists $s \in [T,2T)$ such that $X_0(t) = l$ for every $t \geq s$; therefore

$$V(x) = J(x,0) = \int_T^s e^{-\rho(t-T)} U(X_0(t))dt + \int_s^{+\infty} e^{-\rho(t-T)} U(l)dt.$$

Take $\varepsilon > 0$, $y \in B_{(E,\|\cdot\|_H)}(x,\varepsilon) \cap Clos_{(E,\|\cdot\|_H)}\mathcal{V}$ and set $Y(t) := X(t;T,y,0)$. Of course $0 \in \Theta_{ad}(y)$ and

$$\begin{aligned} V(y) \geq J(y,0) &= \int_T^s e^{-\rho(t-T)} U(Y_0(t))dt + \int_s^{+\infty} e^{-\rho(t-T)} U(Y_0(t))dt \\ &\geq \int_T^s e^{-\rho(t-T)} U(Y_0(t))dt + \int_s^{+\infty} e^{-\rho(t-T)} U(l)dt. \end{aligned}$$

By Corollary 4.16 we get $|X_0(t) - Y_0(t)| \leq K_s\,\varepsilon$, for $t \in [T,s]$, where K_s is the constant given in the same Corollary. Therefore we get the claim by uniform continuity of U.

(ii) As we said, we give only a sketch of the proof of the upper semicontinuity at the boundary. Again, without loss of generality, we can reduce the problem to prove

$$V(x) \geq \limsup_{\substack{y \to x \\ y \in Clos_{(E,\|\cdot\|_H)}(\mathcal{V})}} V(y).$$

The heart of the idea is that our value function is obviously smaller of the value function of the corresponding problem without surplus, i.e. the value function which is the object of [13]. We will call this value function without surplus V^0. In [13] the authors claim the continuity of the value function in the case $U(l) > -\infty$, $rl = q$. Actually the proof of this fact is very technical and they do not give the proof; nevertheless they have the explicity solution in the case when U is a power function, getting in particular the continuity. Let us assume the continuity of V^0.

Let $x \in Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$ be such that $x_0 = l$; then, by Proposition 5.6-(2), we get $V(x) = U(l)/\rho = V^0(x_0)$. On the other hand, since for each $x \in \mathcal{C}$ we have $V(x) \leq V^0(x_0)$,

$$\limsup_{\substack{y \to x \\ y \in Clos_{(E,\|\cdot\|_H)}(\mathcal{V})}} V(y) \leq \limsup_{\substack{y \to x \\ y \in Clos_{(E,\|\cdot\|_H)}(\mathcal{V})}} V^0(y_0) = V^0(x_0) = V(x),$$

so that the claim is proved in this case.

Now let $x \in Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$ be such that $x_0 > l$; then, by Proposition 5.6, we can write, for some $s \in [T, 2T)$,

$$V(x) = \int_T^s e^{-\rho(t-T)} U(X_0(t;T,x,0)dt + \int_s^{+\infty} e^{-\rho(t-T)} U(l)dt.$$

Let us take a sequence $(y_n) \subset Clos_{(E,\|\cdot\|_H)}(\mathcal{V})$ such that $y_n \overset{\|\cdot\|_H}{\longrightarrow} x$; we can write, by Dynamic Programming Principle (see Proposition 6.9 and Remark 6.10),

$$\begin{aligned} V(y_n) &\leq \sup_{\theta(\cdot)\in\Theta_{ad}(y_n)} \mathbb{E}\left[\int_T^s e^{-\rho(t-T)} U(X_0(t;T,x,0)dt + e^{-\rho(s-T)} V(X(s;T,y_n,\theta(\cdot)))\right] \\ &\leq \sup_{\theta(\cdot)\in\Theta_{ad}(y_n)} \mathbb{E}\left[\int_T^s e^{-\rho(t-T)} U(X_0(t;T,x,0)dt + e^{-\rho(s-T)} V^0(X_0(s;T,y_n,\theta(\cdot)))\right]. \end{aligned}$$

Then, using the Girsanov's Theorem, the convexity of the map f_0, the Corollary 4.16 and the continuity of V^0, one could prove that the limsup of the last term in the previous inequality is less than $V(x)$, concluding the proof. #

6 The Hamilton-Jacobi-Bellman equation

In this section we write and study the infinite-dimensional Hamilton-Jacobi-Bellman equation associated with the value function. Unless differently specified we will mean the topological notions in $\|\cdot\|_H$-norm.

For $x \in H$, let us define

$$\begin{array}{rccc} \Sigma(x): & \mathbb{R} & \longrightarrow & H. \\ & a & \longmapsto & a\,(\Phi x) \end{array}$$

For an operator $Q \in \mathcal{L}(H)$, we can use the representation on the components $\mathbb{R}$ and $L^2[-T,0]$,

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix},$$

where $Q_{11} \in \mathcal{L}(\mathbb{R}) \cong \mathbb{R}$, $Q_{12} \in \mathcal{L}(L^2[-T,0];\mathbb{R})$, $Q_{21} \in \mathcal{L}(\mathbb{R}, L^2[-T,0])$ and $Q_{22} \in \mathcal{L}(L^2[-T,0])$.

Let us define, for $x = (x_0, x(\cdot)) \in E$, $p \in D(A^*)$, $Q \in \mathcal{L}(H)$ and $\theta \in [0,1]$, the function

$$\begin{aligned} \mathcal{H}_{cv}(x,p,Q;\theta) &:= U(x_0) + \frac{1}{2}\theta^2\sigma^2 \mathrm{Tr}[Q\Sigma(x)\Sigma(x)^*] + \langle \sigma\lambda\theta\Phi x - F(x), p\rangle \\ &= U(x_0) - f(x)p_0 + \frac{1}{2}\theta^2\sigma^2 x_0^2 Q_{11} + \sigma\lambda\theta x_0 p_0. \end{aligned}$$

Formally the HJB equation associated with V in the space H is

$$\rho v(x) = \langle x, A^* v_x(x)\rangle + \mathcal{H}(x, v_x(x), v_{xx}(x)), \tag{30}$$

where

$$\mathcal{H}(x,p,Q) := \sup_{\theta\in[0,1]} \mathcal{H}_{cv}(x,p,Q;\theta).$$

Notice that we can write

$$\mathcal{H}(x,p,Q) = U(x_0) - f(x)p_0 + \sup_{\theta\in[0,1]} \left\{ \frac{1}{2}\theta^2\sigma^2 x_0^2 Q_{11} + \sigma\lambda\theta x_0 p_0 \right\};$$

in order to calculate the hamiltonian, we observe that the function

$$\mathcal{H}^0_{cv}(x_0,p_0,Q_{11};\theta) := \frac{1}{2}\theta^2\sigma^2 x_0^2 Q_{11} + \sigma\lambda\theta x_0 p_0, \tag{31}$$

when $p_0 \geq 0$, $Q_{11} \leq 0$, $p_0^2 + Q_{11}^2 > 0$, has a unique maximum point over $\theta \in [0,1]$ given by

$$\theta^* = -\frac{\lambda p_0}{\sigma x Q_{11}} \wedge 1$$

(where we mean that, for $Q_{11} = 0$, $\theta^* = 1$) and

$$\mathcal{H}^0(x_0, p_0, Q_{11}) := \sup_{\theta \in [0,1]} \mathcal{H}^0_{cv}(x_0, p_0, Q_{11}; \theta) = \begin{cases} -\frac{\lambda^2 p_0^2}{2Q_{11}}, & \text{if} \quad \theta^* < 1, \\ \sigma \lambda x_0 p_0 + \frac{1}{2}\sigma^2 x_0^2 Q_{11}, & \text{if} \quad \theta^* = 1. \end{cases}$$

When $p_0 = Q_{11} = 0$ each $\theta \in [0,1]$ is a maximum point and $\mathcal{H}^0(x_0, p_0, Q_{11}) = 0$.

Thus (30) can be rewritten as

$$\rho v(x) = \langle x, A^* v_x(x) \rangle_H + U(x_0) - f(x) v_{x_0}(x) + \mathcal{H}^0(x_0, v_{x_0}(x), v_{x_0 x_0}(x)). \tag{32}$$

6.1 Rewriting the problem with a maximal monotone operator

In order to work with a maximal dissipative operator we rewrite the state equation as

$$\begin{cases} dX(t) = \tilde{A} X(t) dt + \left[\left(r + \frac{1}{2}\right) + \sigma \lambda \theta(t)\right] \Phi X(t) dt - F(X(t)) dt + \sigma \theta(t) \Phi X(t) dB(t), \\ X(T) = x, \end{cases} \tag{33}$$

where $\tilde{A} = A - \left(r + \frac{1}{2}\right)\Phi$; of course X is a mild solution of (33) if and only if it is a mild solution of (22). We also rewrite the HJB equation (30) as

$$\rho v(x) = \langle x, \tilde{A}^* v_x(x) \rangle + \tilde{\mathcal{H}}(x, v_x(x), v_{xx}(x)) \tag{34}$$

where, for $x \in E$, $p \in D(\tilde{A}^*)$, $Q \in \mathcal{L}(H)$,

$$\tilde{\mathcal{H}}(x, p, Q) = \sup_{\theta \in [0,1]} \tilde{\mathcal{H}}_{cv}(x, p, Q; \theta),$$

and

$$\tilde{\mathcal{H}}_{cv}(x, p, Q; \theta) = U(x_0) + \left(r + \frac{1}{2}\right) x_0 p_0 - f(x) p_0 + \mathcal{H}^0_{cv}(x_0, p_0, Q_{11}; \theta),$$

where $\mathcal{H}^0_{cv}$ was defined in (31).

Remark 6.1. *We want to underline some specific features of the HJB equation (34):*

- *it is defined on the points of E, because of the presence of f, which is defined on this space;*
- *the linear term is unbounded;*
- *the term $f(\cdot)$ is not continuous with respect to $\|\cdot\|_H$;*
- *the terms associated with the control θ, i.e. in $\mathcal{H}^0_{cv}$, involve only the derivatives with respect to the real component: therefore we can hope to prove a verification theorem giving optimal feedback strategies even only having regularity properties of the value function with respect to the real component; this makes clear the importance of Proposition 5.4 which splits the real and the infinite dimensional component in the argument of the value function, leaving open the possibility of studying the regularity of this function only with respect to the real component.* #

Notice that $D(\tilde{A}^*) = D(A^*)$ and

$$\tilde{A}^* = A^* - \left(r + \frac{1}{2}\right)\Phi^* = A^* - \left(r + \frac{1}{2}\right)\Phi.$$

The following proposition gives the desired properties of the operator $\tilde{A}^*$; we will use the dissipativity of $\tilde{A}^*$ to obtain a Dynkin type formula with inequality for "radial" functions.

Proposition 6.2. *The operator $\tilde{A}$ is maximal dissipative.*

Proof.
(i) We have, for $x \in D(\tilde{A}) = D(A)$, taking into account that $x_1(0) = x_0$,

$$\langle \tilde{A}x, x\rangle = -\frac{1}{2}x_0^2 + \int_{-T}^{0} x_1'(\xi)x_1(\xi)d\xi = -\frac{1}{2}x_0^2 + \left[\frac{x_1(\cdot)^2}{2}\right]_{-T}^{0} = -x_1(-T)^2 \leq 0,$$

so that $\tilde{A}$ is dissipative.
(ii) In order to prove that $\tilde{A}$ is maximal we have to prove that $\mathcal{R}(\tilde{A} - I) = H$; this means that, for each $y = (y_0, y_1(\cdot)) \in H$, we must be able to find $x = (x_0, x_1(\cdot)) \in D(\tilde{A})$ such that

$$\begin{cases} -\frac{3}{2}x_0 = y_0, \\ x_1'(\cdot) - x_1(\cdot) = y_1(\cdot) \quad \text{a.e.}; \end{cases}$$

this means that we must be able to solve the first order ordinary problem of finding $f \in W^{1,2}[-T, 0]$ such that, for given $g \in L^2[-T, 0]$,

$$\begin{cases} f' - f = g, \\ f(0) = \alpha \in \mathbb{R}; \end{cases}$$

the solution, as in the classical case, but meaning here the derivative and the equality only almost everywhere, is given by the variation of constants formula

$$f(t) = \left(\alpha - \int_{-T}^{0} g(\xi)e^{-\xi}d\xi\right)e^t + \int_{-T}^{t} g(\xi)e^{t-\xi}d\xi.$$

#

6.2 Test functions and Dynkin type formulae

In this section we define two sets of functions which will play an important role in the definition of viscosity solution of the HJB equation and we prove Dynkin type formulae for these functions applied to the process X mild solution of

$$\begin{cases} dX(t) = \tilde{A}X(t)dt + \left[\left(r + \frac{1}{2}\right) + \sigma\lambda\theta(t)\right]\Phi X(t)dt - G(X(t))dt + \sigma\theta(t)\Phi X(t)dB(t), \\ X(T) = x \in E, \end{cases} \tag{35}$$

where G is a map $E \to H$, $x \mapsto (g(x), 0)$, and $g : (E, \|\cdot\|_E) \to \mathbb{R}$ is Lipschitz continuous. We have proved in Theorem 4.10, for a given $[0, 1]$-valued and $\mathcal{F}^T$-progressively measurable process $\theta(\cdot)$, existence and uniqueness of mild solution for the equation (35) in the class $C_{\mathcal{P}}\big([T, +\infty); L^2(\Omega, (E, \|\cdot\|_E))\big)$.

Let us start with an approximation result which will be useful for our goal:

Lemma 6.3. *Let $X(\cdot)$ be the mild solution of the equation (35) and let $\tilde{A}_n$ be the Yosida approximations for the operator $\tilde{A}$. Then the stochastic differential equation*

$$\begin{cases} dX_n(t) = \tilde{A}_nX_n(t)dt + \left[\left(r + \frac{1}{2}\right) + \sigma\lambda\theta(t)\right]\Phi X(t)dt - G(X(t))dt + \sigma\theta(t)\Phi X(t)dB(t), \\ X_n(T) = x, \end{cases} \tag{36}$$

admits a unique strong solution in $C_{\mathcal{P}}([T, +\infty); L^2(\Omega, E))$. Moreover, for any $t \in [T, +\infty)$,

$$X_n(t) \to X(t), \quad \text{in } L^2(\Omega; H), \tag{37}$$

and, for any $a \in [T, +\infty)$,

$$X_n \to X, \quad \text{in } \ L^2(\Omega \times [T, a]; H)). \tag{38}$$

Proof. See the Appendix. #

Definition 6.4. *(i) We call $\mathcal{T}_1$ the set of functions $\psi \in C^2(H)$ such that $\psi_x(x) \in D(A^*)$ for any $x \in H$ and $\psi, \psi_x, \tilde{A}^*\psi_x, \psi_{xx}$ are uniformly continuous.*

(ii) We call $\mathcal{T}_2$ the set of functions $g \in C^2(H)$ which are radial and nondecreasing, i.e.

$$g(x) = g_0(\|x\|), \quad g_0 \in C^2([0,+\infty);\mathbb{R}), \quad g_0' \geq 0,$$

and g, g_x, g_{xx} are uniformly continuous. #

Before to proceed we need a functional analysis result and a measure theory result:

Lemma 6.5. *Let $(B, \|\cdot\|)$ be a Banach space and $f : B \to \mathbb{R}$ uniformly continuous. Then f is bounded on the bounded set of E.* #

Lemma 6.6. *Let $(M, \mathcal{M}, \mu)$ be a finite measure space; let $p \geq 1$, $L^p := L^p(M, \mathcal{M}, \mu)$, let $(f_n), f \subset L^p$ be such that $f_n \to f$ in L^p; let $\psi : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Then $\psi(f) \in L^p$ and $\psi(f_n) \to \psi(f)$ in L^p.* #

Let us define, for $\theta \in [0,1]$, the operator $\mathcal{L}^\theta$ on $\mathcal{T}_1$ by

$$\begin{aligned}[\mathcal{L}^\theta\psi](x) \quad &:= \quad -\rho\psi(x) + \langle x, \tilde{A}^*\psi_x(x)\rangle + \left[r + \frac{1}{2} + \sigma\lambda\theta\right]\langle \Phi x, \psi_x(x)\rangle \\ &\quad\quad + \langle G(x), \psi_x(x)\rangle + \frac{1}{2}\sigma^2\theta^2\,\mathrm{Tr}\,[\Sigma(x)\Sigma(x)^*\psi_{xx}(x)]. \end{aligned} \tag{39}$$

Lemma 6.7 (Dynkin's formula (i))**.** *Let $\psi \in \mathcal{T}_1$, $X(\cdot)$ be the solution of (35) and τ a bounded stopping time; then we have*

$$\mathbb{E}\left[e^{-\rho(\tau-T)}\psi(X(\tau)) - \psi(x)\right] = \mathbb{E}\left[\int_T^\tau e^{-\rho(t-T)}[\mathcal{L}^{\theta(t)}\psi](X(t))dt\right].$$

Moreover the same formula holds true also for stopping time almost surely finite such that, for some $r > 0$, $X(t) \in B(x,r)$ for $t \leq \tau$.

Proof. See the Appendix. #

Now, for $\theta \in [0,1]$, let us define the operator $\mathcal{G}^\theta$ on $\mathcal{T}_2$ by

$$\begin{aligned}[\mathcal{G}^\theta g](x) \quad &:= \quad -\rho g(x) + \left[r + \frac{1}{2} + \sigma\lambda\theta\right]\langle \Phi x, g_x(x)\rangle \\ &\quad\quad + \langle G(x), g_x(x)\rangle + \frac{1}{2}\sigma^2\theta^2\,\mathrm{Tr}\,[g_{xx}\Sigma(x)\Sigma(x)^*]. \end{aligned} \tag{40}$$

Lemma 6.8 (Dynkin's formula (ii))**.** *Let $g \in \mathcal{T}_2$, let $X(\cdot)$ be the solution of (35) and τ a stopping time almost surely finite such that, for some $r > 0$, $X(t) \in B(x,r)$ for $t \leq \tau$; then we have*

$$\mathbb{E}[e^{-\rho(\tau-T)}g(X(\tau)) - g(x)] \leq \mathbb{E}\left[\int_T^\tau e^{-\rho(t-T)}[\mathcal{G}^{\theta(t)}g](X(t))dt\right].$$

Proof. See the Appendix. #

6.3 The Dynamic Programming Principle and the value function as viscosity solution of the HJB equation

In this section we will work to investigate the "differential properties" of the value function as viscosity solution of the HJB equation (30) associated with it. The link between the HJB equation and the value function is given by the Dynamic Programming Principle:

Proposition 6.9 (Dynamic Programming Principle)**.** *Let $x \in D(V)$ and let $\left(\tau^{\theta(\cdot)}\right)_{\theta(\cdot)\in\Theta_{ad}(x)}$ be a family of stopping times with respect to $\mathcal{F}^T$ such that $\tau^{\theta(\cdot)} \in [T, +\infty)$ almost surely. Then, setting $X^{\theta(\cdot)}(t) := X(t; T, x, \theta(\cdot))$, we have*

$$V(x) = \sup_{\theta(\cdot)\in\Theta_{ad}(x)} \mathbb{E}\left[\int_T^{\tau^{\theta(\cdot)}} e^{-\rho(t-T)} U\left(X_0^{\theta(\cdot)}(t)\right) dt + e^{-\rho(\tau^{\theta(\cdot)}-T)} V\left(X^{\theta(\cdot)}(\tau^{\theta(\cdot)})\right)\right].$$

#

Remark 6.10. *We do not give the proof of the statement of the previous Proposition. For a proof of this statement in the finite dimensional case without constraint on the state and when the value function is continuous we refer to [44]. For the infinite dimensional case we refer to [23], again when the value function is continuous and the state unconstrained. Similar arguments can be used to prove the result in our case, taking into account the separability of our space H. For a more general setting in which the value function does not need to be continuous, see e.g. [41]: in this case a measurable selection result has to be proved and used.*

Here we want to stress that in Proposition 5.7 we proved the continuity of the value function at the boundary only using the "easy" inequality of the Dynamic Programming Principle, which can be proved without measurable selection arguments, so that we could use the continuity of the value function to prove, without loss of generality, the Dynamic Programming Principle. #

Now we give a definition of viscosity solution for the equation (30); we will prove that the value function solves (30) in this sense. Recall that the set $\mathcal{V}$ was defined in (29) and the set $\mathcal{O}$ was defined in Proposition 5.4. Recall also that $\mathcal{V} = \mathcal{O} \cap E$.

Definition 6.11. *(i) A continuous function $v : \mathcal{O} \longrightarrow \mathbb{R}$ is called a viscosity subsolution of the equation (30) on $\mathcal{V}$ if, for any triple $(x_M, \psi, g) \in \mathcal{V} \times \mathcal{T}_1 \times \mathcal{T}_2$ such that x_M is a local maximum point of $v - \psi - g$, we have*

$$\rho v(x_M) \leq \langle x_M, \tilde{A}^*\psi_x(x_M)\rangle + \tilde{\mathcal{H}}(x_M, \psi_x(x_M) + g_x(x_M), \psi_{xx}(x_M) + g_{xx}(x_M)).$$

(ii) A continuous function $v : \mathcal{O} \longrightarrow \mathbb{R}$ is called a viscosity supersolution of the equation (30) on $\mathcal{V}$ if, for any triple $(x_m, \psi, g) \in \mathcal{V} \times \mathcal{T}_1 \times \mathcal{T}_2$ such that x_m is a local minimum point of $v - \psi + g$, we have

$$\rho v(x_m) \geq \langle x_m, \tilde{A}^*\psi_x(x_m)\rangle + \tilde{\mathcal{H}}(x_m, \psi_x(x_m) - g_x(x_m), \psi_{xx}(x_m) - g_{xx}(x_m)).$$

(iii) A continuous function $v : \mathcal{O} \longrightarrow \mathbb{R}$ is called a viscosity solution of the equation (30) on $\mathcal{V}$ if it is both a viscosity subsolution and a viscosity supersolution. #

Proposition 6.12. *The value function V is a viscosity solution of (30) on $\mathcal{V}$.*

Proof. (i) Let x_m, ψ, g be as in Definition 6.11-(ii); without loss of generality we assume

$$V(x_m) = \psi(x_m) - g(x_m). \tag{41}$$

Let $B_{(H,\|\cdot\|_H)}(x_m, \varepsilon) \subset \mathcal{O}$ be such that

$$V(x) \geq \psi(x) - g(x), \quad \text{for any } x \in B_{(H,\|\cdot\|_H)}(x_m, \varepsilon). \tag{42}$$

Fix a constant control $\theta \in [0, 1]$ and let $X(t) := X(t; T, x_m, \theta)$ be the state solution for our problem associated with the control θ and starting from x_m at time T; set

$$\tau^\theta := \inf\{t \geq T \mid X(t) \notin B_{(H,\|\cdot\|_H)}(x_m, \varepsilon)\};$$

this is of course a stopping time. Moreover the trajectories of $X(\cdot)$ are continuous in $(E, \|\cdot\|_E)$, therefore in $(H, \|\cdot\|_H)$, so that $\tau^\theta > T$ almost surely. By (41) and (42) we get, for $T \leq t \leq \tau^\theta$,

$$e^{-\rho(t-T)} V(X(t)) - V(x_m) \geq e^{-\rho(t-T)} \left(\psi(X(t)) - g(X(t))\right) - \left(\psi(x_m) - g(x_m)\right). \tag{43}$$

Let $h > T$ and set $\tau_h^\theta := \tau^\theta \wedge h$; by the dynamic programming principle we get, for all $\theta \in [0,1]$,

$$V(x_m) \geq \mathbb{E}\left[\int_T^{\tau_h^\theta} e^{-\rho(t-T)} U(X_0(t))dt + e^{-\rho(\tau_h^\theta - T)} V(X(\tau_h^\theta))\right]. \tag{44}$$

So, by (43) and (44) we get, for all $\theta \in [0,1]$,

$$\begin{aligned} 0 &\geq \mathbb{E}\left[\int_T^{\tau_h^\theta} e^{-\rho(t-T)} U(X_0(t))dt + e^{-\rho(\tau_h^\theta - T)} V(X(\tau_h^\theta)) - V(x_m)\right] \\ &\geq E\Big[\int_T^{\tau_h^\theta} e^{-\rho(t-T)} U(X_0(t))dt + e^{-\rho(\tau_h^\theta - T)} \big(\psi(X(\tau_h^\theta)) - g(X(\tau_h^\theta))\big) - (\psi(x_m) - g(x_m))\Big]. \end{aligned} \tag{45}$$

Now we can apply the Dynkin formulae to the function $\varphi(t,x) = e^{-\rho(t-T)}(\psi(x) - g(x))$ and put the result in (45) getting, for all $\theta \in [0,1]$,

$$0 \geq \mathbb{E}\left[\int_T^{\tau_h^\theta} e^{-\rho(t-T)} \big(U(X_0(t)) + [\mathcal{L}^\theta \psi](X(t)) - [\mathcal{G}^\theta g](X(t))\big)\, dt\right],$$

i.e., for all $\theta \in [0,1]$,

$$\begin{aligned} 0 \quad \geq \quad & E\Big[\int_T^{\tau_h^\theta} e^{-\rho(t-T)} \Big(- \rho(\psi(X(t)) - g(X(t))) + \langle x, \tilde{A}^* \psi_x(X(t))\rangle \\ & + \tilde{\mathcal{H}}_{cv}(X(t), \psi_x(X(t)) - g_x(X(t)), \psi_{xx}(X(t)) - g_{xx}(X(t))); \theta\Big) dt\Big]. \end{aligned}$$

Therefore, for all $\theta \in [0,1]$, we can write

$$\begin{aligned} 0 \quad \geq \quad & E\Big[\frac{1}{h-T}\int_T^{h} I_{[T,\tau^\theta]}(t) e^{-\rho(t-T)} \Big(- \rho(\psi(X(t)) - g(X(t))) + \langle x, \tilde{A}^* \psi_x(X(t))\rangle \\ & + \tilde{\mathcal{H}}_{cv}(X(t), \psi_x(X(t)) - g_x(X(t)), \psi_{xx}(X(t)) - g_{xx}(X(t))); \theta\Big) dt\Big]. \end{aligned}$$

Now, using the continuity properties of ψ, g and their derivatives and of $\tilde{\mathcal{H}}_{cv}$, taking into account that $\tau^\theta > T$ almost surely and passing to the limit for $h \to T$, we get by dominated convergence, for all $\theta \in [0,1]$,

$$0 \geq -\rho(\psi(x_m) - g(x_m)) + \langle x_m, \tilde{A}^* \psi_x(x_m)\rangle + \tilde{\mathcal{H}}_{cv}\left(x_m, \psi_x(x_m) - g_x(x_m), \psi_{xx}(x_m) - g_{xx}(x_m); \theta\right),$$

i.e., taking into account (41) and passing to the supremum on $\theta \in [0,1]$,

$$\rho V(x_m) \geq \langle x_m, \tilde{A}^* \psi_x(x_m)\rangle + \tilde{\mathcal{H}}\left(x_m, \psi_x(x_m) - g_x(x_m), \psi_{xx}(x_m) - g_{xx}(x_m)\right).$$

Notice that, passing to the limit in $\tilde{\mathcal{H}}_{cv}$, we have to use that $X(t) \to x_m$ in $(E, \|\cdot\|_E)$ almost surely, as $t \downarrow T$, since $\tilde{\mathcal{H}}_{cv}$ is not continuous with respect to $\|\cdot\|_H$ on the variable x, due to the presence in $\tilde{\mathcal{H}}_{cv}$ of the term f, which is not continuous with respect to $\|\cdot\|_H$.
Therefore we have proved that V is a supersolution on $\mathcal{V}$.

(ii) Let x_M, ψ, g be as in Definition 6.11-(i); without loss of generality we assume

$$V(x_M) = \psi(x_M) + g(x_M). \tag{46}$$

Let $B_{(H,\|\cdot\|_H)}(x_M, \varepsilon') \subset \mathcal{O}$ be such that

$$V(x) \leq \psi(x) + g(x), \quad \text{for any } x \in B_{(H,\|\cdot\|_H)}(x_M, \varepsilon'). \tag{47}$$

We have to prove that

$$\rho V(x_M) \leq \langle x, \tilde{A}^* \psi_x(x_M)\rangle + \tilde{\mathcal{H}}\left(x_M, \psi_x(x_M) + g_x(x_M), \psi_{xx}(x_M) + g_{xx}(x_M)\right).$$

Let us suppose by contradiction that there exists ν such that

$$0 < \nu \leq \rho V(x_M) - \langle x_M, \tilde{A}^* \psi_x(x_M)\rangle - \tilde{\mathcal{H}}\left(x_M, \psi_x(x_M) + g_x(x_M), \psi_{xx}(x_M) + g_{xx}(x_M)\right).$$

By the continuity propreties of ψ, g and their derivatives and of $\tilde{\mathcal{H}}$, we can find $\varepsilon > 0$ such that, for any $x \in B_{(E,\|\cdot\|_E)}(x_M, \varepsilon)$,

$$0 < \nu/2 \leq \rho V(x) - \langle x, \tilde{A}^* \psi_x(x)\rangle - \tilde{\mathcal{H}}\left(x, \psi_x(x) + g_x(x), \psi_{xx}(x) + g_{xx}(x)\right). \tag{48}$$

Notice that to state (48) we have to take the ball in the space $(E, \|\cdot\|_E)$, since $\tilde{\mathcal{H}}$ is not continuous with respect to $\|\cdot\|_H$ on the variable x, due to the presence of the term f, which is not continuous with respect to $\|\cdot\|_H$.
Without loss of generality, since $\|\cdot\|_H \leq (1+T)^{1/2}\|\cdot\|_E$, taking a smaller ε if necessary, we can suppose that $B_{(E,\|\cdot\|_E)}(x_M, \varepsilon) \subset B_{(H,\|\cdot\|_H)}(x_M, \varepsilon')$ and therefore, taking into account (47), that

$$V(x) \leq \psi(x) + g(x), \quad \text{for any } x \in B_{(E,\|\cdot\|_E)}(x_M, \varepsilon). \tag{49}$$

Let us consider a generic control $\theta(\cdot) \in \Theta_{ad}(x_M)$ and set $X(t) := X(t; T, x_M, \theta(\cdot))$; let us define the stopping time

$$\tau^\theta := \inf\left\{t \geq T \mid X(t) \notin B_{(E,\|\cdot\|_E)}(x_M, \varepsilon)\right\} \wedge (2T).$$

The trajectories of $X(\cdot)$ are continuous in $(E, \|\cdot\|_E)$, so that we have $T < \tau^\theta \leq 2T$ almost surely. Now we can apply (48) to $X(t)$, for $t \in [T, \tau^\theta]$, and get

$$\begin{aligned} 0 < \nu/2 \quad \leq \quad & \rho V(X(t)) - \langle X(t), \tilde{A}^* \psi_x(X(t))\rangle \\ & -\tilde{\mathcal{H}}\left(X(t), \psi_x(X(t)) + g_x(X(t)), \psi_{xx}(X(t)) + g_{xx}(X(t))\right); \end{aligned}$$

we multiply by $e^{-\rho(t-T)}$, integrate on $[T, \tau^\theta]$ and take the expectations getting, also taking into account (49),

$$\begin{aligned} 0 \quad < \quad & \frac{\nu}{2}\mathbb{E}\left[\int_T^{\tau^\theta} e^{-\rho(t-T)} dt\right] \\ \leq \quad & E\Big[\int_T^{\tau^\theta} e^{-\rho(t-T)}\Big(\rho(\psi(X(t)) + g(X(t))) - \langle X(t), \tilde{A}^* \psi_x(X(t))\rangle \\ & \qquad -\tilde{\mathcal{H}}\left(X(t), \psi_x(X(t)) + g_x(X(t)), \psi_{xx}(X(t)) + g_{xx}(X(t))\right)\Big) dt\Big]. \end{aligned}$$

We claim that there exists a constant $\delta > 0$, independent on the control $\theta(\cdot)$ chosen, such that

$$\frac{\nu}{2}\mathbb{E}\left[\int_T^{\tau^\theta} e^{-\rho(t-T)} dt\right] \geq \delta;$$

we will prove this fact in Lemma A.1.
So, assuming what claimed above, we can write, taking into account (49),

$$\delta + \mathbb{E}\left[\int_T^{\tau^\theta} e^{-\rho(t-T)} U(X_0(t))\, dt\right] \leq \mathbb{E}\left[\int_T^{\tau^\theta} e^{-\rho(t-T)}\left[-\mathcal{L}^{\theta(t)}\psi(X(t)) - \mathcal{G}^{\theta(t)} g(X(t))\right] dt\right].$$

Now we apply the Dynkin formulae to X on $[T, \tau^\theta]$ with the function $\varphi(t,x) = e^{-\rho(t-T)}(\psi(x) + g(x))$ and, comparing with the previous inequality, we get

$$\psi(x_M)+g(x_M)-\mathbb{E}\left[e^{-\rho(\tau^\theta-T)}\left(\psi(X(\tau^\theta))+g(X(\tau^\theta))\right)\right]\geq\delta+\mathbb{E}\left[\int_T^{\tau^\theta}e^{-\rho(t-T)}U(X(t))\,dt\right];$$

from the previous inequality, taking into account (46) and (49), we get

$$V(x_M)-\mathbb{E}[e^{-\rho(\tau^\theta-T)}V(X(\tau^\theta))]\quad\geq\quad\delta+\mathbb{E}\left[\int_T^{\tau^\theta}e^{-\rho(t-T)}U(X_0(t))\,dt\right];$$

on the other hand, if we choose a $\delta/2$-optimal control $\theta(\cdot)\in\Theta_{ad}(x_M)$, we get

$$\begin{aligned}V(x_M)-\delta/2\quad&\leq\quad\mathbb{E}\left[\int_T^{\tau^\theta}e^{-\rho(t-T)}U(X_0(t))\,dt+\int_{\tau^\theta}^{+\infty}e^{-\rho(t-T)}U(X_0(t))dt\right]\\&\leq\quad\mathbb{E}\left[\int_T^{\tau^\theta}e^{-\rho(t-T)}U(X_0(t))\,dt+e^{-\rho(\tau^\theta-T)}V(X(\tau^\theta))\right].\end{aligned}$$

So we have proved by contradiction that V is even a viscosity subsolution. #

Now we give a definition of constrained viscosity solution on $Clos_{(H,\|\cdot\|_H)}(\mathcal{O})\cap E=Clos_{(E,\|\cdot\|_H)}(\mathcal{V})$. Recall that we have proved in Lemma 5.5-(3) that $Fr_{(E,\|\cdot\|_H)}(\mathcal{V})=Fr_{(H,\|\cdot\|_H)}(\mathcal{O})\cap E$.

Definition 6.13. *A continuous function $v:\mathcal{O}\cup Fr_{(E,\|\cdot\|_H)}(\mathcal{V})\to\mathbb{R}$, is said a constrained viscosity solution of the equation (30) on $Clos_{(E,\|\cdot\|_H)}(\mathcal{V})$ if v is a viscosity solution of (30) on $\mathcal{V}$ and if:*

(i) (supersolution property at the boundary) for any triple $(x_m,\psi,g)\in Fr_{(E,\|\cdot\|_H)}(\mathcal{V})\times\mathcal{T}_1\times\mathcal{T}_2$ such that x_m is a local minimum point for $v-\psi+g$ on $\mathcal{O}\cup Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$, we have

$$\rho v(x_m)\geq\langle x_m,\tilde{A}^*\psi_x(x_m)\rangle+\tilde{\mathcal{H}}_{cv}(x_m,\psi(x_m)-g(x_m),\psi_{xx}(x_m)-g_{xx}(x_m);0);$$

(ii) (subsolution property at the boundary) for any triple $(x_M,\psi,g)\in Fr_{(E,\|\cdot\|_H)}(\mathcal{V})\times\mathcal{T}_1\times\mathcal{T}_2$ such that x_M is a local maximum point for $v-\psi-g$ on $\mathcal{O}\cup Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$, we have

$$\rho v(x_M)\leq\langle x_M,\tilde{A}^*\psi_x(x_M)\rangle+\tilde{\mathcal{H}}(x_M,\psi(x_M)-g(x_M),\psi_{xx}(x_M)-g_{xx}(x_M)).$$

#

We can give the main result:

Theorem 6.14. *Let $U(l)>-\infty$ and $rl=q$. Then the value function V is a constrained viscosity solution of (30) on $Clos_{(E,\|\cdot\|_H)}(\mathcal{V})$.*

Proof. We have proved in Proposition 6.12 that V is a viscosity solution on $\mathcal{V}$. The proof of the viscosity properties at the points of the boundary $Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$ follows the same line of the proof of Proposition 6.12. Notice that in this case the proof is even easier: in this case the stopping times are constant, since we are constrained to choose $\theta=0$ in the case of the supersolution and $\theta(\cdot)\equiv 0$ (see Proposition 5.6-(5)) in the case of the subsolution. #

Remark 6.15. *Usually in the stochastic control problems with state contraints the definition of solution for the HJB equation which works good to get the uniqueness is the definition of constrained viscosity solution. It was introduced by Soner, see [42], in the deterministic case and successfully developed and applied also in the stochastic case in other papers, [31] and [45]. It consists in requiring the subsolution property at the boundary.*

In [29] it is also required the supersolution property at the boundary replacing $\mathcal{H}$ by

$$\mathcal{H}_{in}(\cdot)=\sup_{\theta\in\mathcal{A}}\mathcal{H}_{cv}(\cdot;\theta),$$

where $\mathcal{A}$ is the subset of the control space for which the diffusion term of the state equation vanishes and the correspondent drift term directs inside to the state space (under the assumption that $\mathcal{A}$ is not empty). This last condition is similar to our supersolution condition at the boundary, but in our case the drift term is such that the state remains on the boundary of the state space. #

Remark 6.16. *We want to point out that, if we tried to prove a subsolution viscosity property at the boundary in the general case (when the boundary is not necessarily absorbing) for the upper semicontinuous envelope of the value function, we would be in trouble. Indeed, denoting by V^* this upper semicontinuous envelope, we cannot apply the Dynamic Programming Principle to V^* starting from $x_M \in Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$. Then a possible technique to proceed consists in working on a sequence $(x_n) \subset \mathcal{V}$ approximating x_M, applying the Dynamic Programming Principle starting from these points x_n and passing to the limit. The trouble for this approach consists in the fact that the term f in the equation is not continuous with respect to $\|\cdot\|_H$, whereas the upper semicontinuous envelope is defined with respect to this norm. The same problem would come up if we tried to prove a supersolution viscosity property for the lower semicontinuous envelope. Therefore we need to have semicontinuity properties for the value function at the boundary, in order to be able to apply the Dynamic Programming Principle directly on these points to get viscosity properties there. This consideration motivates the choice to work with an absorbing boundary, in order to get the continuity for the value function at the boundary (Proposition 5.7). Finally our opinion is that the lower semicontinuity of the value function at the boundary holds also in the case when the boundary is not absorbing, but of course the proof of this fact would require a more subtle argument.* #

7 Conclusions

We have investigated a problem strictly related to that studied in [13]. In the context of [13] the surplus term does not appear, so that the problem is one-dimensional without delay. In our context the delay term makes the problem considerably more difficult and an infinite-dimensional approach seems to be necessary in order to make the problem markovian and apply the dynamic programming techniques. The main features of this work are the rewriting of the problem in an infinite-dimensional setting, the proof of the equivalence between the delay one-dimensional problem and the abstract infinite-dimensional one, the proof of the continuity of the value function in the infinite-dimensional setting and the proof that the value function is a constrained viscosity solution (in the sense given in Definition 6.13) of the associated infinite-dimensional HJB equation. The investigation leaves open several topics, such as whether the given definition of constrained viscosity solution is strong enough to guarantee the uniqueness or not, the regularity properties of the value function and a verification theorem giving optimal feedback strategies for the problem. The nearest result seems to be that concerned the uniqueness; however all these topics seem to be not addressable with the standard techniques until now available, so that new techniques seem to be necessary. In conclusion the problem seems to be interesting from a financial point of view and seems to open new interesting questions in the theory of optimal stochastic control in infinite dimension.

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A Appendix

Proof of Proposition 3.15-(4): (i) Let us suppose U integrable in l^+; consider again the null strategy $\theta(\cdot) \equiv 0$, the corresponding state trajectory $x(\cdot)$ and set $\varepsilon = \frac{rl-q-f_0(k)}{2}$; moreover let $\delta > 0$ such that $f_0(k+\delta) < f_0(k) + \varepsilon$; until $l \le x(t) \le l + \delta$, the dynamics of $x(t)$ is given by

$$\begin{aligned} dx(t) &= [rx(t) - q]\, dt - f_0(x(t) - \kappa\, x(t-T))dt \\ &\ge [rl - q]\, dt - f_0(l + \delta - \kappa\, l_0)dt \\ &\ge \varepsilon dt; \end{aligned}$$

so, until $x(t) \leq l+\delta$, we have $x(t) \geq l+\varepsilon(t-s)$ and then $x(t)$ remains up to $l+\delta$. Let $t_0 = \inf\{t \geq s \mid x(t) \geq l+\delta\}$ (of course $t_0 < +\infty$); we can write

$$\begin{aligned} J(\eta;0) &= \int_T^{+\infty} e^{-\rho(t-T)} U(x(t))dt \\ &\geq \int_T^{t_0} e^{-\rho(t-T)} U(l+\varepsilon(t-s))dt + \int_{t_0}^{+\infty} e^{-\rho(t-T)} U(x(t))dt; \end{aligned}$$

the finiteness of the second part of the objective functional (the part $\int_{t_0}^{+\infty}$) is obvious, since there we have $x(t) \geq l+\delta$, so that $\int_{t_0}^{+\infty} e^{-\rho(t-T)} U(x(t))dt \geq U(l+\delta)/\rho$; for the first one we have

$$\int_T^{t_0} e^{-\rho(t-T)} U(x(t))dt \geq \int_T^{t_0} e^{-\rho(t-T)} U(l+\varepsilon(t-T))dt;$$

by the integrability of U and by the change of variable $\xi = l+\varepsilon(t-T)$, we get the finiteness also for this term.

(ii) Let $\theta(\cdot) \in \Theta_{ad}(\eta)$ and $x(t) := x(t;T,\eta,\theta(\cdot))$; we will prove that

$$\mathbb{E}\left[\int_T^{T+1} e^{-\rho(t-T)} U(x(t))dt\right] = -\infty,$$

that is enough to prove the claim. Let us consider the problem in the interval $[T,T+1]$,

$$\begin{cases} dx(t) = [(r+\sigma\lambda\theta(t))x(t)-q]\,dt - f_0(x(t)-\kappa x(t-T))dt + \sigma\theta(t)x(t)dB(t), \\ x(T) = \eta_0 = l, \ x(T+\zeta) = \eta_1(\zeta), \qquad \zeta \in [-T,0); \end{cases}$$

and compare the previous one with the following one:

$$\begin{cases} dy(t) = (r+\sigma\lambda\theta(t))y(t)dt + \sigma\theta(t)y(t)dB(t), \\ y(T) = l; \end{cases} \tag{50}$$

of course, by comparison criterion, if $\theta(\cdot)$ is an admissible control for the problem, then $l \leq x(t;T,\eta,\theta(\cdot)) \leq y(t;T,\eta,\theta(\cdot))$ for $T \leq t \leq T+1$. So

$$\mathbb{E}\left[\int_T^{T+1} e^{-\rho(t-T)} U(x(t))dt\right] \leq \mathbb{E}\left[\int_T^{T+1} e^{-\rho(t-T)} U(y(t))dt\right];$$

by Jensen's inequality

$$\mathbb{E}\left[\int_T^{T+1} e^{-\rho(t-T)} U(y(t))dt\right] \leq \int_T^{T+1} e^{-\rho(t-T)} U(\mathbb{E}[y(t)])dt;$$

but $x(t) \geq l > 0$, so that, passing to the expectations in (50) and considering that $0 \leq \theta(t) \leq 1$, we get

$$\mathbb{E}[y(t)] \leq \left(l e^{(r+\sigma\lambda)(t-T)}\right).$$

In definitive

$$\mathbb{E}\left[\int_T^{T+1} e^{-\rho(t-T)} U(x(t))dt\right] \leq \int_T^{T+1} e^{-\rho(t-T)} U\left(l e^{(r+\sigma\lambda)(t-T)}\right) dt;$$

by the change of variable $\xi = l e^{(r+\sigma\lambda)(t-T)}$, we get the claim. #

Proof of Proposition 3.17: Fix $\eta, \eta' \in \mathcal{C}$; set also $\eta_\gamma := \gamma\eta + (1-\gamma)\eta'$, $\gamma \in [0,1]$; of course $\eta_\gamma \in \mathcal{C}$. We have to prove that

$$V(\eta_\gamma) \geq \gamma V(\eta) + (1-\gamma)V(\eta'), \tag{51}$$

with the convention $0 \cdot \infty = 0$. If $V(\eta) = -\infty$ or $V(\eta') = -\infty$, we have nothing to prove. So let us suppose $V(\eta), V(\eta') > -\infty$ and take $\theta(\cdot) \in \Theta_{ad}(\eta)$ and $\theta'(\cdot) \in \Theta_{ad}(\eta')$ ε-optimal for η, η' respectively and $x(\cdot), x'(\cdot)$ the corresponding state trajectories. Then

$$\begin{aligned}
\gamma V(\eta) + (1-\gamma)V(\eta') &< \gamma\big[J(\eta;\theta(\cdot)) + \varepsilon\big] + (1-\gamma)\big[J(\eta';\theta'(\cdot)) + \varepsilon\big] \\
&= \varepsilon + \gamma J(\eta;\theta(\cdot)) + (1-\gamma)J(\eta';\theta'(\cdot)) \\
&= \varepsilon + \gamma\mathbb{E}\left[\int_T^{+\infty} e^{-\rho(t-T)}U\big(x(t)\big)dt\right] + (1-\gamma)\mathbb{E}\left[\int_T^{+\infty} e^{-\rho(t-T)}U\big(x'(t)\big)dt\right] \\
&= \varepsilon + \mathbb{E}\left[\int_T^{+\infty} e^{-\rho(t-T)}\left[\gamma U\big(x(t)\big) + (1-\gamma)U\big(x'(t)\big)\right]dt\right].
\end{aligned}$$

The concavity of U implies that

$$\gamma U\big(x(t)\big) + (1-\gamma)U\big(x'(t)\big) < U\big(\gamma x(t) + (1-\gamma)x'(t)\big), \qquad \forall t \geq T.$$

Consequently, if we set $x_\gamma(\cdot) := \gamma x(\cdot) + (1-\gamma)x'(\cdot)$, we get

$$\gamma V(\eta) + (1-\gamma)\,V(\eta') < \varepsilon + \mathbb{E}\left[\int_T^{+\infty} e^{-\rho(t-T)}U\big(x_\gamma(t)\big)dt\right].$$

If there exists $\theta_\gamma(\cdot) \in \Theta(\eta_\gamma)$ such that $x_\gamma(\cdot) \leq x(\cdot;T,\eta_\gamma,\theta_\gamma(\cdot))$, then we would have

$$\varepsilon + \mathbb{E}\left[\int_T^{+\infty} e^{-\rho(t-T)}U\big(x_\gamma(t)\big)dt\right] \leq \varepsilon + J(\eta_\gamma;\theta_\gamma(\cdot)) \leq \varepsilon + V(\eta_\gamma),$$

i.e.

$$\gamma V(\eta) + (1-\gamma)V(\eta') < \varepsilon + V(\eta_\gamma)$$

and therefore, by the arbitrariness of ε, the claim (51) would be proved. We will show that $\theta_\gamma(t) := a(t)\theta(t) + d(t)\theta'(t)$, where $a(\cdot) = \gamma\frac{x(\cdot)}{x_\gamma(\cdot)}$ and $d(\cdot) = (1-\gamma)\frac{x'(\cdot)}{x_\gamma(\cdot)}$ is good; the admissibility of θ_γ is clear since:

(i) for every $t \geq T$ we have $\theta(t), \theta'(t) \in [0,1]$, and $a(t) + d(t) = 1$ so by convexity of $[0,1]$ we get $\theta_\gamma(t) \in [0,1]$;

(ii) by construction $x_\gamma(t) \geq l$ for any $t \geq s$, so that it will have to be also $x_\gamma(t;T,\eta_\gamma,\theta_\gamma(\cdot)) \geq l$.

We can write for the dynamics of x_γ

$$\begin{aligned}
dx_\gamma(t) &= \gamma\,dx(t) + (1-\gamma)\,dx'(t) \\
&= \gamma\big[\,[(r+\sigma\lambda\theta(t))x(t) - q]\,dt - f_0(x(t) - \kappa x(t-T))dt + \theta(t)\sigma x(t)dB(t)\big] \\
&\quad +(1-\gamma)\big[\,[(r+\sigma\lambda\theta'(t))x'(t) - q]\,dt - f_0(x'(t) - \kappa x'(t-T))dt + \theta'(t)\sigma x'(t)dB(t)\big] \\
&= \big[rx_\gamma(t) - q + [\gamma\theta(t)x(t) + (1-\gamma)\theta'(t)x'(t)]\,\big]dt + \sigma\,[\gamma\theta(t)x(t) + (1-\gamma)\theta'(t)x'(t)]\,dB(t) \\
&\quad - [\gamma f_0(x(t) - \kappa x(t-T)) + (1-\gamma)f_0(x'(t) - \kappa x'(t-T))]\,dt \\
&= \left[rx_\gamma(t) - q + \left[\gamma\theta(t)\frac{x(t)}{x_\gamma(t)} + (1-\gamma)\theta'(t)\frac{x'(t)}{x_\gamma(t)}\right]x_\gamma(t)\right]dt \\
&\quad +\sigma\left[\gamma\theta(t)\frac{x(t)}{x_\gamma(t)} + (1-\gamma)\theta'(t)\frac{x'(t)}{x_\gamma(t)}\right]x_\gamma(t)dB(t) \\
&\quad - [\gamma f_0(x(t) - \kappa x(t-T)) + (1-\gamma)f_0(x'(t) - \kappa x'(t-T))]\,dt \\
&\leq [(r+\sigma\lambda\theta_\gamma(t))x_\gamma(t) - q]\,dt - f_0(x_\gamma(t) - \kappa x_\gamma(t-T))dt + \sigma\theta_\gamma(t)x_\gamma(t)dB(t)
\end{aligned}$$

(where the inequality follows by the convexity of the function f_0), with initial condition

$$x_\gamma(T) = (\eta_\gamma)_0;\ x_\gamma(T+\zeta) = (\eta_\gamma)_1(\zeta), \qquad \zeta \in [-T, 0).$$

Instead by definition $x_\gamma(t; T, \eta_\gamma, \theta_\gamma(\cdot))$ satisfies the previous one with the equality, so that by comparison criterion $x(t; s, \eta_\gamma, \theta_\gamma(\cdot)) \geq x_\gamma(t)$ and the claim follows. #

Proof of Lemma 4.7: Let $X \in C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big)$; for $a \leq t \leq b$, taking into account Lemma 4.5 and the equality $I_{[0,+\infty)}(t+\zeta-\tau) = I_{[\tau,+\infty)}(t+\zeta)$, we have, for a.e. $\zeta \in [-T, 0]$, the following equalities:

$$\begin{aligned}
\begin{pmatrix} \gamma_\theta(X)_0(t) \\ \gamma_\theta(X)_1(t)(\zeta) \end{pmatrix} &= \int_a^t \theta(\tau) S(t-\tau)\left[\Phi X(\tau)\right] d\tau = \int_a^t \theta(\tau) S(t-\tau)\left(X_0(\tau), 0\right) d\tau \\
&= \int_a^t \begin{pmatrix} \theta(\tau) X_0(\tau) e^{r(t-\tau)} \\ I_{[\tau,+\infty)}(t+\zeta)\theta(\tau) X_0(\tau) e^{r(t+\zeta-\tau)} \end{pmatrix} d\tau \\
&= \begin{pmatrix} \int_a^t \theta(\tau) X_0(\tau) e^{r(t-\tau)} d\tau \\ \\ \begin{cases} 0, & \text{if } t+\zeta \leq a \\ e^{r\zeta} \int_a^{t+\zeta} \theta(\tau) X_0(\tau) e^{r(t-\tau)} d\tau, & \text{if } t+\zeta \geq a \end{cases} \end{pmatrix};
\end{aligned}$$

this shows that, for any $t \in [a,b]$ and $X \in C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big)$, we can take the random variable $\gamma_\theta(X)(t)$ taking values in E. Now we have to show that $\gamma_\theta(X)(t) \in L^2(\Omega; E)$. Taking into account the Holder's inequality, we have

$$\begin{aligned}
\mathbb{E}\left[\|\gamma_\theta(X)(t)\|_E^2\right] &\leq 2\mathbb{E}\left[\left|\int_a^t \theta(\tau) X_0(\tau) e^{r(t-\tau)} d\tau\right|^2 + \sup_{\zeta \in [-T,0)} \left|\int_a^{t+\zeta} \theta(\tau) X_0(\tau) e^{r(t-\tau)} d\tau\right|^2\right] \\
&\leq 4(b-a) e^{2r(b-a)} \int_a^t \mathbb{E}[|X_0(\tau)|^2] d\tau < +\infty. \qquad (52)
\end{aligned}$$

By an estimate like the previous one we can get, for $t_0, t \in [a,b]$,

$$\mathbb{E}\left[\|\gamma_\theta(X)(t) - \gamma_\theta(X)(t_0)\|_E^2\right] \leq 4(b-a) e^{2r(b-a)} \int_{t \wedge t_0}^{t \vee t_0} \mathbb{E}[|X_0(\tau)|^2] d\tau.$$

getting $\gamma_\theta(X) \in C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big)$, by mean-square continuity of $X_0(\cdot)$. The last statement follows arguing as in the estimate (52), but taking the supremum on $t \in [a,b]$ before to pass to the expectations. #

Proof of Lemma 4.8: Let $X \in C_{\mathcal{P}}\big([a,b]; L^2(\Omega; E)\big)$; for $a \leq t \leq b$, taking into account Lemma 4.5 and the equality $I_{[0,+\infty)}(t+\zeta-\tau) = I_{[\tau,+\infty)}(t+\zeta)$, we have, for a.e. $\zeta \in [-T, 0]$, the following equalities:

$$\begin{aligned}
\begin{pmatrix} \gamma(X)_0(t) \\ \gamma(X)_1(t)(\zeta) \end{pmatrix} &= \int_a^t S(t-\tau) G(X(\tau)) d\tau = \int_a^t S(t-\tau)\left(g(X(\tau)), 0\right) d\tau \\
&= \int_a^t \begin{pmatrix} g(X(\tau)) e^{r(t-\tau)} \\ I_{[\tau,+\infty)}(t+\zeta) g(X(\tau))) e^{r(t+\zeta-\tau)} \end{pmatrix} d\tau \\
&= \begin{pmatrix} \int_a^t g(X(\tau)) e^{r(t-\tau)} d\tau \\ \\ \begin{cases} 0, & \text{if } t+\zeta \leq a \\ e^{r\zeta} \int_a^{t+\zeta} g(X(\tau)) e^{r(t-\tau)} d\tau, & \text{if } t+\zeta \geq a \end{cases} \end{pmatrix};
\end{aligned}$$

this shows that, for any $t \in [a,b]$ and $X \in C_{\mathcal{P}}\big([a,b];L^2(\Omega;E)\big)$, we can take the random variable $\gamma(X)(t)$ taking values in E. Now we have to show that, for any $t \in [a,b]$, $\gamma(X)(t) \in L^2(\Omega;E)$; indeed we have

$$\begin{aligned}
\mathbb{E}\left[\|\gamma(X)(t)\|_E^2\right] &\leq 2\mathbb{E}\left[\left|\int_a^t g(X(\tau))e^{r(t-\tau)}d\tau\right|^2 + \sup_{\zeta\in[-T,0)}\left|\int_a^{t+\zeta} g(X(\tau))e^{r(t-\tau)}d\tau\right|^2\right] \\
&\leq 8e^{2r(b-a)}(b-a)\mathbb{E}\left[\int_a^t \left(C_g^2\|X(\tau)\|_E^2 + |g(0)|^2\right)d\tau\right] < +\infty.
\end{aligned}$$

We can prove that $\gamma(X) \in C_{\mathcal{P}}\big([a,b];L^2(\Omega;E)\big)$ arguing as in the proof of the previous Lemma and the last statement arguing as in the previous estimate, but taking the supremum on $t \in [a,b]$ before to pass to the expectations. #

Proof of Lemma 4.9: Let $X \in C_{\mathcal{P}}\big([a,b];L^2(\Omega;E)\big)$; for $a \leq t \leq b$, taking into account Lemma 4.5 and the equality $I_{[0,+\infty)}(t+\zeta-\tau) = I_{[\tau,+\infty)}(t+\zeta)$, we have, for a.e. $\zeta \in [-T,0]$, the following equalities:

$$\begin{aligned}
\begin{pmatrix} \tilde{\gamma}_\theta(X)_0(t) \\ \tilde{\gamma}_\theta(X)_1(t)(\zeta) \end{pmatrix} &= \int_a^t \theta(\tau)S(t-\tau)\left[\Phi X(\tau)\right]dB(\tau) = \int_a^t S(t-\tau)\left(\theta(\tau)X_0(\tau),0\right)dB(\tau) \\
&= \int_a^t \begin{pmatrix} \theta(\tau)X_0(\tau)e^{r(t-\tau)} \\ I_{[\tau,+\infty)}(t+\zeta)\theta(\tau)X_0(\tau)e^{r(t+\zeta-\tau)} \end{pmatrix} dB(\tau) \\
&= \begin{pmatrix} \int_a^t \theta(\tau)X_0(\tau)e^{r(t-\tau)}dB(\tau) \\ \\ \begin{cases} 0, & \text{if } t+\zeta \leq a \\ e^{r\zeta}\int_a^{t+\zeta}\theta(\tau)X_0(\tau)e^{r(t-\tau)}dB(\tau), & \text{if } t+\zeta \geq a \end{cases} \end{pmatrix};
\end{aligned}$$

this shows that, for any $t \in [a,b]$ and $X \in C_{\mathcal{P}}\big([a,b];L^2(\Omega;E)\big)$, we can take the random variable $\tilde{\gamma}_\theta(X)(t)$ taking values in E. Now we have to show that $\tilde{\gamma}_\theta(X)(t) \in L^2(\Omega;E)$. Taking into account the Doob's inequality for continuous square-integrable martingales and the Ito's isometry, we have

$$\begin{aligned}
\mathbb{E}\left[\|\tilde{\gamma}_\theta(X)(t)\|_E^2\right] &\leq 2\mathbb{E}\left[\left|\int_a^t \theta(\tau)X_0(\tau)e^{r(t-\tau)}dB(\tau)\right|^2 + \sup_{\zeta\in[-T,0)}\left|\int_a^{t+\zeta}\theta(\tau)X_0(\tau)e^{r(t-\tau)}dB(\tau)\right|^2\right] \\
&\leq 10\,\mathbb{E}\left[\left|\int_a^t \theta(\tau)X_0(\tau)e^{r(t-\tau)}dB(\tau)\right|^2\right] \leq 10\,e^{2r(b-a)}\int_a^t \mathbb{E}[|X_0(\tau)|^2]d\tau < +\infty.
\end{aligned}$$

We can prove that $\tilde{\gamma}_\theta(X) \in C_{\mathcal{P}}\big([a,b];L^2(\Omega;E)\big)$ and the last statement arguing as in Lemma 4.7. #

Proof of Proposition 4.13: Let $x(\cdot)$ be the solution of (28) and let $X(\cdot) := (x(\cdot), x(\cdot+\zeta)|_{\zeta\in[-T,0]})$. Then X belongs to the space $C_{\mathcal{P}}([T,+\infty);L^2(\Omega;E))$ because the function $[T,+\infty) \to L^2(\Omega;\mathbb{R})$, $t \mapsto x(t)$ is continuous and therefore uniformly continuous on the compact subsets of $[T,+\infty)$. So we have to prove that $X(t) = (X_0(t), X_1(t)) = (x(t), x(t+\zeta)|_{\zeta\in[-T,0]})$ satisfies (22) on both the components. For the first one we have to verify that, for any $t \geq T$,

$$\begin{aligned}
X_0(t) &= e^{r(t-T)}x_0 + \sigma\lambda\int_T^t e^{r(t-\tau)}\theta(\tau)X_0(\tau)d\tau \\
&\quad -\int_T^t e^{r(t-\tau)}\left[f_0\left(X_0(\tau) - \kappa X_1(\tau)(-T)\right) + q\right]d\tau \\
&\quad +\sigma\int_T^t e^{r(t-\tau)}\theta(\tau)X_0(\tau)dB(\tau),
\end{aligned}$$

i.e. that

$$\begin{aligned} x(t) &= e^{r(t-T)}x_0 + \sigma\lambda \int_T^t e^{r(t-\tau)}\theta(\tau)x(\tau)d\tau \\ &\quad - \int_T^t e^{r(t-\tau)} \left[f_0\left(x(\tau) - \kappa x(\tau - T)\right) + q\right] d\tau \\ &\quad + \sigma \int_T^t e^{r(t-\tau)}\theta(\tau)x(\tau)dB(\tau); \end{aligned}$$

but this comes from the assumption that $x(\cdot)$ is a solution of (28).

For the second component, taking into account that $I_{[0,+\infty)}(t + \cdot - \tau) = I_{[\tau,+\infty)}(t + \ \cdot)$ and taking into account Lemma 4.4 and Lemma 4.5, we have to verify, for any $t \geq T$, the equalities in $L^2(\Omega;\mathbb{R})$ for a.e. $\zeta \in [-T, 0]$

$$\begin{aligned} X_1(t)(\zeta) &= I_{[0,T]}(t+\zeta)x_1(t+\zeta-T) + I_{[T,+\infty)}(t+\zeta)x_0 e^{r(t+\zeta-T)} \\ &\quad + \sigma\lambda \int_T^t I_{[\tau,+\infty)}(t+\zeta)\theta(\tau)X_0(\tau)e^{r(t+\zeta-\tau)}d\tau \\ &\quad - \int_T^t I_{[\tau,+\infty)}(t+\zeta)\left[f_0\left(X_0(\tau) - \kappa X_1(\tau)(-T)\right) + q\right] e^{r(t+\zeta-\tau)}d\tau \\ &\quad + \sigma \int_T^t I_{[\tau,+\infty)}(t+\zeta)\theta(\tau)X_0(\tau)e^{r(t+\zeta-\tau)}dB(\tau); \end{aligned}$$

i.e., for any $t \geq T$, the equalities in $L^2(\Omega, \mathbb{R})$ for a.e. $\zeta \in [-T, 0]$

$$\begin{aligned} x(t+\zeta) &= I_{[0,T]}(t+\zeta)x_1(t+\zeta-T) + I_{[T,+\infty)}(t+\zeta)x_0 e^{r(t+\zeta-T)} \\ &\quad + \sigma\lambda \int_T^t I_{[\tau,+\infty)}(t+\zeta)\theta(\tau)x(\tau)e^{r(t+\zeta-\tau)}d\tau \\ &\quad - \int_T^t I_{[\tau,+\infty)}(t+\zeta)\left[f_0\left(x(\tau) - \kappa x(\tau-T)\right) + q\right] e^{r(t+\zeta-\tau)}d\tau \\ &\quad + \sigma \int_T^t I_{[\tau,+\infty)}(t+\zeta)\theta(\tau)x(\tau)e^{r(t+\zeta-\tau)}dB(\tau). \end{aligned} \tag{53}$$

For $\zeta \in [-T, 0]$ such that $t + \zeta \in [0, T]$, (53) reduces to

$$x(t+\zeta) = x_1(t+\zeta-T)$$

and this is true by the initial condition of (28); instead for $\zeta \in [-T, 0]$ such that $t + \zeta \geq T$, (53) reduces to

$$\begin{aligned} x(t+\zeta) &= x_0 e^{r(t+\zeta-T)} + \sigma\lambda \int_T^{t+\zeta} \theta(\tau)x(\tau)e^{r(t+\zeta-\tau)}d\tau \\ &\quad - \int_T^{t+\zeta} \left[f_0\left(x(\tau) - \kappa x(\tau-T)\right) + q\right] e^{r(t+\zeta-\tau)}d\tau \\ &\quad + \sigma \int_T^{t+\zeta} \theta(\tau)x(\tau)e^{r(t+\zeta-\tau)}dB(\tau); \end{aligned}$$

setting $u := t + \zeta$ this equality becomes, for $u \geq T$,

$$\begin{aligned} x(u) &= x_0 e^{r(u-T)} + \sigma\lambda \int_T^u \theta(\tau)x(\tau)e^{r(u-\tau)}d\tau \\ &\quad - \int_T^u \left[f_0\left(x(\tau) - \kappa x(\tau-T)\right) + q\right] e^{r(u-\tau)}d\tau \\ &\quad + \sigma \int_T^u \theta(\tau)x(\tau)e^{r(u-\tau)}dB(\tau); \end{aligned}$$

again this is true because $x(\cdot)$ solves (28). #

Proof of Proposition 4.14: Notice that we cannot proceed as in Proposition 4.14 because the function F is not Lipschitz continuous with respect to the H-norm. Let $T \le a < b$ such that $b - a \le T$ and γ the map defined by

$$\begin{array}{rccc} \gamma: & C([a,b];(E,\|\cdot\|_H)) & \longrightarrow & C([a,b];(E,\|\cdot\|_H) \\ & X(\cdot) & \longmapsto & \int_a^{\cdot} S(\cdot-\tau)F(X(\tau))d\tau. \end{array}$$

As in the proof of Lemma 4.8, for $t \in [a,b]$,

$$\begin{pmatrix} \gamma(X)_0(t) \\ \gamma(X)_1(t)(\zeta) \end{pmatrix} = \begin{pmatrix} \int_a^t f(X(\tau))e^{r(t-\tau)}d\tau \\ \begin{cases} 0, & \text{if } t+\zeta \le a, \\ e^{r\zeta}\int_a^{t+\zeta} f(X(\tau))e^{r(t-\tau)}d\tau, & \text{if } t+\zeta \ge a; \end{cases} \end{pmatrix};$$

let K_0 be the Lipschitz constant of f_0; for generic $X, Y \in C([a,b];(E,\|\cdot\|_H))$ we have the following estimate with respect to the H-norm:

$$\begin{aligned} \sup_{t\in[a,b]} \|\gamma(X)(t)-\gamma(Y)(t)\|_H^2 &= \sup_{t\in[a,b]} \Bigg[\left| \int_a^t e^{r(t-\tau)} \left(f(X(\tau)) - f(Y(\tau))\right) d\tau \right|^2 \\ &+ \int_{(a-t)}^0 \left| e^{r\zeta} \int_a^{t+\zeta} e^{r(t-\tau)} \left(f(X(\tau))) - f(Y(\tau))\right) d\tau \right|^2 d\zeta \Bigg] \\ &\le \sup_{t\in[a,b]} \Bigg[(t-a) \int_a^t e^{2r(t-\tau)} \left|f(X(\tau)) - f(Y(\tau))\right|^2 d\tau \\ &+ \int_{(a-t)}^0 \left[e^{2r\zeta}(t+\zeta-a) \int_a^{t+\zeta} e^{2r(t-\tau)} \left|f(X(\tau)) - f(Y(\tau))\right|^2 d\tau \right] d\zeta \Bigg] \\ &\le 2(b-a)e^{2r(b-a)}K_0^2 \int_a^b \left[|X_0(\tau)-Y_0(\tau)|^2 + |X_1(\tau)(-T) - Y_1(\tau)(-T)|^2 \right] d\tau \\ &+ 2T(b-a)e^{2r(b-a)}K_0^2 \int_a^b \left[|X_0(\tau)-Y_0(\tau)|^2 + |X_1(\tau)(-T) - Y_1(\tau)(-T)|^2 \right] d\tau \\ &= 2(b-a)e^{2r(b-a)}(1+T)K_0^2 \int_a^b \left[|X_0(\tau)-Y_0(\tau)|^2 + |X_1(\tau)(-T) - Y_1(\tau)(-T)|^2 \right] d\tau. \end{aligned}$$

Now, if we set $X := X(x)$, $Y := X(y)$, taking into account that $-T \le \tau - a \le b - a - T \le 0$, by Proposition 4.13 it results $X_1(\tau)(-T) = X_1(a)(\tau - a - T)$ and $Y_1(\tau)(-T) = Y_1(a)(\tau - a - T)$, so that we get

$$\begin{aligned} \sup_{t\in[a,b]} \|\gamma(X)(t)-\gamma(Y)(t)\|_H^2 &\le 2(b-a)^2e^{2r(b-a)}(1+T)K_0^2\|X-Y\|^2_{C([a,b];H)} \\ &\quad +2(b-a)e^{2r(b-a)}(1+T)\,K_0^2\,\|X(a)-Y(a)\|_H^2 \\ &\le 2(b-a)^2e^{2r(b-a)}(1+T)\,K_0^2\,\|X-Y\|^2_{C([a,b];H)} \\ &\quad +2(b-a)e^{2r(b-a)}(1+T)\,K_0^2\,\|X-Y\|^2_{C([a,b];H)}; \end{aligned}$$

thus, for small enough $b - a$,

$$\|\gamma(X)-\gamma(Y)\|_{C([a,b];(E,\|\cdot\|_H))} \le C\|X-Y\|_{C([a,b];(E,\|\cdot\|_H))}$$

for some $0 < C < 1$. Now the claim follows arguing as in the proof of Proposition 4.14, taking into account that, if χ is the map defined in Lemma 4.6 and $\psi, \psi' \in E$, then by (23)

$$\|\chi(\psi-\psi')\|_{C([a,b];(E,\|\cdot\|_H))} \le (3+2T)^{1/2}e^{r(b-a)}\|\psi-\psi'\|_H.$$

#

Proof of Proposition 5.4: 1. Let $x \in \mathcal{V}$; we know that V is continuous at x in E, so that, by concavity, it is Lipschitz continuous on $B_{(E,\|\cdot\|_H)}(x,\rho_x) = \{y \in E \mid \|y-x\|_H < \rho_x\} \subset \mathcal{V}$ for suitable $\rho_x > 0$ (see again Corollary 2.4, Chapter 1, of [16]). $B_{(E,\|\cdot\|_H)}(x,\rho_x)$ is dense in $B_{(H,\|\cdot\|_H)}(x,\rho_x)$, since $B_{(E,\|\cdot\|_H)}(x,\rho_x) = B_{(H,\|\cdot\|_H)}(x,\rho_x) \cap E$. Thus we can extend by continuity V to a continuous function V_x on $B_{(H,\|\cdot\|_H)}(x,\rho_x) = \{y \in H \mid \|y-x\|_H < \rho_x\}$. Of course we can repeat this construction for all the points of $\mathcal{V}$. These extensions are compatible each other, in the sense that, if $x, x' \in \mathcal{V}$ and $y \in B_{(H,\|\cdot\|_H)}(x,\rho_x) \cap B_{(H,\|\cdot\|_H)}(x',\rho_{x'})$, then $V_x(y) = V_{x'}(y)$. Indeed, by density, we can take a sequence $(x_n) \subset B_{(E,\|\cdot\|_H)}(x,\rho_x) \cap B_{(E,\|\cdot\|_H)}(x',\rho_{x'})$ such that $x_n \to y$; on this sequence it results $V_x(x_n) = V(x_n) = V_{x'}(x_n)$, therefore, taking the limit for $n \to \infty$, we get $V_x(y) = V_{x'}(y)$ by continuity of $V_x, V_{x'}$. Let us define the open set $\mathcal{O}$ of $(H, \|\cdot\|_H)$ by

$$\mathcal{O} := \bigcup_{x \in \mathcal{V}} B_{(H,\|\cdot\|_H)}(x,\rho_x); \tag{54}$$

thanks to the compatibility argument it remains defined on $\mathcal{O}$ a continuous function $\bar{V}$. Of course

$$\mathcal{V} = \bigcup_{x \in \mathcal{V}} B_{(E,\|\cdot\|_H)}(x,\rho_x) \subset \mathcal{O} \tag{55}$$

and, by construction, $\bar{V}|_{\mathcal{V}} = V$.

2. Let $x \in \mathcal{V}$; then, of course $x \in \mathcal{O}$ and $x \in E$, so that $\mathcal{V} \subset \mathcal{O} \cap E$. Conversely let $x \in \mathcal{O} \cap E$; then, since $x \in \mathcal{O}$, from (54) we have $x \in B_{(H,\|\cdot\|_H)}(z,\rho_z)$ for some $z \in \mathcal{V}$ and, on the other hand, since $x \in E$, we have $x \in B_{(H,\|\cdot\|_H)}(z,\rho_z) \cap E = B_{(E,\|\cdot\|_H)}(z,\rho_z)$, so that, by (55), we have $x \in \mathcal{V}$ and therefore we can conclude that also $\mathcal{O} \cap E \subset \mathcal{V}$.

About the second statement, thanks to the fact that E is dense in H and that $\mathcal{O}$ is open in $(H, \|\cdot\|_H)$, we can write

$$Int_{(H,\|\cdot\|_H)}\left(Clos_{(H,\|\cdot\|_H)}(\mathcal{V})\right) = Int_{(H,\|\cdot\|_H)}\left(Clos_{(H,\|\cdot\|_H)}(\mathcal{O} \cap E)\right) = Int_{(H,\|\cdot\|_H)}\left(Clos_{(H,\|\cdot\|_H)}(\mathcal{O})\right) = \mathcal{O}. \tag{56}$$

3. The convexity of $\mathcal{O}$ follows by (56) and by the fact that $\mathcal{V}$ is convex. The concavity of $\bar{V}$ follows by its continuity and by the concavity of V on $\mathcal{V}$. #

Proof of Lemma 5.5: 1. By Proposition 5.4-(2) we know that $\mathcal{V} = \mathcal{O} \cap E$. Thus we can write

$$(H \setminus \mathcal{O}) \cap E = (H \cap E) \setminus (\mathcal{O} \cap E) = E \setminus \mathcal{V}.$$

2. Let $x \in Clos_{(E,\|\cdot\|_H)}(\mathcal{V})$; then we can find a sequence $(x_n) \subset \mathcal{V}$ such that $x_n \stackrel{\|\cdot\|_H}{\longrightarrow} x$; of course $(x_n) \subset \mathcal{O}$, so that $x \in Clos_{(H,\|\cdot\|_H)}(\mathcal{O})$; on the other hand $x \in E$, since $x \in Clos_{(E,\|\cdot\|_H)}(\mathcal{V})$, so that $x \in Clos_{(H,\|\cdot\|_H)}(\mathcal{O}) \cap E$.

Conversely, let $x \in Clos_{(H,\|\cdot\|_H)}(\mathcal{O}) \cap E$; we know (see Proposition 5.4) that $Clos_{(H,\|\cdot\|_H)}(\mathcal{O}) = Clos_{(H,\|\cdot\|_H)}(\mathcal{V})$, thus there exists a sequence $(x_n) \subset \mathcal{V}$ such that $x_n \stackrel{\|\cdot\|_H}{\longrightarrow} x$; together with the assumption $x \in E$, this shows that $x \in Clos_{(E,\|\cdot\|_H)}(\mathcal{V})$.

3. Notice that $E \setminus \mathcal{V}$ and $H \setminus \mathcal{O}$ are closed respectively in $(E, \|\cdot\|_H)$ and $(H, \|\cdot\|_H)$. Let $x \in Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$; this means that $x \in Clos_{(E,\|\cdot\|_H)}(\mathcal{V}) \cap (E \setminus \mathcal{V})$. Thanks to the point (2) of this proposition we get $x \in Clos_{(H,\|\cdot\|_H)}(\mathcal{O}) \cap (H \setminus \mathcal{O}) \cap E$, i.e. $x \in Fr_{(H,\|\cdot\|_H)}(\mathcal{O}) \cap E$.

Conversely, let $x \in Fr_{(H,\|\cdot\|_H)}(\mathcal{O}) \cap E$; this means that $x \in \left(Clos_{(H,\|\cdot\|_H)}(\mathcal{O}) \cap E\right) \cap \left((H \setminus \mathcal{O}) \cap E\right)$, so that, by the point (2), $x \in Clos_{(E,\|\cdot\|_H)}(\mathcal{V}) \cap (E \setminus \mathcal{V})$, i.e. $x \in Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$. #

Proof of Proposition 5.6: 1. Recall that, by definition, $D(V) = \{x \in \mathcal{C} \mid V(x) > -\infty\}$. By the assumption $U(l) > -\infty$, we get $x \in D(V)$ if and only if $x \in \mathcal{C}$ and $\Theta_{ad}(x) \neq \emptyset$ and, thanks to Lemma 3.4, this occurs if and only if $x \in \mathcal{C}$ and $0 \in \Theta_{ad}(x)$, so that $D(V) = \{x \in \mathcal{C} \mid 0 \in \Theta_{ad}(x)\}$.

Since by definition $\mathcal{V} = Int_{(E,\|\cdot\|_H)}(D(V))$, we can say that $\{x \in \mathcal{C} \mid 0 \in \Theta_{ad}(x)\} = D(V) \subset Clos_{(E,\|\cdot\|_H)}(\mathcal{V})$, so that it remains to prove the inclusion $Clos_{(E,\|\cdot\|_H)}(\mathcal{V}) \subset D(V) = \{x \in \mathcal{C} \mid 0 \in \Theta_{ad}(x)\}$. So let us take $x \in Clos_{(E,\|\cdot\|_H)}(\mathcal{V}) = \mathcal{V} \cup Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$. If $x \in \mathcal{V}$, then, by definition of $\mathcal{V}$, $x \in D(V)$. So let us suppose $x \in Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$; we want to prove that $x \in \{x \in \mathcal{C} \mid 0 \in \Theta_{ad}(x)\}$, i.e. $X_0(\cdot;T,x,0) \geq l$. Let us suppose, by contradiction, that, for some $t \geq T$, $\varepsilon > 0$, we have $X_0(t;T,x,0) \leq l - \varepsilon$. By definition of $Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$, there exists $y \in \mathcal{V} \subset D(V)$ such that $\|x-y\|_H < \varepsilon/2K_t$, where K_t is the constant in the estimate of Corollary 4.16; so, by the same Corollary, we would have $X_0(t;T,y,0) \leq l - \varepsilon/2$, i.e., by Lemma 3.4, $\Theta_{ad}(y) = \emptyset$ and thus $y \notin D(V)$. Therefore the contradiction arises and so the claim is proved.

2. Let $x \in D(V)$. If $x_0 = l$, then we have $X(T;T,x,0) = l$ and we are in the second case. So let $x \in D(V)$ be such that $x_0 > l$. Of course, since $x_1(\zeta) \to x_0$ when $\zeta \to 0^-$, we can find $\varepsilon > 0$ such that $x_1(\zeta) \geq l$ for $\zeta \in [-\varepsilon, 0)$. Let us suppose that we are not in the second case, i.e. $X_0(t;T,x,0) > 0$ for every $t \in [T,2T)$; then there exists some $\alpha > 0$ such that $X_0(t;T,x,0) \geq l+\alpha$, for $t \in [T, 2T-\varepsilon]$. We want to show that then we are in the first case. By the semigroup property of the mild solution $X(\cdot;T,x,0)$ we have, for $t \geq 2T-\varepsilon$, $X_0(t;T,x,0) = X_0(t;2T-\varepsilon, X(2T-\varepsilon;T,x,0),0)$; since $X_1(2T-\varepsilon;T,x,0)(\cdot) \geq l$ and $X_0(2T-\varepsilon;T,x,0) \geq l+\alpha$, we have $X(2T-\varepsilon;T,x,0) \in \mathcal{D}_0$. Therefore, by Lemma 3.7-(2), there exists $\alpha' > 0$ such that $X_0(t;2T-\varepsilon, X(2T-\varepsilon;T,x,0),0) \geq l+\alpha'$, for all $t \geq 2T-\varepsilon$, so that we get what desired taking $\beta = \alpha \wedge \alpha'$.

About the second part of the statement, notice that, when $X_0(t;T,x,0) = l$, thanks to Proposition 4.13, we have from the delay state equation $dX_0(t;T,x,0) \leq rl - q = 0$. On the other hand $x \in D(V)$, so that we know, by the point (1), that $0 \in \Theta_{ad}(x)$; thus also $dX_0(t;T,x,0) \geq 0$. Therefore we get $dX_0(t;T,x,0) = 0$ and the proof is complete.

3. If $x \in D(V)$ is such that $X_0(\cdot;T,x,0) \geq l+\beta$ for some $\beta > 0$, then, arguing as in Lemma 5.1, we get $x \in \mathcal{V}$. Conversely let $x \in \mathcal{V}$; of course $x \in D(V)$; thus we have to prove that $X_0(t;T,x,0) > 0$ for $t \in [T,2T)$ and then, by the point (2), we get the claim. So let us suppose by contradiction that, for some $s \in [T,2T)$, we have $X_0(s;T,x,0) = l$; since $x \in \mathcal{V}$, there exists $\varepsilon > 0$ such that $B_{(E,\|\cdot\|_H)}(x,\varepsilon) \subset \mathcal{V} \subset D(V) = \{x \in \mathcal{C} \mid 0 \in \Theta_{ad}(x)\}$ and in particular it has to be $x_0 > l$. On the other hand we can choose $y \in B_{(E,\|\cdot\|_H)}(x,\varepsilon)$ such that $y_1(\zeta) = x_1(\zeta)$, for $\zeta \in [-T, s+\frac{2T-s}{2} - 2T]$, and $y_0 < x_0$. Working in the interval $[T, s+\frac{2T-s}{2}]$ we can forget to be concerned with a delay equation and consider the term $x(t-T)$ in the equation as a datum. Thus, thanks to Proposition 4.13, we have for the dynamics of $X_0(\cdot;T,x,0)$ and $X_0(\cdot;T,y,0)$ in the interval $[T, s+\frac{2T-s}{2}]$

$$\begin{cases} dX_0(t;T,x,0)) = [rX_0(t;T,x,0) - q]\,dt - f_0\left(X_0(t;T,x,0) - \kappa x_1(t-2T)\right)dt, \\ X_0(T;T,x,0) = x_0; \end{cases}$$

$$\begin{cases} dX_0(t;T,y,0)) = [rX_0(t;T,y,0) - q]\,dt - f_0\left(X_0(t;T,y,0) - \kappa y_1(t-2T)\right)dt, \\ X_0(T;T,y,0) = y_0. \end{cases}$$

These two dynamics refer to the same ordinary differential equation on $[T, s+\frac{2T-s}{2}]$, since $y_1(\zeta) = x_1(\zeta)$, for $\zeta \in [-T, s+\frac{2T-s}{2} - 2T]$, and this differential equation satisfies the classic hypothesis of the Cauchy's Theorem for ordinary differential equations. Therefore, by uniqueness, since $y_0 < x_0$, the solution starting at y_0 has to stay strictly below the solution starting at x_0; in particular, since $X_0(s;T,x,0) = l$, we get $X_0(s;T,y,0) < l$, i.e. $0 \notin \Theta_{ad}(y)$ and the contradiction arises.

4. If we denote by the symbol $\mathring{\cup}$ the disjoint union, we have $Clos_{(E,\|\cdot\|_H)}(\mathcal{V}) = \mathcal{V} \mathring{\cup} Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$, since $\mathcal{V}$ is open. By the point (2) we can split $D(V)$ as

$$D(V) = \left\{x \in D(V) \mid \exists\, \beta > 0 \text{ s.t. } X_0(\cdot;T,x,0) \geq l+\beta\right\} \mathring{\cup} \left\{x \in D(V) \mid \exists\, s \in [T,2T) \text{ s.t. } X_0(s;T,x,0) = l\right\}.$$

On the other hand, by the point (3), we can say that $\mathcal{V} = \{x \in D(V) \mid \exists\, \beta > 0 \text{ s.t. } X_0(\cdot;T,x,0) \geq l+\beta\}$, so that, by the point (1) it has to be $Fr_{(E,\|\cdot\|_H)}(\mathcal{V}) = \{x \in D(V) \mid \exists\, s \in [T,2T) \text{ s.t. } X_0(s;T,x,0) = l\}$, i.e. the claim.

5. Let us suppose that $x \in Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$. We know, by the point (1), that $0 \in \Theta_{ad}(x)$; moreover, by the point (4), there exists $s \in [T,2T)$ such that $X_0(s;T,x,0) = l$. Let us suppose that

$\theta(\cdot) \in \Theta_{ad}(x)$. Then, arguing as in the proof of Lemma 3.4, we get, using the same notation therein, that $\tilde{\mathbb{E}}_{2T}\left[X_0(s;T,x,\theta(\cdot))\right] \leq X_0(s;T,x,0) = l$. On the other hand, since $\theta(\cdot) \in \Theta_{ad}(x)$, it has to be $X_0(s;T,x,\theta(\cdot)) \geq l$ almost surely; therefore we can say that $X_0(s;T,x,\theta(\cdot)) = l$ almost surely, so that $\mathrm{Var}\left[X_0(s,T,x,\theta(\cdot))\right] = 0$ and this fact can occur only if $\theta(t) \equiv 0$ for $t \in [T,s]$. Afterwards, for $t \geq s$, of course it must be $\theta(t) \equiv 0$. Therefore we have proved that $Fr_{(E,\|\cdot\|_H)}(\mathcal{V}) \subset \{x \in D(V) \mid \Theta_{ad}(x) = \{0\}\}$.

Conversely let us suppose that $x \in D(V)$ is such that $\Theta_{ad}(x) = \{0\}$. If, by contradiction, $x \in \mathcal{V}$, then we can find $\varepsilon > 0$ such that $B_{(E,\|\cdot\|_H)}(x,\varepsilon) \subset \mathcal{V}$. Let us consider the constant strategy $\theta \equiv 1$ and let us define the stopping time $\tau := \inf\{t \geq T \mid X(t;T,x,1) \notin B_{(E,\|\cdot\|_H)}(x,\varepsilon)\}$. The trajectories of $X(\cdot;T,x,1)$ are $\|\cdot\|_E$-continuous and therefore they are also $\|\cdot\|_H$-continuous, so that $\tau > T$ almost surely. Now define the strategy

$$\theta(t) = \begin{cases} 1, & \text{if } t \leq \tau, \\ 0, & \text{if } t > \tau. \end{cases}$$

By definition $X(\tau;T,x,1) \in \mathcal{V}$, so that in particular $0 \in \Theta_{ad}(X(\tau;T,x,1))$; therefore we have $\theta(\cdot) \neq 0$ and $\theta(\cdot) \in \Theta_{ad}(x)$, so that a contradiction arises.

6. Let $t \geq T$; by the point (5), if $x \in Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$, then the only admissible strategy is the null one; therefore it has to be also $\Theta_{ad}(X(t;T,x,0)) = \{0\}$. Applying again the point (5) we get $X(t;T,x,0) \in Fr_{(E,\|\cdot\|_H)}(\mathcal{V})$, i.e. the claim. #

Proof of Lemma 6.3: First of all notice that, in the equation (36), $X(\cdot)$ is a given process. Let $\tilde{S}_n$ be the uniformly continuous semigroups on H genenerated by the bounded operators $\tilde{A}_n$ and $\tilde{S}$ that generated by the unbounded operator $\tilde{A}$.
$\tilde{A}$ is maximal dissipative, so that it generates a contractive semigroup (see Corollary II.3.20 in [18]); therefore, for any $t \geq 0$,

$$\|\tilde{S}(t)\|_{\mathcal{L}(H)} \leq 1. \tag{57}$$

Moreover also the semigroups $\tilde{S}_n$ are contractive (see Theorem II.3.5 again in [18]), so that, for any $t \geq 0$,

$$\|\tilde{S}_n(t)\|_{\mathcal{L}(H)} \leq 1. \tag{58}$$

Moreover, for any $t \geq 0$ and $x \in H$, we have the convergence, for $n \to \infty$,

$$\|\tilde{S}(t)x - \tilde{S}_n(t)x\|_H \to 0. \tag{59}$$

The proof of the existence and uniqueness for the strong solution of (36) follows e.g. by Proposition 6.4 of [9], because of the boundedness of $\tilde{A}_n$. Of course a strong solution is also a mild solution, therefore we can write

$$\begin{aligned} X_n(t) &= \tilde{S}_n(t-T)x + \int_T^t \left[\left(r + \frac{1}{2}\right) + \sigma\lambda\theta(\tau)\right] \tilde{S}_n(t-\tau)\Phi X(\tau)d\tau \\ &\quad - \int_T^t \tilde{S}_n(t-\tau)G(X(\tau))d\tau + \sigma \int_T^t \theta(\tau)\tilde{S}_n(t-\tau)\Phi X(\tau)dB(\tau). \end{aligned}$$

Therefore, setting $K := r + \frac{1}{2} + \sigma(\lambda+1)$,

$$\begin{aligned} \|X(t) - X_n(t)\|_H &\leq \|\tilde{S}(t-T)x - \tilde{S}_n(t-T)x\|_H \\ &\quad + K\int_T^t \|(\tilde{S}(t-\tau) - \tilde{S}_n(t-\tau))\Phi X(\tau)\|_H d\tau \\ &\quad + \int_T^t \|(\tilde{S}(t-\tau) - \tilde{S}_n(t-\tau))\, G(X(\tau))\|_H d\tau. \end{aligned} \tag{60}$$

Let C_g be the Lipschitz constant of the map $g : (E, \|\cdot\|_E) \to \mathbb{R}$; then

$$\begin{aligned} \mathbb{E}\left[\int_T^t \|G(X(\tau))\|_H^2 d\tau\right] &\leq \mathbb{E}\left[\int_T^t \|G(X(\tau))\|_E^2 d\tau\right] \leq \mathbb{E}\left[\int_T^t |g(X(\tau))|^2 d\tau\right] \\ &\leq 2\mathbb{E}\left[\int_T^t \left(C_g^2\|X(\tau)\|_E^2 + |g(0)|^2\right) d\tau\right] < +\infty. \end{aligned} \tag{61}$$

The first and the second term of the right-handside of (60) can be dominated in $L^2(\Omega;\mathbb{R})$ thanks to (57), (58) and by Holder inequality; the third one can be dominated thanks to (57), (58), (61) and by Holder inequality. Moreover the right-handside of (60) converges pointwise to 0, when $n \to \infty$, thanks to (59). Therefore (37) follows by dominated convergence from (60) taking the expectations and letting $n \to \infty$. Integrating (60) on $[T, a]$ and taking the expectations, letting $n \to \infty$, in the same way we can get (38) by pointwise and dominated convergence. #

Proof of Lemma 6.7: *First step.* Let us suppose that τ takes a finite number of finite values; we can apply the Dynkin's formula to the approximating processes X_n of Lemma 6.3 (see Theorem 4.7 of [9]) with the function $e^{-\rho(t-T)}\psi(x)$ to get

$$\begin{aligned}\mathbb{E}\left[e^{-\rho(\tau-T)}\psi(X_n(\tau))-\psi(x)\right] \quad = \quad & E\Big[\int_T^\tau e^{-\rho(t-T)}\Big(-\rho\psi(X_n(t))+\langle\tilde{A}_nX_n(t),\psi_x(X_n(t))\rangle\\ &+\Big[r+\frac{1}{2}+\sigma\lambda\theta(t)\Big]\langle\Phi X(t),\psi_x(X_n(t))\rangle-\langle G(X(t)),\psi_x(X_n(t))\rangle\\ &+\frac{1}{2}\sigma^2\theta(t)^2\mathrm{Tr}\left[\Sigma(X(t))\Sigma(X(t))^*\psi_{xx}(X_n(t))\right]\Big)dt\Big];\end{aligned}$$

we want to get the first claim letting $n \to \infty$ and taking into account Lemma 6.3 and the continuity properties of ψ and its derivatives. By (37) $X_n(\tau) \to X(\tau)$ in $L^2(\Omega; H)$, so that we have the desired convergence in the left handside thanks to Lemma 6.6. For the right handside we have the desired convergences of the bounded terms thanks to (38) and Lemma 6.6, taking also into account the estimate (61) for the term containing G.
The only non trivial convergence in the right hand-side is that one concerned with the unbounded linear term, i.e.

$$\mathbb{E}\left[\int_T^\tau e^{-\rho(t-T)}\langle\tilde{A}_nX_n(t),\psi_x(X_n(t))\rangle dt\right]\longrightarrow\mathbb{E}\left[\int_T^\tau e^{-\rho(t-T)}\langle X(t),\tilde{A}^*\psi_x(X(t))\rangle dt\right].$$

Without loss of generality we can suppress in the following argument the term $e^{-\rho(t-T)}$ which is bounded. Let $x \in H$ and $(y_n) \subset D(\tilde{A}^*)$, $y \in D(\tilde{A}^*)$; we have

$$\begin{aligned}|\langle\tilde{A}_nx_n,y_n\rangle-\langle x,\tilde{A}^*y\rangle| \quad &\le \quad |\langle\tilde{A}_nx_n,y_n\rangle-\langle\tilde{A}_nx,y_n\rangle|+|\langle\tilde{A}_nx,y_n\rangle-\langle\tilde{A}_nx,y\rangle|+|\langle\tilde{A}_nx,y\rangle-\langle x,\tilde{A}^*y\rangle|\\ &= \quad |\langle(x_n-x),\tilde{A}_n^*y_n\rangle|+|\langle x,\tilde{A}_n^*(y_n-y)\rangle|+|\langle x,(\tilde{A}_n^*-\tilde{A}^*)y\rangle|.\end{aligned}$$

let us indicate by $\|\cdot\|_{\mathcal{L}}$ the operator norm for the linear continuous operators

$$\Big(D(\tilde{A}^*),\|\cdot\|_{D(\tilde{A}^*)}\Big)\to(H,\|\cdot\|_H).$$

We have the convergence $\tilde{A}_n^*v \to \tilde{A}^*v$ for any $v \in D(\tilde{A}^*)$ (see Proposition 4.13 in [26], Lemma 3.4-(ii) and Corollary B.12 in [18]), so that, by Banach-Steinhaus Theorem, $\|\tilde{A}_n^*\|_{\mathcal{L}} \le C$, for some $C > 0$. Thus

$$|\langle\tilde{A}_nx_n,y_n\rangle-\langle x,\tilde{A}^*y\rangle|\le C\Big[\|x_n-x\|_H\,\|y_n\|_{D(\tilde{A}^*)}+\|x\|_H\,\|y_n-y\|_{D(\tilde{A}^*|}\Big]+|\langle x,(\tilde{A}_n^*-\tilde{A}^*)y\rangle|.$$

Therefore, by Holder inequality,

$$\begin{aligned}&\mathbb{E}\left[\int_T^\tau\Big|\langle\tilde{A}_nX_n(t),\psi_x(X_n(t))\rangle-\langle X(t),\tilde{A}^*\psi_x(X(t))\rangle\Big|dt\right]\\ &\le C\,\mathbb{E}\left[\int_T^\tau\|X_n(t)-X(t)\|_H^2dt\right]\mathbb{E}\left[\int_T^\tau\|\psi_x(X_n(t))\|^2_{D(\tilde{A}^*)}dt\right]\\ &+C\,\mathbb{E}\left[\int_T^\tau\|X(t)\|_H^2dt\right]\mathbb{E}\left[\int_T^\tau\|\psi_x(X_n(t))-\psi_x(X(t))\|^2_{D(\tilde{A}^*)}dt\right]\\ &+\mathbb{E}\left[\int_T^\tau\|X(t)\|_H\|(\tilde{A}_n^*-\tilde{A}^*)\psi_x(X(t))\|_{D(\tilde{A}^*)}dt\right].\end{aligned}$$

The first term and the second term of the right handside go to 0 thanks to (38) and Lemma 6.6 (recall that we are assuming $\psi_x, \tilde{A}^*\psi_x$ uniformly continuous).
The third one converges pointwise to 0 and the integrand is dominated by

$$(C + \|\tilde{A}^*\|_{\mathcal{L}})\, \|X(\cdot)\|_H\, \|\psi_x(X(\cdot))\|_{D(\tilde{A}^*)},$$

which is integrable thanks to Holder inequality, so that we can conclude by dominated convergence.

Second step. Now let τ be a bounded stopping time; of course we may find a suquence (τ_n) of stopping time taking a finite number of finite values such that $\tau_n \uparrow \tau$ almost surely; by the first step for these stopping time the claim holds true, therefore we can pass to the limit and again conclude the proof in this case by dominated convergence.

Third step. Let τ be a stopping time almost surely finite such that, for some $r > 0$, $X(t) \in B(x, r)$ for $t \leq \tau$; we can find a sequence of bounded stopping times (τ_n) such that $\tau_n \uparrow \tau$; for these stopping times the claim holds true by the second step; moreover notice that, by the equivalence result of Proposition 4.13, if $X(t) \in B(x, r)$ for $t \leq \tau$, then $X(t) \in B_{(E, \|\cdot\|_E)}(x, r')$ for some r'; therefore, taking also into account the estimate (61), we can get the claim for τ passing to the limit by dominated convergence thanks to Lemma 6.5. #

Proof of Lemma 6.8: The proof follows the same line of the proof of Lemma 6.7; but in this case we cannot have the convergence for the term $\langle \tilde{A}_n X_n(u), g_x(X_n(u)) \rangle$; nevertheless we have

$$\langle \tilde{A}_n x_n, g_x(x_n) \rangle = \frac{g_0'(\|x_n\|)}{\|x_n\|} \langle \tilde{A}_n x, x_n \rangle;$$

since $\tilde{A}$ is dissipative we have $\langle \tilde{A}_n x_n, x_n \rangle \leq 0$ for any $n \in \mathbb{N}$ and so the claim follows taking the limsup. #

Lemma A.1. *Let τ^θ defined as in the part (ii) of the proof of Proposition 6.12 (here, in order to semplify the notation, we write $\bar{x}$ for x_M). Then there exists $\alpha > 0$ such that, for each $\theta(\cdot) \in \Theta_{ad}(\bar{x})$,*

$$\mathbb{E}\left[\int_T^{\tau^\theta} e^{-\rho(t-T)} dt\right] \geq \alpha.$$

Proof.
First step. Let $\theta(\cdot) \in \Theta_{ad}(\bar{x})$ and define the stopping time

$$\tau_\theta := \inf\{t \geq T \mid |X_0(t) - \bar{x}_0| \geq \varepsilon/4\} \wedge (2T);$$

in this step we will show that there exists $\beta > 0$ such that, for each $\theta(\cdot) \in \Theta_{ad}(\bar{x})$,

$$\mathbb{E}\left[\int_T^{\tau_\theta} e^{-\rho(t-T)} dt\right] \geq \beta.$$

For the controls such that $P\{\tau_\theta < 2T\} < 1/2$, we have the estimate

$$\mathbb{E}\left[\int_T^{\tau_\theta} e^{-\rho(t-T)} dt\right] \geq \frac{1}{2}\left[\frac{1 - e^{-\rho T}}{\rho}\right].$$

Therefore we can suppose without loss of generality that $P\{\tau_\theta < 2T\} \geq 1/2$. Recalling Proposition (4.13) and setting $x(t) := x(t; T, \bar{x}, \theta(\cdot))$ for the solution of the stochastic delay differential equation (15), we have

$$\tau_\theta = \inf\{t \geq T \mid |x(t) - \bar{x}_0| \geq \varepsilon/4\} \wedge (2T).$$

Now we can apply the classical Dynkin's formula to the one-dimensional process $y(\cdot) := x(\cdot) - \bar{x}_0$ with the function $\psi(t,y) = e^{-\rho(t-T)}y^2$ on $[T, \tau_\theta]$ and get

$$\begin{aligned}
\mathbb{E}\left[e^{-\rho(\tau_\theta - T)}\left(x(\tau_\theta) - \bar{x}_0\right)^2\right] &= \mathbb{E}\left[\int_T^{\tau_\theta}\left(-\rho e^{-\rho(t-T)}(x(t) - \bar{x}_0)^2\right)dt\right] \\
&+\mathbb{E}\left[\int_T^{\tau_\theta} e^{-\rho(t-T)}(x(t) - \bar{x}_0)(r + \sigma\lambda\theta(t))x(t)dt\right] \\
&-\mathbb{E}\left[\int_T^{\tau_\theta} e^{-\rho(t-T)}(x(t) - \bar{x}_0)[f_0\left(x(t) - \kappa x(t-T)\right) + q]dt\right] \\
&+\mathbb{E}\left[\int_T^{\tau_\theta} e^{-\rho(t-T)}\sigma^2\theta(t)^2 x(t)^2 dt\right];
\end{aligned}$$

now, taking into account that before τ_θ we have $|x(t) - \bar{x}_0| < \varepsilon/4$, $|x(t-T)| \leq \|\bar{x}\|_E + \varepsilon/4$ and that $|\theta(t)| \leq 1$, we can write, passing to the modulus on the right hand-side and taking into account that f_0 is Lipschitz continuous with Lipschitz constant K_0,

$$\begin{aligned}
\mathbb{E}\left[e^{-\rho(\tau_\theta - T)}\left(x(\tau_\theta) - \bar{x}_0\right)^2\right] &\leq \mathbb{E}\left[\int_T^{\tau_\theta}\rho e^{-\rho(t-T)}\frac{\varepsilon^2}{16}dt\right] \\
&+\mathbb{E}\left[\int_T^{\tau_\theta} e^{-\rho(t-T)}\frac{\varepsilon}{4}(r + \sigma\lambda)\left(|\bar{x}_0| + \frac{\varepsilon}{4}\right)dt\right] \\
&+\mathbb{E}\left[\int_T^{\tau_\theta} e^{-\rho(t-T)}\frac{\varepsilon}{4}\left[K_0\left(|\bar{x}_0| + \frac{\varepsilon}{4} + \kappa\left(\frac{\varepsilon}{4} + \|\bar{x}\|_E\right)\right) + q\right]dt\right] \\
&+\mathbb{E}\left[\int_T^{\tau_\theta} e^{-\rho(t-T)}\sigma^2\left(|\bar{x}_0| + \frac{\varepsilon}{4}\right)^2 dt\right],
\end{aligned}$$

so that, for some $K > 0$,

$$\mathbb{E}\left[e^{-\rho(\tau_\theta - T)}\left(x(\tau_\theta) - \bar{x}_0\right)^2\right] \leq K\,\mathbb{E}\left[\int_T^{\tau_\theta} e^{-\rho(t-T)}dt\right]. \tag{62}$$

Recalling that $(x(\tau_\theta) - \bar{x}_0)^2 \leq \varepsilon^2/16$ on $[T, \tau_\theta]$ and $(x(\tau_\theta) - \bar{x}_0)^2 = \varepsilon^2/16$ on $\{\tau_\theta < 2T\}$, and considering that

$$e^{-\rho(\tau_\theta - T)} = 1 - \rho\int_T^{\tau_\theta} e^{-\rho(t-T)}dt,$$

we can write by (62)

$$\frac{\varepsilon^2}{16}P\{\tau_\theta < 2T\} - \frac{\rho\varepsilon^2}{16}\mathbb{E}\left[\int_T^{\tau_\theta} e^{-\rho(t-T)}dt\right] \leq K\mathbb{E}\left[\int_T^{\tau_\theta} e^{-\rho(t-T)}dt\right],$$

so that, by the assumption $P\{\tau_\theta < +\infty\} \geq 1/2$,

$$\frac{1}{32}\varepsilon^2 \leq \left(K + \frac{\rho\varepsilon^2}{16}\right)\mathbb{E}\left[\int_T^{\tau_\theta} e^{-\rho(t-T)}dt\right]$$

and we get what claimed with $\beta = \frac{\varepsilon^2}{32}\left(K + \frac{\rho\varepsilon^2}{16}\right)^{-1}$.

Second step. The function $(\bar{x})_1(\cdot)$ is uniformly continuous and $\bar{x}_0 = \lim_{\zeta\to 0^-}\bar{x}_1(\zeta)$; let $\omega(\cdot)$ be its modulus of uniform continuity. Let $x(t) := x(t; T, \bar{x}, \theta(\cdot))$ be the solution of the stochastic delay differential equation (15). If we denote by $\omega'(\cdot)$ the modulus of uniform continuity of the trajectory $s \mapsto x(s)$ on $[0, \tau_\theta]$ (thus depending on the trajectory, i.e. on the point of the probability space), we have

$$\omega'(\eta) = \sup_{\substack{|s-s'|<\eta \\ s,s'\in[0,\tau_\theta]}}|x(s) - x(s')|;$$

but, by definition of τ_θ, if $|s-s'|<\eta$,

$$|x(s)-x(s')| \leq \begin{cases} \omega(\eta), & \text{if } s,s' \in [0,T], \\ \varepsilon/2, & \text{if } s,s' \in [T,\tau_\theta], \\ \omega(\eta)+\varepsilon/2, & \text{if } 0 \leq s < T < s' \leq \tau_\theta; \end{cases}$$

therefore $\omega'(\eta) \leq \omega(\eta)+\varepsilon/2$ without regard to the trajectory. Thus take $c>0$ such that $\omega(c)<\varepsilon/4$; we get, for $T \leq t \leq \tau_\theta \wedge (T+c)$,

$$\begin{aligned} \|X(t)-\bar{x}\|_E &= \left[\sup_{\zeta\in[-T,0)} \left| X_1(t)(\zeta)-\bar{x}_1(\zeta)\right|\right] + \left|X_0(t)-\bar{x}_0\right| \\ &\leq \left[\sup_{\zeta\in[-T,0)} \left| x(t+\zeta)-x(T+\zeta)\right|\right] + \varepsilon/4 \leq \omega(c)+\frac{\varepsilon}{2}+\frac{\varepsilon}{4} < \varepsilon. \end{aligned}$$

Therefore we have $\tau^\theta \geq \tau_\theta \wedge (T+c)$. Thus we can write

$$\begin{aligned} \mathbb{E}\left[\int_T^{\tau^\theta} e^{-\rho(t-T)}dt\right] &\geq \mathbb{E}\left[\int_T^{T+c} I_{\{\tau_\theta \geq T+c\}} e^{-\rho(t-T)}dt\right] \\ &\quad +\mathbb{E}\left[\int_T^{\tau_\theta} I_{\{\tau_\theta < T+c\}} e^{-\rho(t-T)}dt\right] \\ &= \frac{1-e^{-\rho c}}{\rho}\, P\{\tau_\theta \geq T+c\} \\ &\quad +\mathbb{E}\left[\int_T^{\tau_\theta} I_{\{\tau_\theta < T+c\}} e^{-\rho(t-T)}dt\right]. \end{aligned}$$

We claim that the last term is greater than a strictly positive number independent on $\theta(\cdot)$ (that is enough to make our proof complete); indeed let us suppose by contradiction that there exists a sequence $(\theta_n(\cdot))$ such that

$$\frac{1-e^{-\rho c}}{\rho} P\{\tau_{\theta_n} \geq T+c\} + \mathbb{E}\left[\int_T^{\tau_{\theta_n}} I_{\{\tau_{\theta_n} < T+c\}} e^{-\rho(t-T)}dt\right] \longrightarrow 0;$$

then, of course, we would have

$$P\{\tau_{\theta_n} \geq T+c\} \longrightarrow 0 \tag{63}$$

and also

$$\mathbb{E}\left[\int_T^{\tau_{\theta_n}} I_{\{\tau_{\theta_n} < T+c\}} e^{-\rho(t-T)}dt\right] \longrightarrow 0. \tag{64}$$

We consider

$$\mathbb{E}\left[\int_T^{\tau_{\theta_n}} e^{-\rho(t-T)}dt\right] - \mathbb{E}\left[\int_T^{\tau_{\theta_n}} I_{\{\tau_{\theta_n} < T+c\}} e^{-\rho(t-T)}dt\right] \tag{65}$$

and rewrite it as

$$\mathbb{E}\left[\int_T^{2T} I_{[T,\tau_{\theta_n}]}(t)\left(1-I_{\{\tau_{\theta_n} < T+c\}}\right) e^{-\rho(t-T)}dt\right];$$

by (63) the integrand converges to 0 in measure $P \times dt$ on $\Omega \times [T,2T]$; so, by dominated convergence,

$$\mathbb{E}\left[\int_T^{2T} I_{[T,\tau_{\theta_n}]}(t)\left(1-I_{\{\tau_{\theta_n} < T+c\}}\right) e^{-\rho(t-T)}dt\right] \longrightarrow 0,$$

i.e. also the expression in (65) goes to 0. Taking into account (64), we should conclude that also

$$\mathbb{E}\left[\int_T^{\tau_{\theta_n}} e^{-\rho(t-T)}dt\right] \longrightarrow 0,$$

but this convergence contradicts the first step. #

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