COMPUTATIONAL ISSUES IN INTERNAL MODELS: THE CASE OF PROFIT-SHARING LIFE INSURANCE POLICIES

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ABSTRACT

In this paper we focus on the computational issues in the development of *internal models*, according to the technical requirements established by the Solvency II project. In particular, we consider the valuation of *profit-sharing life insurance policies*. Numerical simulations must provide reliable estimates of the relevant quantities involved in the contracts; therefore, valuation processes have to be performed by accurate algorithms able to provide solutions in a suitable turnaround time. To gain in accuracy we propose a change of numéraire in the stochastic processes for risks sources, thus providing estimates under the forward risk-neutral measure. To speed-up the simulation process we use high performance computing environments. We develop algorithms based on the parallelization of Monte Carlo method, using both the risk-neutral and the forward measure, thus providing numerical procedures capable to trade-off accuracy and efficiency.

KEYWORDS

Asset-liability management portfolio, Life insurance policies, Forward risk-neutral measure, Monte Carlo method, High performance computing

1 INTRODUCTION

The new rules of the Solvency II Directive Proposal are increasing more and more the request of stochastic asset-liability management (ALM) models for insurance companies.
In the Proposal ([6], Articles 118-123) the established requirements mainly concern the statistical quality standards of the models. Nevertheless, in order to make the models of "effective" use, it is fundamental to obtain responses in a suitable turnaround time, therefore, the computational performance of the solution process plays a crucial role in this framework. For the internal models validation it is a challenging matter to focus on their numerical solution, with the aim of obtaining adaptive solution processes, that is, capable of being properly scaled in order to balance accuracy and computational efficiency on demand, depending on the evaluation context.
In this paper we study computational issues related to the valuation and validation of internal models; in particular, the analysis is carried out on *profit-sharing life insurance policies* (PS policies). In these contracts, the benefits which are credited to the policyholder are indexed to the annual return of an specified investment portfolio, called the *segregated fund*. A profit-sharing policy is then a derivative contract, with underlying the segregated fund. In Italian insurance market, the crediting mechanism typically guarantees a minimum to the policyholder. The PS policies are complex contracts because they rely on non-marketable indexes and on management actions, and because complex options are embedded in them; their valuation and risk management require the use of stochastic models and Monte Carlo (MC) simulation techniques.
The quantities in a single contract, defined in condition of financial and actuarial uncertainty, have to be computed with great accuracy and in "adequate" time in order to effectively support companies in coming to decisions and to allow them to quickly act in an appropriate way. The complexity of the contracts and the great number of contracts in the portfolio lead to a complex overall valuation process, thus requiring both accurate and efficient numerical algorithms and *High Performance Computing* (HPC) methodologies and resources.
In order to improve accuracy, and then reliability of the estimates, we apply a change of *numéraire* in the stochastic processes for risk sources, since the flexibility of this approach can be particularly valuable in a model with stochastic interest rates. In particular, we analyse the use of the numéraire which defines the *forward risk-neutral measure*. Pricing under the forward measure can provide considerable gains in terms of accuracy, since it allows to discount at a deterministic price deflator, even though the short rate is stochastic [13]. On the other hand, pricing under the forward measure can decrease computational efficiency with respect to the classical approach based on the risk-neutral measure.
With the aim of obtaining efficient simulation processes, we use high performance computer systems. They nowadays appear the only ones able to provide the computational power needed to have timely and reliable risk estimates; for this reason, parallel computers, mostly cluster of Personal Computers and blade servers, are increasingly spreading in financial companies. Further, accurate numerical techniques that have an high computational cost can be then applied thanks to the reachable computational performances by advanced architectures. The use of these environments nevertheless strongly affects the development of numerical algorithms; a few parallel algorithms and software for financial applications are at the present available. In [8] we developed a parallel algorithm for evaluation of participating life insurance policies by using an implicit Euler scheme for the simulation of the market spot rate and the antithetic variates variance reduction technique to compute multidimensional integrals; the impact of these numerical methods in the solution of the financial problems has been investigated in [7].

Now we propose algorithms for the valuation of asset-liability portfolios of PS policies, under both risk-neutral and forward risk-neutral measure, based on the parallelization of Monte Carlo method. We point out that the algorithms exhibit different behaviors in terms of both accuracy and efficiency; thus they can be alternatively used in dependence of the specific needs.

This paper is organized as follows. In section 2 we outline the asset-liability framework for evaluation of PS policies and in section 3 we describe the stochastic processes for the risk sources. In section 4 we introduce the change of numéraire and we illustrate the mathematical framework under the forward risk-neutral measure. In section 5 we describe the developed parallel Monte Carlo algorithms and in section 6 we show the numerical results of a valuation of a real portfolio, in terms of accuracy and efficiency, obtained implementing the proposed algorithms in the ALM software described in [5]. We test both sequential algorithms based on risk-neutral and forward measure respectively, and the parallel ones implemented on a blade server with twelve processors. Finally, in section 7 we give some conclusions.

2 VALUATION FRAMEWORK

In a typical asset-liability framework, the basic elements are:

- the evaluation date t;
- the payment dates $\mathbf{t} := \{t_1, t_2, \cdots, t_m\}$;
- the stream of premiums $\mathbf{X} := \{X_1, X_2, \cdots, X_m\}$;
- the stream of benefits $\mathbf{Y} := \{Y_1, Y_2, \cdots, Y_m\}$;
- the cash-flow stream generated by the segregated fund $\mathbf{Z} := \{Z_1, Z_2, \cdots, Z_m\}$.

From the insurance company point of view, vectors $\mathbf{X},\ \mathbf{Z}$ are on the asset side, while $\mathbf{Y}$ is on the liability one. The core of the evaluation problem is the computation of the assets and the liabilities at time t, for the control of the financial equilibrium between them. Then, at time t, the value of the assets

$$V(t;\mathbf{Z}) + V(t,\mathbf{X}),$$

and the value of the liabilities

$$V(t;\mathbf{Y})$$

must be computed. For providing estimates which are reliable and market-consistent, a mark-to-market stochastic model has to be considered. In such a framework, the value

$$V_t := V(t;\mathbf{Y}) - V(t,\mathbf{X})$$

that is, the value in t of the difference between the obligations of the company and the obligations of the policyholders, gives the *market value* in t of the outstanding net liabilities of the company: then, it actually represents the amount required to the company at time t in order to meet the future liabilities. For this reason this quantity, called *stochastic reserve*, plays a crucial role for the insurance company.

In order to describe a simplified evaluation framework, we consider a single premium pure endowment insurance contract. Let the policy be written in $t_0 = 0$ for a life of age x; we denote with T the term in years of the contract and with I_t the rate of return earned by the segregated fund in the period $[t-1, t]$. According to a typical interest crediting mechanism, the benefits are readjusted at the end of year t according to

$$C_t = C_{t-1}(1+\rho_t), \qquad t = 1, \cdots, T, \tag{1}$$

where ρ_t is the readjustment rate defined as

$$\rho_t := \frac{\max\{\beta I_t, i\} - i}{1+i}, \tag{2}$$

and i is the technical interest rate, both contractually defined. The number $\beta \in (0, 1]$ is the so called *participation coefficient*: the product βI_t in (2) represents the portion of the fund return which is credited to the policyholder, the remaining portion $(1-\beta)I_t$ represents the company gain. The final benefit is given by the insured sum C_0, raised at the financial readjustment factor

$$C_T = C_0 \Phi_T. \tag{3}$$

Therefore, from (1), (2) and (3), it follows

$$\Phi_T = \prod_{t=1}^{T}(1+\rho_t) = (1+i)^{-T}\prod_{t=1}^{T}(1+\max\{\beta I_t, i\}).$$

At the purpose of discussing the evaluation of V_t, we consider, for instance, $V(t; \mathbf{Y})$ in t_0 since the generalization is straightforward. If we denote by $\varepsilon(x, T)$ the event "the aged x insured is alive at time T", then the liability of the company in T is given by

$$Y_T = C_0 \Phi_T I_{\varepsilon(x,T)},$$

where $I_{\varepsilon(x,T)}$ is the indicator function of $\varepsilon(x, T)$, expressing actuarial uncertainty. Both actuarial and financial uncertainty have to be taken into account, anyway these risk sources can be supposed to be mutually independent, thus they can be treated separately. Under the risk-neutral probability measure Q, the functional V is expressed by the conditional expectation

$$V(0, Y_T) = C_0 E^Q[e^{-\int_0^T r(t)\,dt}\Phi_T]\,{}_Tp_x, \tag{4}$$

where $r(t)$ is the spot rate and, following the usual actuarial notation, the symbol ${}_Tp_x$ denotes the technical expectation. Therefore, it is required to compute the expected value of the readjustment factor, discounted at the risk-free deflator.
For an exhaustive analysis of the asset-liability valuation framework we address to [9, 10].

3 STOCASTHIC PROCESSES FOR RISKS

This section is devoted to the description of the risk sources involved in the portfolio of life insurance policies we consider in this paper. Risk sources time evolution is modelled by stochastic differential equations, under a certain probability measure. It is well-known that any positive martingale, with initial value one, defines a change of probability measure, thus, through the Radon-Nikodym derivative we can switch between suitable probability measures. This process is usually referred to as a change of *numéraire* [1, 12], where the numéraire is a non-dividend paying asset with respect to which a probability measure is defined. In this paper, we compare the classical approach based on the risk-neutral measure to an approach based on the forward risk-neutral one. We start by introducing the models for risks in a general framework, then we deal with the specific choices of the numéraire.

The segregated fund is typically composed of stocks and bonds, thus, risk sources related to them both have to be taken into account. We refer to a market model with three sources of uncertainty: interest rate, stock market and inflation risk. In the following of this section, we denote by

$$\mathrm{d}\mathbf{W} = (dW_r, dW_p, dW_S) \tag{5}$$

a three-dimensional standard Brownian motion driving the time evolution processes of the short rate, inflation and stock market respectively.
For the interest rate risk, we refer to the one-factor CIR model. Denoted by $r(t)$ the market short rate at time t, it is assumed to follow the square-root mean-reverting diffusion process

$$dr(t) = \alpha[\gamma - r(t)]\, dt + \eta\sqrt{r(t)}\, dW_r(t) \ \ , \tag{6}$$

where $\alpha,\ \gamma$ and η are positive constant parameters, with $2\alpha\gamma > \eta^2$, that ensures the positivity of the process. The function q which gives the market price of interest rate risk is supposed to satisfy the relation

$$q(r(t), t) = \frac{\pi\sqrt{r(t)}}{\eta},$$

with $\pi \in R$ a constant parameter. Stock market and inflation uncertainties are described via log-normal processes; in particular, for the inflation risk we consider the stochastic process described in [4]

$$\frac{dp(t)}{p(t)} = y_t dt + \sigma_p dW_p(t), \tag{7}$$

where

$$y_t = y_\infty + (y_0 - y_\infty)e^{-\alpha t},$$

with $y_0,\ y_\infty$ the levels of current and long-period expected inflation respectively. Finally, we refer to the Black and Scholes model for the stock market risk

$$\frac{dS(t)}{S(t)} = (\mu - \lambda)dt + \sigma_S dW_S(t), \tag{8}$$

where λ is the dividend yield.

4 IMPROVING ACCURACY: FORWARD RISK-NEUTRAL MEASURE

The numéraire corresponding to the risk-neutral measure is the money market account

$$\beta(t, T) = e^{-\int_t^T r(u)\, du}. \tag{9}$$

In this framework, pricing at time t a security with payoff $V(T)$ requires the computation of the expected value of the payoff, discounted at a stochastic price deflator given by the related numéraire (9). The risk-neutral dynamics of the state variables (6), (7) and (8) is

$$\begin{aligned} dr(t) &= \tilde{\alpha}[\tilde{\gamma} - r(t)]\, dt + \eta\sqrt{r(t)}\, d\tilde{W}_r(t), \\ \frac{dp(t)}{p(t)} &= \tilde{y}_t dt + \sigma_p d\tilde{W}_p(t), \\ \frac{dS(t)}{S(t)} &= (r(t) - \lambda)dt + \sigma_S d\tilde{W}_S(t), \end{aligned}$$

where the risk-adjusted parameters satisfy the following

$$\tilde{\alpha} = \alpha - \pi, \quad \tilde{\gamma} = \frac{\alpha\gamma}{\tilde{\alpha}}, \quad \tilde{y}_t = y_t - \sigma_p^2.$$

The vector

$$\mathrm{d}\tilde{\mathbf{W}} = (d\tilde{W}_r, d\tilde{W}_p, d\tilde{W}_S)$$

contains the risk-neutral Girsanov transformations of the Brownian motions (5). If we consider a constant correlation among the state variables and denote by $\mathbf{L}$ the Cholesky factor of the correlation matrix, then introducing correlation has the effect of modifying $\mathrm{d}\tilde{\mathbf{W}}$ into $\mathbf{L}\cdot\mathrm{d}\tilde{\mathbf{W}}$.
The forward risk-neutral measure (the expression has been proposed by Jamshidian in [13]) is the probability measure associated to the zero-coupon bond maturing at T, with unitary face value [1, 11, 12, 13]. In this case, it can be shown that the pricing formula of a security becomes

$$V(t) = B(t,T)E^{F_T}\left[V(T)\right],$$

where $B(t,T)$ is the value in t of the numéraire bond and E^{F_T} denotes expectation under the forward measure. As a consequence, the stochastic value of liabilities in (4), under the forward measure becomes

$$V(0, Y_T) = C_0 B(0,T) E^{F_T}[\Phi_T]_T p_x. \tag{10}$$

Pricing under the forward measure can provide considerable gains in accuracy, since it allows to discount at the deterministic price deflator $B(t,T)$, even though the short rate is stochastic; indeed the forward measure is considered the right probability measure when evaluating a future random cash-flow in a stochastic interest rate environment [11].
The bond price dynamics is given, when r evolves according to the CIR model, by [12]

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt - A(t,T)\eta\sqrt{r(t)}dW_r(t),$$

where, as usual, W denotes a standard Brownian motion,

$$A(t,T) = \frac{2(e^{d(T-t)} - 1)}{(d+\alpha)(e^{d(T-t)} - 1) + 2d},$$

and $d = \sqrt{\alpha^2 + 2\eta^2}$. Applying Girsanov's Theorem, it can be shown that the process $W_r^{F_T}$ defined by

$$dW_r^{F_T} = dW_r + \eta\sqrt{r(t)}A(t,T)dt \tag{11}$$

is a standard Brownian motion under the forward measure.
The dynamics of the short rate thus becomes

$$dr(t) = \tilde{\alpha}[\tilde{\gamma} - (1 + \eta^2 A(t,T))r(t)]\,dt + \eta\sqrt{r(t)}\,dW_r^{F_T}(t). \tag{12}$$

Equation (12) can be written in the following form:

$$dr(t) = \tilde{\alpha}(1 + \eta^2 A(t,T))\Big[\frac{\tilde{\gamma}}{1 + \eta^2 A(t,T)} - r(t)\Big]\,dt + \eta\sqrt{r(t)}\,dW_r^{F_T}(t),$$

which shows that, under the forward measure, the spot rate is again a square root diffusion, but in this case both the level to which the process reverts and the speed with which it reverts are functions of time [12]. It can be proved that, in general, the change of numéraire affects only the drift of the processes ([1] - p. 30, [12] - p. 35), that is, neither the volatilities nor the

```
begin ALM procedure
% N is the number of trajectories in MC scheme
% T_end is the end of the simulation period
for i = 1 to N do
    simulate risk factors
    for each month:
        evaluate assets and liquidity
        apply the investment strategy
        revaluate the insured capitals according to (1)
    for each year:
        evaluate the balance sheet
        evaluate the statutory reserve
endfor
compute the averages over the trajectories
end
```

Figure 1: sketch of a procedure for the numerical simulation of asset-liability portfolio of PS policies.

correlation matrix are modified when moving from the risk-neutral measure to the forward one. Stochastic processes for inflation and stock-market are modified, under the forward measure, according to (11) and (12) and following a change of numéraire scheme shown in [1].
Note that the dynamics of the state variables under the forward measure depends on the bond maturity T. In the considered asset-liability framework, where the valuation of $V(0, Y_T)$, by means of (10), has to be performed at least for $T \in \mathbf{t}$, that is at least at all the payment dates, this dependence implies that the simulation process of the dynamics of state variables changes with respect to the specific date of payment. This affects the computational complexity of the numerical evaluation process, as it will be discussed in section 6.

5 IMPROVING EFFICIENCY: PARALLEL ALGORITHMS

The simulation of an asset-liability portfolio results in a large-scale computational problem. Let $t = 0$ and $t = T$ be respectively the beginning and the end of the simulation period, expressed in years. The interval $[0, T]$ is typically decomposed into K periods, the period length T/K is often equal to one month. Then, every month the investment strategy is reviewed in dependence from assets and liabilities current value; as a consequence, all the involved quantities, expressed by complex functions of random variables, have to be evaluated at each month up to year T. Moreover, at least at the end of each year the balance sheet and the stochastic reserve have to be computed. Given the complexity of the profit-sharing rule, the expectations in (4) and (10) are computed by means of Monte Carlo method; in figure 1 a simplified outline of the procedure described in [5] for asset-liability management of a portfolio is shown.

In order to develop a parallel version of the simulation algorithm that allows to gain in efficiency in high performance computing environments, we introduce parallelism in Monte Carlo method. Monte Carlo method for multivariate integration is based on the replacement of a continuous average with a discrete one over randomly selected points. Denoted by

$$I(f) = \int_{[0,1]^d} f(x)dx$$

Procedure PMC(**in:** N; p; f; id; **out:** $I(f)$)
%initialize parallel processors
%processor id, generates its local stream
generate $Z_1^{id}, ..., Z_{N/p}^{id}$
%processor id computes the average over local trajectories
`for` $i = 1, N/P$ `do`
$I_{N/P}^{id}(f) = \frac{1}{N/P} \sum_{i=1}^{N/P} f(Z_i^{id})$
`endfor`
%combine local averages
$I_N(f) = \frac{1}{P} \sum_{j=0}^{P-1} I_{N/P}^{j}(f)$
End Procedure

Figure 2: parallel Monte Carlo algorithm based on cycle parametrization technique for the generation of pseudo-random streams.

the integral of function f over the d-dimensional unit cube, then

$$I(f) \approx I_N(f) = \frac{1}{N} \sum_{i=1}^{N} f(Z_i),$$

with $Z_i,\ i = 1, ..., N$ uniform in $[0, 1]^d$. Note that in Monte Carlo evaluation of an expectation involving a stochastic process with K time periods, the resulting integral is K-dimensional [3]. In our application, we have $d = 3K$, since we consider three risk sources.
The natural strategy for parallelizing Monte Carlo method is to distribute trajectories among processors; processors work concurrently to compute the local averages that are afterwards combined to obtain the overall sample value. Therefore, Monte Carlo method is generally considered "naturally parallel". The core of MC algorithm is the generation of pseudo-random sequences capable to mimic random samples drawn from uniform distribution. The effectiveness of a parallel MC algorithm is strongly related to the underlying parallel pseudo-random generator (PPRG) as well. Generation of pseudo-random sequences in a parallel setting must deal with both inter-processor and intra-processor correlations. Parallelization schemes based on *cycle parametrization* are mostly employed. In particular, we use an parallel additive Lagged-Fibonacci generator based on a cycle parametrization scheme [17, 19].
In figure 2 we report the outline of the developed parallel MC algorithm. Communication among processors is limited to the initialization phase, where basic common information are to be exchanged, and the final phase, when partial results are to be combined to compute the global average which gives the MC method result.

6 NUMERICAL SIMULATIONS

In the following we show some of the numerical experiments we performed on a real portfolio. The asset-liability framework for a real portfolio is obviously much more complex than the one we described in section 2. Anyway, we point out that the strategies we propose here, to improve accuracy and efficiency, are related to the valuation of the risk factors and to the parallelization of Monte Carlo method - basic kernels of asset-liability models - thus allowing to apply them also to more complex situations. In order to test the performances of these strategies we integrated the algorithms that we developed in the ALM software described in [5].
We simulate a real portfolio containing about 78000 policies aggregated in 5600 fluxes. The

time horizon of simulation we consider is 40 years. The segregated fund includes about 100 assets, both bonds and equities. We solve the SDEs for the risk sources by means of the Euler method [14] with a monthly discretization step; as a consequence the dimension of the involved integrals is $3 \cdot 480$.
The valuation is performed using a single-factor CIR model calibrated on market data at December 30th 2005 ($r(t) = 0.0249$, $\alpha = 0.5739$, $\gamma = 0.0222$, $\eta = 0.0534$, $\pi = 0.2499$). Stock market volatility is $\sigma_S = 0.1$ and the dividend yield is $\lambda = 0.027$, the parameters related to inflation are $y_0 = 0.0236,\ y_\infty = 0.0211,\ \sigma_p = 0.0089$. The values of the expectation of life have been computed by means of the life tables SIM92. Finally, the correlation factor between dW_r and dW_S is set at 0.07, while we suppose inflation to be uncorrelated to the other risk sources.
We carried out our experiments on an IBM Bladecenter installed at the University of Naples ”Parthenope”. It consists of 6 Blade LS 21, each one of which is equipped with 2 AMD Opteron 2210 and with 4 GB of RAM. The implemented software is written in Fortran 90 language, using the Message Passing Interface (MPI) communication system. We use the freely available SPRNG package [18] to generate parallel pseudo-random sequences. SPRNG provides different generators, among which Additive and Multiplicative lagged-Fibonacci; in all versions, message passing relies on MPI. It allows to obtain low inter-processor correlations as well as reproducible results, both desirable properties in applications requiring the generation of pseudo-random sequences in parallel settings. Moreover, SPRNG passed all the statistical tests of the DieHard [16] and TESTU01 [15] packages.
The pseudo-random streams have been mapped to values drawn from standard normal random variables via the routine dinvnr of the package dcdflib [2], available through Netlib repository. The routine approximates the inverse normal cumulative function via Newton's method [12].

6.1 AN ANALYSIS OF ACCURACY

We start our discussion from the analysis of the impact of the change of numéraire on the accuracy of the estimates.
In table 1 we compare the 95% confidence intervals obtained in the estimation of the market value of the outstanding liabilities of the company when stochastic processes for risks are modelled under the risk-neutral measure and the forward measure respectively, for different values of Monte Carlo simulated trajectories. We denote by $\hat{V}$ the sample mean, and, as usual, by $z_{0.05/2}$ the 95% quantile of the standard normal distribution, by s the sample standard deviation, so that $s/\sqrt{N}$ is the standard error. In all the cases, we observe that the confidence intervals obtained via the forward measure are contained into the corresponding ones estimated under the risk-neutral measure; the half-width of the confidence intervals estimated in the risk-neutral setting is about ten times the half-width observed working under the forward measure, thus we gain one order of magnitudo in terms of accuracy.
In figure 3 we show the relative standard error

$$RSE = \frac{s}{\hat{V}\sqrt{N}}.$$

We observe that the estimated RSE is of order 10^{-4} when simulations are performed under the risk-neutral measure, 10^{-5} in the forward measure case. The two lines representing the RSE exhibit almost the same slope, that is, an almost constant reduction factor in RSE is observed when simulating under the forward measure, coherently to values reported in table 1. On the other hand, we recall that, as already pointed out in section 3, the dynamics of the state variables

95% confidence intervals				
N	measure	$\hat{V} - z_{0.05/2}\frac{s}{\sqrt{N}}$	$\hat{V} + z_{0.05/2}\frac{s}{\sqrt{N}}$	$z_{0.05/2}\frac{s}{\sqrt{N}}$
1000	R-N	[2,305,030,474	2,311,939,670]	3,454,598
	FW	[2,308,799,862	2,309,475,980]	338,059
2000	R-N	[2,305,212,676	2,310,184,153]	2,485,739
	FW	[2,308,871,978	2,309,341,618]	234,820
4000	R-N	[2,306,464,218	2,310,001,768]	1,768,775
	FW	[2,308,776,199	2,309,104,117]	163,959
5000	R-N	[2,306,272,289	2,309,443,908]	1,585,810
	FW	[2,308,784,101	2,309,076,514]	146,206

Table 1: Column 1: number of MC trajectories; column 2: probability measure; column 3: 95% confidence interval for outstanding company liabilities; column 4: half-width of the interval.

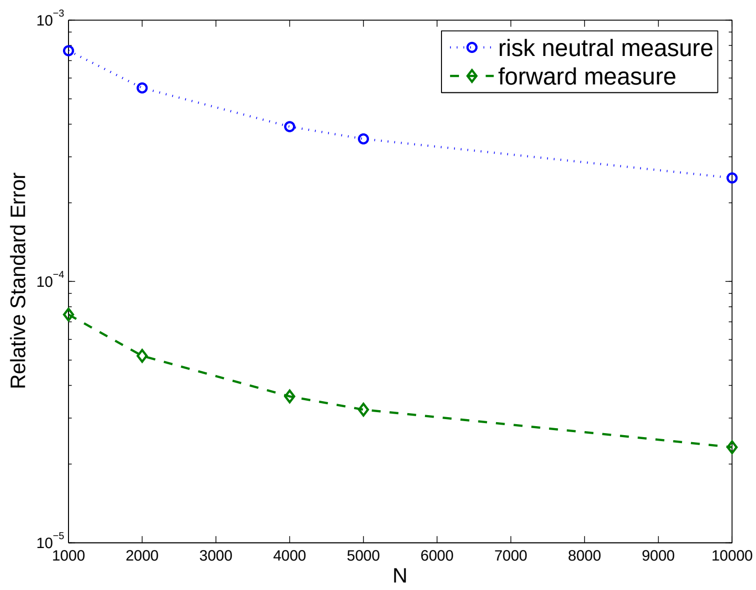

Figure 3: RSE versus number of MC trajectories.

under the forward measure depends on the bond maturity T. Therefore, for each evaluation date a different simulation has to be carried out for all the risk sources. This obviously results in a time overhead, as it can be seen in figure 4, where the execution time, expressed in hours, for different values of simulated trajectories is reported.

For a deeper analysis, in table 2 we report the values of RSE obtained in the risk-neutral setting, in the forward setting and the ratio between them, for different numbers of MC simulated trajectories. We note that, as already observed, the ratio between the RSE values is always about ten; the RSE obtained when working under the forward measure with $N = 1000$ MC trajectories is smaller than the one concerning the MC simulation under the risk-neutral measure with $N = 5000$, thus confirming that modelling risks under the forward measure results in a considerable improvement in terms on accuracy. In table 2 we also report the execution time, expressed in hours, required in the two cases and the ratio between these values, which gives the time overhead related to the forward measure approach. We note that the simulation under the forward measure almost doubles the execution time with respect to the one performed under the risk-neutral measure, but we observe that the simulation corresponding to $N = 1000$

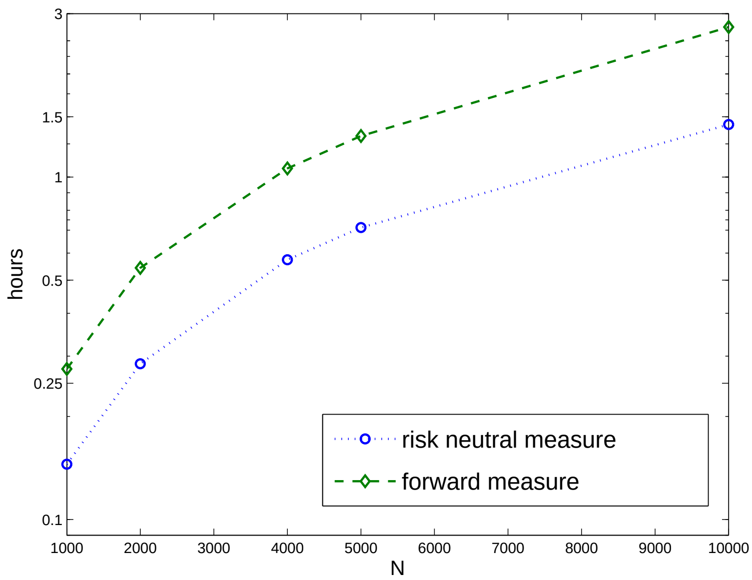

Figure 4: Execution time in hours versus number of MC trajectories.

trajectories under the forward measure provides, in about 20 minutes, more accurate estimates than the one corresponding to $N = 10000$ trajectories under the risk-neutral measure as well, which requires about one hour and a half.

	RSE			**execution time**		
N	**R-N**	**FW**	**ratio**	**R-N**	**FW**	**ratio**
1000	7.635E-04	7.469E-05	10.22	0.15	0.28	1.90
2000	5.496E-04	5.188E-05	10.59	0.29	0.54	1.90
4000	3.910E-04	3.623E-05	10.79	0.57	1.06	1.85
5000	3.506E-04	3.231E-05	10.85	0.71	1.32	1.85
10000	2.486E-04	2.310E-05	10.74	1.42	2.74	1.92

Table 2: Column 1: number of MC trajectories; column 2: RSE under the risk-neutral measure; column 3: RSE under the forward measure; column 4: RSE under the risk-neutral measure over RSE under the forward measure. Column 5: execution time in hours under the risk-neutral measure; column 6: execution time in hours under the forward measure; column 7: overhead of the forward measure approach.

6.2 AN ANALYSIS OF PARALLEL EFFICENCY

We now turn to analyse the performances of the parallel algorithms we developed. Since the change of numéraire does not affect parallel performances, we confine our discussion to the forward measure.

In figure 5 the execution time in hours versus the number of processors involved in the computation is represented. Here the global number of trajectories is fixed; that is, if N is the number of simulated trajectories and $procs$ is the number of processors involved in the computation, then each processor simulates $N/procs$ trajectories. Moreover, we report for each simulation time the corresponding RSE value. We observe that the RSE basically does not vary with respect to the number of processors, thus confirming the scalability of the chosen parallel pseudo-random numbers generator. In this case, if we fix a target accuracy for estimates, then parallelism allows to realize the target accuracy in a strongly reduced time: this is clearly meaningful to insurance companies.

In figure 6 the RSE versus the number of processors involved in the computation is shown. The results refer to simulations in which we fixed the local number of simulated trajectories: thus, for instance, the point corresponding to 4 processors in the line referring to $N = 1000$ trajectories is the value of the RSE obtained with a global number of 4000 trajectories.

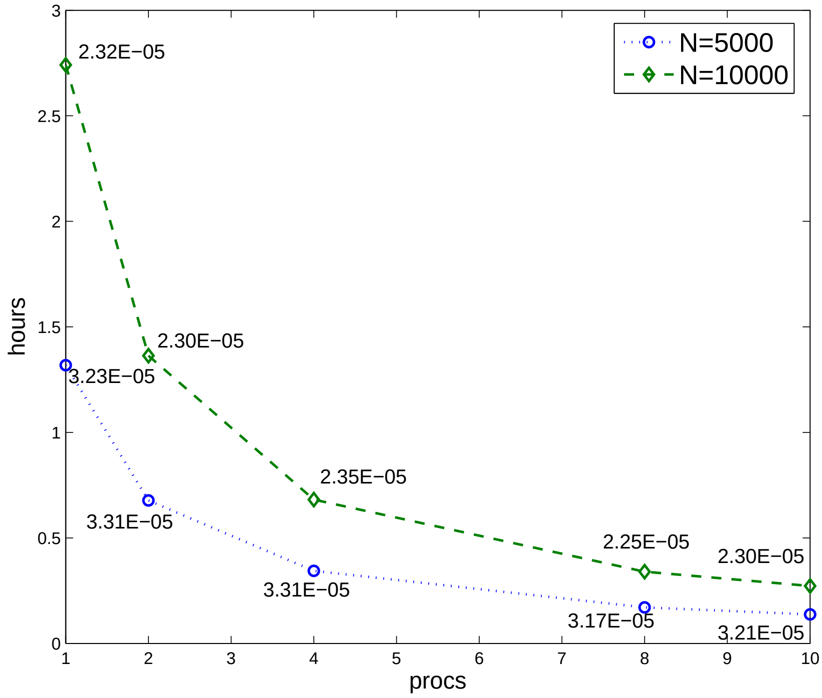

Figure 5: Execution time in hours versus number of processors involved in the simulation. The global number of simulated trajectories is fixed. For each simulation, the value of the RSE is also reported.

Moreover, we report for each estimated RSE value the related execution time in hours. Looking at the two lines, we observe that the execution times do not vary, up to two digits, with respect to processors, thus confirming the scalability of the parallel MC algorithm, and, obviously, the RSE is reduced. Therefore, if we fix a target time for responses, then, parallelism allows to improve the estimates reliability within the target time.

To evaluate the parallel performance of the algorithm, we finally show, in figure 7, the speed-up for two values of global number of simulated trajectories, $N = 5000$ and $N = 10000$. The graph reveals the good scalability properties of the algorithm. Indeed, speed-up is almost linear. The same behavior was observed in all our experiments.

7 CONCLUSIONS

The accuracy and the efficiency of a procedure for asset-liability management of portfolios of PS policies are nowadays two relevant targets for insurance companies. Here we showed two different approaches respectively based on a change of numéraire and on the use of high performance computing environments. The former, focusing on accuracy, allows to significantly reduce the standard error in the estimate of the stochastic reserve of the portfolio, as numerical simulations show; the latter improves the computational efficiency by decreasing the time of response by a factor almost equal to p on p processors. The combined use of the two ap-

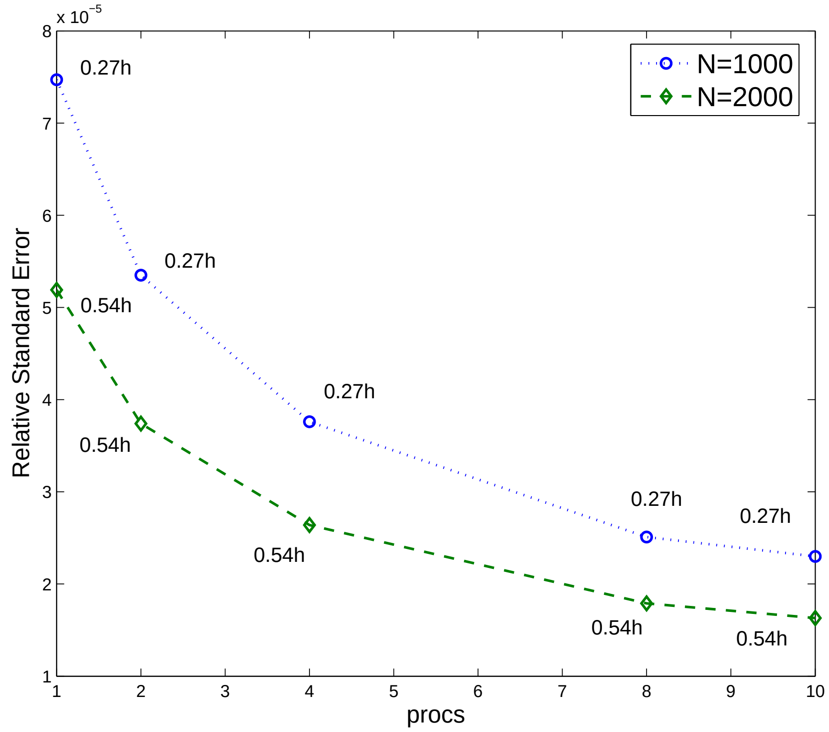

Figure 6: RSE versus number of processors involved in the simulation. The local number of simulated trajectories is fixed. For each simulation, the value of the execution time in hours is also reported.

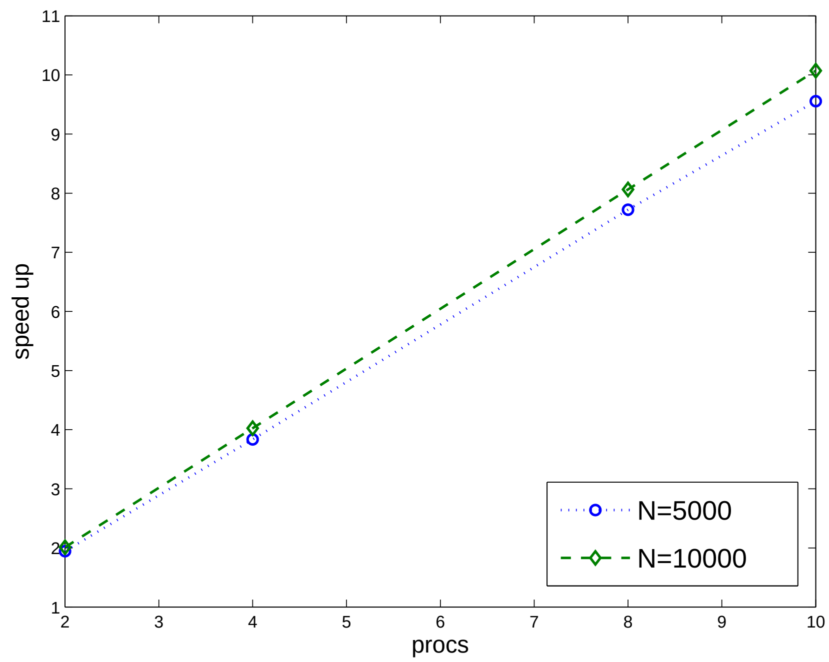

Figure 7: Speed-up versus number of processors.

proaches allows to develop parallel algorithms for the valuation of asset-liability portfolios that are accurate and efficient. We point out that advanced architectures can be used to speed-up the valuation process also in a standard risk-neutral setting; moreover they pull down the time overhead related to the forward measure based approach, thus allowing to obtain much more accurate estimates in a suitable turnaround time. The selection of one between the two probability measures can then be made on demand depending on the specific valuation needs.

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