Constrained portfolio choices in the decumulation phase of a pension plan

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Abstract

This paper deals with a constrained investment problem for a defined contribution pension fund where retirees are allowed to defer purchase of annuity at some future time after retirement. This flexibility, sometimes referred to as "income drawdown option", reduces the annuity risk borne by members of the schemes, since it allows to postpone the conversion of the accumulated capital into pension within a given period of time beyond the retirement. During this period the capital is allocated dynamically while the pensioner withdraws periodic amounts of money to provide for daily life in accordance with restrictions imposed by the scheme's rules or by law.

The aim of this work is to find the optimal portfolio choice to be adopted by the retiree until the purchase of the annuity becomes compulsory. The financial market is composed by a risky and a riskless asset and the allocation has constraints on the amount of money invested in the risky asset.

The mathematical problem is naturally formulated as a stochastic control problem with constraints on the control variable, representing the investment on the risky asset, and is approached by the tool of dynamic programming. We explicitly compute the value function for the problem and give the optimal strategy in feedback form.

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1 Introduction

In countries where immediate annuitization is the only option available to retiring members of defined contribution pension schemes, members who retire at a time of low bond yield rates have to accept a pension lower than the one available with higher bond yields. In UK and US the retiree is allowed to defer annuitization at some time after retirement, withdraw periodic income from the fund and invest the rest of it in the period between retirement and annuitization. This allows the retiree to postpone the decision to purchase an annuity until a more propitious time. In UK there are limits imposed on both the consumption (which must be between 35% and 100% of the annuity which would have been purchasable immediately on retirement) and on how long the annuity purchase can be deferred (the fund must be used to purchase an annuity at age 75, if this has not been done earlier). On the other hand, there is virtually unlimited freedom to invest the fund in a broad range of assets.

The three degrees of freedom of the retiree (amount of consumption, investment allocation and time of annuitization), together with the important issue of ruin possibility, have been investigated in the actuarial and financial literature in many papers. Among others, [Albrecht & Maurer, 2002], [Blake, Cairns & Dowd, 2003], [Gerrard, Haberman & Vigna, 2004], [Gerrard, Haberman & Vigna, 2006], [Milevsky, 2001], [Milevsky, Moore & Young, 2006] and [Milevsky & Young, 2007].

In this paper we consider the position of a representative participant to a defined contribution pension fund who retires and compulsorily has to purchase an annuity within a certain period of time after retirement. In the interim the accumulated capital is dynamically allocated while the pensioner withdraws periodic amounts of money to provide for daily life in accordance with restrictions imposed by the scheme's rules or by legislation. in particular we assume that an individual who retires at time $t = 0$ acquires control of a fund of size x_0 which is invested in a market that consists of a risky and a riskless asset. The value of the risky asset is assumed to follow a geometric Brownian motion model. At age T the entire fund must be invested in an annuity. The retiree is given only one degree of freedom, namely the investment allocation. The income withdrawn from the fund in the unit time is assumed to be fixed and equal to b_0 and the retiree is obliged to annuitize at future time T.

Due to the difficulty that arises by the inclusion of restrictions on the controls, we do not consider here the first and the third degrees of freedom of the pensioner, namely the optimal consumption strategy and the optimal annuitization time, treated – without restrictions on the controls – e.g. in [Gerrard, Haberman & Vigna, 2006] and [Gerrard, Højgaard & Vigna, 2008]. On the other hand, differently from the previous literature, we analyze the problem in the presence of short selling constraints, extending the work done by [Gerrard, Haberman & Vigna, 2004]. Ongoing research aims to characterize the optimal policy in the presence of borrowing constraints and the conditions granting the nonnegativity of current wealth. Furthermore, the extension to a model allowing for the three possible choices outlined, as well as restrictions on the controls, is in the agenda for future research.

The remainder of the paper is organized as follows. In Section 2 we introduce the model. In Section 3 we develop the dynamic programming approach proving various results about the value function, the solution of the hamilton jacobi bellman equation, the verification theorem and the optimal feedback policies.

2 The model

In our model we consider the position of an individual who chooses the drawdown option at retirement, i.e. withdraws a certain income until she/he achieves the age at which the purchase of the annuity is compulsory. The fund is invested in two assets, a riskless asset, with constant instantaneous rate of return $r \geq 0$, and a risky asset, whose price follows a geometric Brownian motion with constant volatility $\sigma > 0$ and drift $\mu := r + \sigma\lambda$, where $\lambda \geq 0$ is the risk premium. The pensioner withdraws an amount b_0 in the unit of time. Therefore, according to [Merton, 1969] the state equation that describes the dynamics of the fund wealth is the following

$$\begin{cases} dX(s) = [rX(s) + \sigma\lambda\pi(s) - b_0]\, ds + \sigma\pi(s)dB(s), \\ X(0) = x_0, \end{cases}$$

where x_0 is the fund wealth at the retirement date $t = 0$, $B(\cdot)$ is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and $\pi(\cdot)$ is the progressively measurable process (with respect to the filtration generated by $B(\cdot)$) representing the amount of money invested in the risky asset at time $t \in [0, T]$. We impose the constraint $\pi(\cdot) \geq 0$, i.e. short selling of risky asset is not allowed.

We introduce the loss function

$$L(s, x) = (F(s) - x)^2\,, \tag{1}$$

where $F(\cdot)$ is the target function, i.e. the target that the agent wishes to track at any time $s \in [0, T]$.

In [Højgaard & Vigna, 2007] is proved that the quadratic loss function applied to the defined contribution pension schemes is a particular case of mean variance portfolio optimization approach. in other words, the optimal portfolio found via the quadratic loss function (1) is an efficient portfolio in the mean-variance setting. Namely, there is no other portfolio that provides a higher expected return with the same variance, and no other portfolio that provides a lower variance with the same mean.

According to [Gerrard, Haberman & Vigna, 2004], we choose

$$F(s) = \frac{b_0}{r} + \left(F(T) - \frac{b_0}{r}\right) e^{-r(T-s)},$$

where $F(T)$ can be chosen arbitrarily. The interpretation of the target function is straightforward. Should the fund hit $F(t)$ at time $t \leq T$, the pensioner would be able, by investing the whole portfolio in the riskless asset, to consume b_0 from t to T and receive the desired target $F(T)$ at the time T of compulsory annuitization. Clearly, in this case the loss function would be 0 at any time $t \leq s \leq T$.

As it is shown in [Gerrard, Haberman & Vigna, 2004], with this choice of the target function the fund never reaches the target, provided that at initial time $t = 0$ the fund x_0 is lower than the target $F(0)$.

The optimization problem consists in minimizing, over the set of the strategies $\pi(\cdot) \geq 0$, the functional

$$E\left[\int_0^T e^{-\rho s} L(s, X(s)) ds + \kappa e^{-\rho T} L(T, X(T))\right],$$

where ρ is the individual discount factor, $\kappa > 0$ is a weighting constant which measures the importance of the final cost due to a deviation from the target at time T relative to the running costs for deviation experienced before then.

3 Dynamic Programming

We want to study the problem by the tool of the dynamic programming. In order to do this we define the problem for generic initial data $(t, x) \in [0, T] \times \mathbb{R}$. Thus we consider the state equation

$$\begin{cases} dX(s) = [rX(s) + \sigma\lambda\pi(s) - b_0]\, ds + \sigma\pi(s)dB(s), \\ X(t) = x; \end{cases}$$

this equation admits, for given $\pi(\cdot)$, a unique strong solution on $(\Omega, \mathcal{F}, P)$ and we denote it by $X(\cdot; t, x, \pi(\cdot))$. The objective functional is given by

$$J(t, x; \pi(\cdot)) := E\left[\int_t^T e^{-\rho s} L(s, X(s; t, x, \pi(\cdot)))ds + \kappa e^{-\rho T} L(T, X(T; t, x, \pi(\cdot)))\right].$$

We define the value function

$$V(t, x) := \inf_{\pi(\cdot)\geq 0} J(t, x; \pi(\cdot)).$$

As usual in the context of optimal control problems with finite horizon the value function is associated to a nonlinear parabolic PDE with terminal boundary condition, which is the so called HJB equation.

Let us define the function *hamiltonian current value*

$$\begin{array}{rccc} \mathcal{H}_{cv}: & \mathbb{R} \times (0, +\infty) \times [0, +\infty) & \longrightarrow & \mathbb{R}, \\ & (p, P, \pi) & \longmapsto & \frac{1}{2}\sigma^2 P\pi^2 + \sigma\lambda p\pi. \end{array}$$

The function $\pi \mapsto \mathcal{H}_{cv}(p, P; \pi)$ has a unique minimum point on $[0, +\infty)$ given by

$$\pi^* = -\frac{\lambda\, p}{\sigma\, P} \vee 0;$$

thus we can compute the minimum

$$\mathcal{H}_{cv}(p, P, \pi^*) = \begin{cases} -\frac{\lambda^2 p^2}{2P}, & \text{if } p < 0, \\ 0, & \text{if } p \geq 0. \end{cases} \tag{2}$$

Let us define also the function *hamiltonian*

$$\begin{array}{rccc} \mathcal{H}: & \mathbb{R} \times (0, +\infty) & \longrightarrow & \mathbb{R}, \\ & (p, P) & \longmapsto & \inf_{\pi\geq 0} \mathcal{H}_{cv}(p, P, \pi) = \mathcal{H}_{cv}(p, P, \pi^*). \end{array}$$

The HJB equation associated with our problem is

$$\begin{cases} e^{-\rho t}(F(t) - x)^2 + \dfrac{\partial v}{\partial t}(t, x) + (rx - b_0)\dfrac{\partial v}{\partial x}(t, x) + \mathcal{H}\left(\dfrac{\partial v}{\partial x}(t, x), \dfrac{\partial^2 v}{\partial x^2}(t, x)\right) = 0, \ \text{ on } [0, T] \times \mathbb{R}, \\ v(T, x) = \kappa e^{-\rho T}(F(T) - x)^2, \quad x \in \mathbb{R}, \end{cases} \tag{3}$$

which, if seen in classical sense, in particular requires for the definition $\frac{\partial^2 v}{\partial x^2}(t, x) > 0$.

3.1 Properties of the value function

In this section we want to prove some properties for the value function. We start with a lemma which analyze the behaviour of the state trajectory under the null control.

Lemma 3.1. *Let $t \in [0,T]$, $x = F(t)$. Then $X(s;t,x,0) = F(s)$ for all $s \in [t,T]$.*

Proof. Let $t \in [0,T]$, $x \in \mathbb{R}$, and set $X(\cdot) := X(\cdot;t,x,0)$. The dynamics of $X(\cdot)$ is given by

$$\begin{cases} dX(s) = (rX(s) - b_0)\, ds, \\ X(t) = x. \end{cases}$$

The "dynamics" of the target $F(\cdot)$ after t is given by

$$\begin{cases} dF(s) = (rF(s) - b_0)\, ds, \\ F(t) = F(t). \end{cases}$$

Therefore $X(\cdot)$ and $F(\cdot)$ solve the same ordinary differential equation. If $x = F(t)$ they have also the same initial condition, so that they coincide. #

Lemma 3.1 shows that the null strategy $\pi(\cdot) \equiv 0$ is optimal for the initial datum $(t, F(t))$, since we have $J(t, F(t); 0) = 0$ and, on the other hand, $V(\cdot,\cdot) \geq 0$. In particular $V(t, F(t)) = 0$ for each $t \in [0,T]$. Note also that, if $x \neq F(t)$, then it has to be $V(t,x) > 0$.
This suggests that the graph of $F(\cdot)$ works as a barrier for the problem, so that we are led to separate the space $[0,T] \times \mathbb{R}$ in the two regions

$$U_1 := \{(t,x) \mid t \in [0,T],\ x \leq F(t)\}, \quad U_2 := \{(t,x) \mid t \in [0,T],\ x \geq F(t)\};$$

notice that

$$U_1 \cup U_2 = [0,T] \times \mathbb{R}, \qquad U_1 \cap U_2 = \{(t, F(t)) \mid t \in [0,T]\}.$$

Lemma 3.2. *Let $(t,x) \in [0,T] \times \mathbb{R}$, $\pi(\cdot) \geq 0$ a strategy; set $X(\cdot) := X(\cdot;t,x,\pi(\cdot))$ and define, with the convention $\inf \emptyset = T$, the stopping time*

$$\tau := \inf\{s \geq t \mid X(s) = F(s)\};$$

define the strategy

$$\pi^\tau(s) := \begin{cases} \pi(s), & \text{if } s < \tau, \\ 0, & \text{if } s \geq \tau. \end{cases}$$

Then $J(t,x;\pi^\tau(\cdot)) \leq J(t,x;\pi(\cdot))$.

Proof. It follows straightly from Lemma 3.1. #

Definition 3.3. *Let $(t,x) \in [0,T] \times \mathbb{R}$, $\delta > 0$; a strategy $\pi^\delta(\cdot) \geq 0$ is called δ-optimal if*

$$J(t,x;\pi^\delta(\cdot)) \leq V(t,x) + \delta.$$

#

Proposition 3.4. *Let $t \in [0,T]$. The function $\mathbb{R} \to \mathbb{R}$, $x \mapsto V(t,x)$ is convex.*

Proof. *Step 1.* In this step we will prove that $x \mapsto V(t,x)$ is convex on $(-\infty, F(t)]$. Let us suppose $x, y \leq F(t)$. Let $\delta > 0$ and let $\pi_x^\delta(\cdot), \pi_y^\delta(\cdot)$ two controls δ-optimal for x, y respectively, i.e.

$$J(t,x;\pi_x^\delta(\cdot)) \leq V(t,x)+\delta, \qquad J(t,y;\pi_y^\delta(\cdot)) \leq V(t,y)+\delta.$$

Set $X(s) := X(s;t,x,\pi_x^\delta(\cdot))$, $Y(s) := X(s;t,y,\pi_y^\delta(\cdot))$. Without loss of generality, thanks to Lemma 3.2, we can suppose $X(s), Y(s) \leq F(s)$ for all $s \in [t,T]$. We want to prove that, for all $\gamma \in [0,1]$,

$$V(t,\gamma x + (1-\gamma)y) \leq \gamma V(t,x) + (1-\gamma)V(t,y).$$

Fix $\gamma \in [0,1]$ and set $Z(s) := \gamma X(s) + (1-\gamma)Y(s)$; of course $Z(s) \leq F(s)$, for all $s \in [t,T]$. We have

$$\begin{aligned}
\gamma V(t,x) + (1-\gamma)V(t,y) + \delta &\geq \gamma J(t,x;\pi_x^\delta(\cdot)) + (1-\gamma)J(t,y;\pi_y^\delta(\cdot)) \\
&= \gamma E\left[\int_t^T e^{-\rho s}(F(s)-X(s))^2 ds + \kappa e^{-\rho T}(F(T)-X(T))^2\right] \\
&\quad +(1-\gamma)E\left[\int_t^T e^{-\rho s}(F(s)-Y(s))^2 ds + \kappa e^{-\rho T}(F(T)-Y(T))^2\right] \\
&\geq E\left[\int_t^T e^{-\rho s}(F(s)-Z(s))^2 ds + \kappa e^{-\rho T}(F(T)-Z(T))^2\right] \qquad (4)
\end{aligned}$$

where the last inequality follows by convexity of $\xi \mapsto (F(s)-\xi)^2$. Let us write the dynamics for $Z(\cdot)$:

$$\begin{aligned}
dZ(s) &= \gamma dX(s) + (1-\gamma)dY(s) \\
&= \gamma\left[rX(s) + \sigma\lambda\pi_x^\delta(s) - b_0\right]ds + (1-\gamma)\left[rY(s) + \sigma\lambda\pi_y^\delta(s) - b_0\right]ds \\
&\quad +\gamma\sigma\pi_x^\delta(s)dB(s) + (1-\gamma)\sigma\pi_y^\delta(s)dB(s) \\
&= \left[rZ(s) + \sigma\lambda(\gamma\pi_x^\delta(s) + (1-\gamma)\pi_y^\delta(s)) - b_0\right]ds + \sigma\left(\gamma\pi_x^\delta(s) + (1-\gamma)\pi_y^\delta(s)\right)dB(s).
\end{aligned}$$

Thus, if we define $\pi_z(\cdot) := \gamma\pi_x^\delta(\cdot) + (1-\gamma)\pi_y^\delta(\cdot) \geq 0$, we get $Z(s) = X(s;t,\gamma x + (1-\gamma)y, \pi_z(\cdot))$. Therefore we get

$$E\left[\int_t^T e^{-\rho s}(F(s)-Z(s))^2 ds + \kappa e^{-\rho T}(F(T)-Z(T))^2\right] \geq V(t,\gamma x + (1-\gamma)y)) \qquad (5)$$

and comparing (4) with (5) we get the claim in this case by the arbitrariness of δ.

Step 2. We can argue exactly as in the step 1 and conclude that $x \mapsto V(t,x)$ is convex on $[F(t), +\infty)$.

Step 3. We can notice that $V(t,\cdot)$ is nonnegative and that, thanks to Lemma 3.1, $V(t,F(t)) = 0$, so that $F(t)$ is a minimum for $V(t,\cdot)$. Thus the global convexity of $V(t,\cdot)$ follows from the convexity on the two half lines $(-\infty, F(t)]$, $[F(t), +\infty)$ and from the fact that it has a minimum in $F(t)$. #

Since, for $t \in [0,T]$, $V(t,\cdot)$ is convex and admits a minimum at $x = F(t)$, we have directly the following result. However we give a proof of the statement independent on Proposition 3.4.

Proposition 3.5. *Let $t \in [0,T]$; the function $x \mapsto V(t,x)$ is decreasing on $(-\infty, F(t)]$ and increasing on $[F(t), +\infty)$.*

Proof. We prove the statement on $(-\infty, F(t)]$; the other one follows as well. So let $x \le y \le F(t)$, let $\delta > 0$ and $\pi^\delta(\cdot) \ge 0$ a δ-optimal strategy for x, so that

$$V(t,x) + \delta \ge J(t,x;\pi^\delta(\cdot)). \tag{6}$$

Set $X(s) := X(s;t,x,\pi^\delta(\cdot))$ and $Y(s) := X(s;t,y,\pi^\delta(\cdot))$. Again, thanks to Lemma 3.2, we can suppose without loss of generality that $X(s) \le F(s)$ for all $s \in [t,T]$. By comparison criterion (see, e.g., [Karatzas & Shreve, 1991]) we have $X(s) \le Y(s)$ for all $s \in [t,T]$. Let us define the strategy

$$\tilde{\pi}(s) := \begin{cases} \pi^\delta(s), & \text{if } Y(s) < F(s), \\ 0, & \text{if } Y(s) = F(s); \end{cases}$$

of course $\tilde{\pi}(\cdot) \ge 0$ and, if we set $\tilde{Y}(s) := X(s;t,y,\tilde{\pi}(\cdot))$, again thanks to Lemma 3.1 we get $X(s) \le \tilde{Y}(s) \le F(s)$, for all $s \in [t,T]$. Thus, by monotonicity of $L(s,\cdot)$,

$$J(t,x;\pi^\delta(\cdot)) \ge J(t,y;\tilde{\pi}(\cdot)) \tag{7}$$

and of course

$$J(t,y;\tilde{\pi}(\cdot)) \ge V(t,y). \tag{8}$$

Comparing (6), (7) and (8) we get the claim by the arbitrariness of δ. #

3.2 The HJB equation and the optimal feedback map

Recall that we have set

$$U_1 := \{(t,x) \mid t \in [0,T],\ x \le F(t)\}, \quad U_2 := \{(t,x) \mid t \in [0,T],\ x \ge F(t)\}.$$

If we suppose that the value function is regular on U_1 and on U_2, then, inspired by the previous section, which gives information on the signs of V_x, V_{xx} on the regions U_1 and U_2, we can split the HJB equation in these two regions. We get that V should satisfy the equation

$$\begin{cases} e^{-\rho t}(F(t)-x)^2 + v_t(t,x) + (rx - b_0)v_x(t,x) - \dfrac{\lambda^2 v_x^2(t,x)}{2v_{xx}(t,x)} = 0, & \text{on } U_1 \setminus \{(t,F(t)) \mid t \in [0,T]\}, \\ e^{-\rho t}(F(t)-x)^2 + v_t(t,x) + (rx - b_0)v_x(t,x) = 0, & \text{on } U_2 \setminus \{(t,F(t)) \mid t \in [0,T]\}, \end{cases} \tag{9}$$

with boundary conditions

$$\begin{cases} v_x(t,F(t)) = 0, & t \in [0,T], \\ v(T,x) = \kappa e^{-\rho T}(F(T)-x)^2, & x \in \mathbb{R}. \end{cases} \tag{10}$$

Lemma 3.6. *1. Let $v_1(t,x) = e^{-\rho t}A_1(t)(F(t)-x)^2$, where $A_1(\cdot)$ is the unique solution of the ordinary differential equation*

$$\begin{cases} A_1'(t) = \left(\rho + \lambda^2 - 2r\right) A_1(t) - 1, \\ A_1(T) = \kappa. \end{cases}$$

i.e., setting $a_1 := \rho + \lambda^2 - 2r$.

$$A_1(t) = \begin{cases} \left(\kappa - \frac{1}{a_1}\right) e^{-a_1(T-t)} + \frac{1}{a_1}, & \text{if } a_1 \ne 0, \\ -t + T + \kappa, & \text{if } a_1 = 0. \end{cases}$$

Then:

(a) $v_{1_x} \leq 0$ *on* U_1;

(b) $v_{1_{xx}} > 0$ *on* U_1;

(c) v_1 *solves*

$$\begin{cases} e^{-\rho t}(F(t)-x)^2 + v_t(t,x) + (rx - b_0)v_x(t,x) - \dfrac{\lambda^2 v_x^2(t,x)}{2v_{xx}(t,x)} = 0, \quad \text{on } [0,T]\times\mathbb{R}, \\ v(T,x) = \kappa e^{-\rho T}(F(T)-x)^2, \quad x \in \mathbb{R}. \end{cases} \tag{11}$$

2. *Let* $v_2(t,x) = e^{-\rho t}A_2(t)(F(t)-x)^2$, *where* $A_2(\cdot)$ *is the unique solution of the ordinary differential equation*

$$\begin{cases} A_2'(t) = (\rho - 2r)\, A_2(t) - 1, \\ A_2(T) = \kappa, \end{cases}$$

i.e., setting $a_2 := \rho - 2r$.

$$A_2(t) = \begin{cases} \left(\kappa - \frac{1}{a_2}\right) e^{-a_2(T-t)} + \frac{1}{a_2}, & \text{if } a_2 \neq 0, \\ -t + T + \kappa, & \text{if } a_2 = 0. \end{cases}$$

Then:

(a) $v_{2_x} \geq 0$ *on* U_2;

(b) $v_{2_{xx}} > 0$ *on* U_2;

(c) v_2 *solves*

$$\begin{cases} e^{-\rho t}(F(t)-x)^2 + v_t(t,x) + (rx - b_0)v_x(t,x) = 0, \quad \text{on } [0,T]\times\mathbb{R}, \\ v(T,x) = \kappa e^{-\rho T}(F(T)-x)^2, \quad x \in \mathbb{R}. \end{cases} \tag{12}$$

3. *For* $t \in [0,T]$, *we have*

(a) $v_1(t,F(t)) = v_2(t,F(t)) = 0$;

(b) $v_{1_t}(t,F(t)) = v_{2_t}(t,F(t)) = 0$;

(c) $v_{1_x}(t,F(t)) = v_{2_x}(t,F(t)) = 0$.

Moreover

- *if* $\lambda = 0$, *then* $v_{1_{xx}}(t,F(t)) = v_{2_{xx}}(t,F(t))$ *for* $t \in [0,T]$;
- *if* $\lambda > 0$, *then* $v_{1_{xx}}(t,F(t)) \neq v_{2_{xx}}(t,F(t))$ *for* $t \in [0,T)$.

Proof. Let us consider, for $A(\cdot) \in C^1([0,T];\mathbb{R})$, the function

$$v(t,x) = e^{-\rho t}A(t)(F(t)-x)^2.$$

We have

$$v_t(t,x) = -\rho e^{-\rho t}A(t)(F(t)-x)^2 + e^{-\rho t}A'(t)(F(t)-x)^2 + 2e^{-\rho t}A(t)(F(t)-x)F'(t),$$

$$v_x(t,x) = -2e^{-\rho t}A(t)(F(t)-x),$$

$$v_{xx}(t,x) = 2e^{-\rho t}A(t).$$

Notice also that

$$F'(t) = rF(t) - b_0.$$

Finally it is immediate to check, splitting the proofs in the cases $a_1 < 0$, $a_1 = 0$, $a_1 > 0$ and $a_2 < 0$, $a_2 = 0$, $a_2 > 0$, that the functions $A_1(\cdot), A_2(\cdot)$ are strictly positive on $[0,T]$. Therefore all the statements follow by simple computations taking $A(\cdot) = A_1(\cdot)$ and $A(\cdot) = A_2(\cdot)$. #

Let us define on $[0,T]\times\mathbb{R}$ the function v by

$$v(t,x) := \begin{cases} v_1(t,x), & \text{if } (t,x)\in U_1, \\ v_2(t,x), & \text{if } (t,x)\in U_2; \end{cases} \tag{13}$$

thanks to Lemma 3.6 this function is well defined on $U_1\cap U_2$.

Lemma 3.7 (Fundamental identity)**.** *1. Let $(t,x)\in U_1$, let v_1 be the function defined in Lemma 3.6-(1) and let $\pi(\cdot)\geq 0$ be a strategy such that $X(\cdot;t,x,\pi(\cdot))\in U_1$; then*

$$\begin{aligned} v_1(t,x) &= J(t,x;\pi(\cdot)) \\ &+E\left[\int_t^T \Big(\mathcal{H}\big(v_{1_x}(s,X(s)), v_{1_{xx}}(s,X(s))\big) - \mathcal{H}_{cv}\big(v_{1_x}(s,X(s)), v_{1_{xx}}(s,X(s));\pi(s)\big)\Big)\,ds\right]. \end{aligned}$$

2. Let $(t,x)\in U_2$, let v_2 be the function defined in Lemma 3.6-(2) and let $\pi(\cdot)\geq 0$ be a strategy such that $X(\cdot;t,x,\pi(\cdot))\in U_2$; then

$$\begin{aligned} v_2(t,x) &= J(t,x;\pi(\cdot)) \\ &+E\left[\int_t^T \Big(\mathcal{H}\big(v_{2_x}(s,X(s)), v_{2_{xx}}(s,X(s))\big) - \mathcal{H}_{cv}\big(v_{2_x}(s,X(s)), v_{2_{xx}}(s,X(s));\pi(s)\big)\Big)\,ds\right]. \end{aligned}$$

Proof. (1) Let v_1 be the function defined in Lemma 3.6-(1); by the same lemma v_1 solves (11) on U_1. Since $v_{1_x}\leq 0, v_{1_{xx}}>0$ on U_1, we get

$$\mathcal{H}(v_{1_x}(t,x), v_{1_{xx}}(t,x)) = -\frac{\lambda^2 v_x^2(t,x)}{2v_{xx}(t,x)}, \qquad (t,x)\in U_1,$$

so that v_1 solves the originary HJB equation (3) on U_1.
Let us take $\pi(\cdot)$ such that the correspondent state trajectory $X(\cdot) := X(\cdot;t,x,\pi(\cdot))$ remains in U_1 and apply the Dynkin formula to $X(\cdot)$ with the function v_1; we get

$$\begin{aligned} v_1(T,X(T)) - v_1(t,x) &= E\Big[\int_t^T \Big(v_{1_t}(s,X(s)) + (rX(s)-b_0)\,v_{1_x}(s,X(s)) \\ &\qquad + \mathcal{H}_{cv}\big(v_{1_x}(s,X(s)), v_{1_{xx}}(s,X(s));\pi(s)\big)\Big)\,ds\Big], \end{aligned}$$

i.e.

$$\begin{aligned} v_1(t,x) &= E\Big[\kappa e^{-\rho T}(F-X(T))^2 \\ &- \int_t^T \Big(v_{1_t}(s,X(s)) + (rX(s)-b_0)\,v_{1_x}(s,X(s)) + \mathcal{H}_{cv}\big(v_{1_x}(s,X(s)), v_{1_{xx}}(s,X(s));\pi(s)\big)\Big)\,ds\Big]. \end{aligned}$$

Taking into account the assumption on $\pi(\cdot)$ and the fact that, as showed, v_1 solves the originary HJB equation on U_1, we can write,

$$\begin{aligned} v_1(t,x) &= E\Big[\kappa e^{-\rho T}(F-X(T))^2 + \int_t^T e^{-\rho s}(F(s)-X(s))^2\,ds \\ &\quad + \int_t^T \Big(\mathcal{H}\big(v_{1_x}(s,X(s)), v_{1_{xx}}(s,X(s))\big) - \mathcal{H}_{cv}\big(v_{1_x}(s,X(s)), v_{1_{xx}}(s,X(s)); \pi(s)\big)\Big) ds\Big] \\ &= J(t,x;\pi(\cdot)) \\ &\quad + E\left[\int_t^T \Big(\mathcal{H}\big(v_{1_x}(s,X(s)), v_{1_{xx}}(s,X(s))\big) - \mathcal{H}_{cv}\big(v_{1_x}(s,X(s)), v_{1_{xx}}(s,X(s)); \pi(s)\big)\Big) ds\right]. \end{aligned}$$

(2) Let v_2 be the function defined in Lemma 3.6-(2); by the same lemma v_2 solves (12) on U_2. Since $v_{2_x} \geq 0, v_{2_{xx}} > 0$ on U_1, we get

$$\mathcal{H}(v_{2_x}(t,x), v_{2_{xx}}(t,x)) = 0, \qquad (t,x) \in U_2,$$

so that v_2 solves the originary HJB equation (3) on U_2. Now the proof follows the same line of the proof of the previous statement. #

Lemma 3.8. *1. Let $(t,x) \in U_1$. There exists a unique process $X(\cdot)$ solution of the equation*

$$\begin{cases} dX(s) = \left[rX(s) + \lambda^2(F(s)-X(s)) - b_0\right] ds + \lambda(F(s)-X(s))dB(s), \\ X(t) = x. \end{cases}$$

Moreover the process $X(\cdot)$ is such that $(s, X(s))$, $s \in [t,T]$, lives in U_1.

2. Let $(t,x) \in U_2$. The (deterministic) process $X(\cdot) := X(\cdot; t, x, 0)$ is such that $(s, X(s))$, $s \in [t,T]$, lives in U_2.

Proof. **(1)** The proof of the existence and uniqueness of X is standard. About the second part of the statement, notice that, if write the dynamics of $Z(\cdot) := F(\cdot) - X(\cdot)$ in the interval $[t,T]$, we get

$$\begin{cases} dZ(s) = \left(r - \lambda^2\right) Z(s)ds - \lambda Z(s)dB(s), \\ Z(t) = F(t) - x. \end{cases}$$

Therefore $Z(\cdot)$ is a geometric Brownian motion with positive starting point, so that it has to be positive and this claim is proved.
(2) This statement follows arguing as in the proof of Lemma 3.1 and taking into account that $x \geq F(t)$. #

Let us define the feedback map

$$(s,y) \mapsto \Pi^*(s,y) := \begin{cases} \frac{\lambda}{\sigma}(F(s)-y), & \text{if } (s,y) \in U_1, \\ 0, & \text{if } (s,y) \in U_2. \end{cases} \tag{14}$$

Notice that, thanks to Lemma 3.6-(1a), we have

$$F(s) - y = -\frac{v_{1_x}(s,y)}{v_{1_{xx}}(s,y)}. \tag{15}$$

Moreover, by definition of Π^*, the strategy $\pi^*(\cdot)$ defined by the feedback map (14), i.e. $\pi^*(s) := \Pi^*(s, X(s))$, satisfies the constraint $\pi^*(\cdot) \geq 0$.

Theorem 3.9 (Verification)**.** *Let $(t, x) \in [0, T] \times \mathbb{R}$. Then the feedback strategy $\pi^*(\cdot)$ is optimal for the problem starting from (t, x), i.e. $V(t, x) = J(t, x; \pi^*(\cdot))$, and $V(t, x) = v(t, x)$, where v is the function defined in (13).*

Proof. Let $(t, x) \in U_1$ and $\pi(\cdot) \geq 0$ and set $X^*(\cdot) := X(\cdot; t, x, \pi^*(\cdot))$, $X(\cdot) := X(\cdot; t, x, \pi(\cdot))$. Let us suppose $X(\cdot) \in U_1$. Thus we can apply the fundamental identity to $X(\cdot)$ with v_1 getting

$$\begin{aligned} v_1(t,x) &= J(t,x;\pi(\cdot)) \\ &\quad + E\left[\int_t^T \Big(\mathcal{H}\big(v_{1_x}(s,X(s)), v_{1_{xx}}(s,X(s))\big) - \mathcal{H}_{cv}\big(v_{1_x}(s,X(s)), v_{1_{xx}}(s,X(s)); \pi(s)\big)\Big)\, ds\right] \\ &\leq J(t,x;\pi(\cdot)). \end{aligned}$$

Taking into account Lemma 3.2, this shows that $v_1(t, x) \leq V(t, x)$. On the other hand, we know from Lemma 3.8-(1) that also $X^*(\cdot) \in U_1$, so that we can apply the fundamental identity also to $X^*(\cdot)$ with v_1. Taking into account (15) we see that, by (2) and by Lemma 3.6-(1b), the feedback map minimizes at any time s the hamiltonian current value. Thus we get in this case $v_1(t, x) = J(t, x; \pi^*(\cdot))$, which shows that $v_1(t, x) = V(t, x) = J(t, x; \pi^*(\cdot))$.

If $(t, x) \in U_2$ we can argue exactly in the same way getting $v_2(t, x) = V(t, x) = J(t, x; \pi^*(\cdot))$, which completes the proof. #

Remark 3.10. *Theorem 3.9 and Lemma 3.6-(3) say that*

- *if $\lambda = 0$, then the value function is C^2;*
- *if $\lambda > 0$, then the value function is C^1, but not C^2.* #

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