GAUGING RISK WITH HIGHER MOMENTS: HANDRAILS IN MEASURING AND OPTIMIZING CONDITIONAL VALUE AT RISK

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ABSTRACT
The aim of the paper is to study empirically the influence of higher moments of the return distribution on conditional value at risk (CVaR). To be more exact, we try to reveal to which extent the risk given by CVaR can be estimated relying on the mean, standard deviation, skewness and kurtosis. Furthermore, it is intended to study how this relationship can be utilized in portfolio optimization. First, based on a database of 600 individual equities from 22 emerging markets of the world, factor models incorporating the first four moments of the return distribution have been constructed at different confidence levels for CVaR, and the contribution of the identified factors in explaining CVaR was determined. After that the influence of higher moments was examined in portfolio context, i.e. asset allocation decisions were simulated by creating emerging market portfolios from the viewpoint of US investors. In our analysis we compare different approaches that take higher moments into account with the standard mean-variance framework. Throughout the work special attention is given to implied preferences to the different higher moments in optimizing CVaR.

KEYWORDS
Emerging markets, Skewness, Kurtosis, Factor Model, Mean-CVaR versus Higher-Moment Portfolio Allocation

1. INTRODUCTION

There were several trials in the literature of asset allocation to incorporate higher moments of the return distribution into portfolio optimization. Markowitz (1952, 1991) in his pioneering work has used the first two moments, i.e. the mean and the standard deviation in ranking the different investment options. In the mean-variance model developed by Markowitz the mean return and the variance[1] of return served as a measure of expected profitability and risk, respectively.

The main technical advantage of the mean-variance approach comes from the fact that it only requires the estimation of the first two moments of the return distribution. This along with the estimated linear correlation coefficients between each pair of assets makes its application very effective also in portfolio context.

However, the model of Markowitz got a lot of criticism because of applying the standard deviation (and equivalently the variance) for measuring risk. It is well-known, if the return distribution is non-normal, especially negatively skewed and leptokurtic, the above mentioned approach is rather problematic. Instead of the normal distribution Eftekhari-Pedersen-Satchell (2000) mentioned the class of elliptic distribution[2] as a precondition so that standard deviation can be used as an exact measure of risk. Szegö (2002, 2005) concluded that the elliptic distribution of returns is necessary for the applicability of any risk measure which uses the linear correlation coefficient as a measure of dependence between the random returns. Szegö has also drawn attention to that if the model formulated by Markowitz is used for non-elliptic distributions one can highly underestimate those extreme events that cause the most severe losses.

Using the standard deviation as a risk measure at least one further problem emerges. As the standard deviation is a symmetric measure of risk, its application is in contradiction with the notion that the investors only regard those returns “risky” which are lower than an expected target value.

Emerging stock market returns may exhibit significant skewness and kurtosis which makes it necessary to involve these parameters in asset allocation. The idea of polynomial goal programming (PGP) was originally introduced by Tayi and Leonard (1988) to solve portfolio selection problems with skewness. Subsequently, it has been used by Lai (1991) for the portfolio of domestic assets and later by Chunhachinda et al. (1997) for internationally diversified portfolios to incorporate the first three moments of the return distribution, namely the expected return, variance and skewness into investment decision making. Davies et al. (2006) have extended PGP to the first four moments, including the kurtosis of the return distribution as well.

Usually, a multi-objective portfolio selection model incorporates a two-step procedure. In the first step, the conflicting and competing objectives are optimized independently in order to get a set of non-dominated solutions. In the second step, taking investor’s preferences for the various objectives into account a PGP problem is solved. The aim here is to minimize a polynomial containing the deviations of the different objectives from their optimum level determined in the first step as well as the particular preference parameters for the various objectives.

The main problem with the PGP portfolio selection model is that the necessity of choosing (fixing) the investor preference parameter values makes the mapping and interpreting the optimal portfolios peremptory and sophisticated. If we want to use this model to support investment decision making and ask the investor about his/her marginal rates of substitution between the different pairs of higher moments, maybe he/she seems to be more puzzled as if we go in for his/her utility function.

CVaR as a measure of risk has very attractive properties. First of all, it is a downside measure of risk, i.e. it is consistent with the intuitive notion of risk as it takes only the unfavourable part of the return/loss distribution into account. Furthermore, it is a coherent risk measure in the sense of Artzner–Delbaen–Eber–Heath (1999) axioms. It also accounts for risk beyond Value at Risk (VaR), which is especially important in case of fat tail distributions. In addition, CVaR has two favourable technical properties: it is continuous with respect to the confidence level and convex with respect to the control variables. The latter property is relevant in portfolio optimization. In order to optimize within the Mean-CVaR framework, as it was shown by Rockafellar and Uryasev (2000), one has to solve a linear programming problem. This makes CVaR very appealing in asset allocation.

Considering that the computation of CVaR is based on the return/loss distribution[3], this raises the question whether applying CVaR as a risk measure, to which extent the information given by the higher moments of the return distribution can be utilized. In order to answer this question, based on cross-section return data of 600 individual equities from 22 emerging markets of the world, factor models incorporating the first four moments of the return distribution will be constructed at different confidence levels for CVaR, and the contribution of the identified factors in explaining CVaR will be determined. Furthermore, we intend to reveal some characteristics of this prosperous risk measure, which can be utilized in portfolio optimization.

In fact, our main contribution to the literature is as follows. We show that when CVaR is optimized (minimized) we can count on some implied preferences in favor of higher skewness (and mean) and to lower kurtosis (and standard deviation). This property of CVaR makes it possible to apply the linear programming proposed by Rockafellar and Uryasev for CVaR optimization as a simple and effective alternative to PGP in portfolio allocation.

The remainder of the paper is organised as follows. Section 2 gives an insight into the methodology of conditional value-at-risk as well as higher moment portfolio optimization. The results of the empirical analysis are discussed in Section 3. First the specification of data and the research design are described. It is followed by the results provided by the different factor models at different significance levels for CVaR. Finally, the results of portfolio optimization are presented and analyzed, both on an ex post and ex ante basis. Section 4 offers some concluding remarks.

2. METHODOLOGY

2. 1. Conditional value at risk as a risk measure

For continuous loss distributions conditional value-at-risk (CVaR) on a given confidence level (α) is defined as the expected loss given that the loss (L) is higher than or equal to value-at-risk (VaR)[4] on the same confidence level:

$$CVaR_\alpha = E\{L \mid L \geq VaR_\alpha\} \tag{1}$$

The definition is more sophisticated in the case of discontinuous distributions. They are of special importance in applications such as the present study when we rely on finite sampling, i.e. the series of past returns. In these cases we should differentiate (see Rockafellar/Uryasev (2002a)) the upper CVaR ($CVaR^+$) and lower CVaR ($CVaR^-$). The difference between the two is that $CVaR^+$ measures the expected value of losses strictly exceeding VaR[5] whereas $CVaR^-$ determines the expected value of losses higher than or equal to VaR (see formula (1)).

Rockafellar and Uryasev (2002, p.1452) proved that in discontinuous cases CVaR can be expressed as the weighted average of VaR and $CVaR^+$, so with $\lambda = \dfrac{\Psi(VaR_\alpha) - \alpha}{1-\alpha}$ $(0 \leq \lambda \leq 1)$ the following equation holds:

$$CVaR_\alpha = \lambda VaR_\alpha + (1-\lambda) CVaR_\alpha^+ \tag{2}$$

where Ψ is the cumulative probability distribution of L, so that $\Psi(VaR_\alpha) = P(L \leq VaR_\alpha)$.

2.2. Optimization by conditional value-at-risk

CVaR was introduced into portfolio optimization quite recently by Rockafellar and Uryasev (2000, 2002) as an alternative to VaR.

Let $\underline{R}_1, \underline{R}_2, ..., \underline{R}_q$ [6] be a sample set of return vectors. For a particular realisation of portfolio returns, i.e. for a specific return vector the loss on a portfolio can be determined as:

$$L_{p_k} = -\underline{x}^T \underline{R}_k = -\sum_{i=1}^{n} x_i R_{k,i}$$

where $\underline{x}^T$ is the transpose of the vector of portfolio weights $\underline{x}$.

In order to identify the portfolio with the minimum CVaR, as it is shown by Rockafellar and Uryasev (2000), the following linear programming problem has to be solved:

$$\min_{\underline{x},\zeta} CVaR(\underline{x},\zeta) = \zeta + \frac{1}{q(1-\alpha)}\sum_{k=1}^{q} u_k$$

$$\text{subject to} \qquad (3)$$

$$\underline{x}^T \underline{R}_k + \zeta + u_k \geq 0,$$
$$u_k \geq 0, \;\; k = 1,2,...,q$$
$$\sum_{i=1}^{n} x_i = 1, \; x_i \geq 0, \quad i = 1, 2,..., n$$

By solving (*) we find the optimal portfolio weights ($\underline{x}^*$) as well as the corresponding VaR (ζ^*).

2.3. Higher moment portfolio optimization

According to an idea proposed by Lai (1991) the portfolio selection can be rescaled and restricted to unit variance space, namely for $\{\underline{x} \in \boldsymbol{R^n} | \underline{x}^T \underline{V}\,\underline{x} = 1\}$, where $\underline{V}$ is a variance-covariance matrix of the rates of return on the risky assets in the portfolio. This was utilized by Davies et al. (2006) in extending the polynomial goal programming (PGP) approach to the first four moments of the return distribution.

Incorporating the first four moments of the return distribution into the investment decision making process, the portfolio selection model can be formulated as follows. In the first step, the following multiple objectives have to be optimized independently:

$$Max\ Z_1 = E(\underline{x}^T \underline{R}) + x_{n+1} r_f$$
$$Max\ Z_3 = E\left[\underline{x}^T (\underline{R} - E(\underline{R}))\right]^3$$
$$Min\ Z_4 = E\left[\underline{x}^T (\underline{R} - E(\underline{R}))\right]^4$$

$$\text{subject to} \qquad (4)$$

$$\underline{x}^T \underline{V}\,\underline{x} = 1$$
$$\sum_{i=1}^{n} x_i = 1 - x_{n+1}, \quad x_i \geq 0, \quad i = 1, 2,..., n$$

where x_{n+1} indicates the proportion of money invested in the riskless asset with the promised rate of return r_f.

In the model above we separately optimize the mean return (Z_1), the skewness (Z_3) and kurtosis (Z_4), in particular we try to find the portfolio with the highest expected return (Z_1^*) in the mean-variance space and the portfolio with the highest skewness (Z_3^*) in the skewness-variance space as well as the portfolio with the lowest kurtosis (Z_4^*) in the kurtosis-variance space. In all cases we restrict our choice to unit-variance portfolios. It is important to note, that usually we do not find a single portfolio which is optimal with respect to all the three criteria. In fact, non-dominated portfolios are found for which one cannot find a more favourable portfolio in a sense that it cannot possess a higher mean return at the same level of skewness and kurtosis or a higher skewness at the same level of mean return and kurtosis or a lower kurtosis at the same level of mean return and skewness.

In the second step, given the investor's preferences $[\alpha, \beta, \gamma]$ among the different objectives the following PGP model has to be solved:

$$\begin{aligned}
&Min\ Z = (1+d_1)^{\alpha} + (1+d_3)^{\beta} + (1+d_4)^{\gamma} \\
&\text{subject to} \\
&\qquad E(\underline{x}^T \underline{R}) + x_{n+1} r_f + d_1 = Z_1^* \\
&\qquad E\left[\underline{x}^T (\underline{R} - E(\underline{R}))\right]^3 + d_3 = Z_3^* \\
&\qquad -E\left[\underline{x}^T (\underline{R} - E(\underline{R}))\right]^4 + d_4 = -Z_4^* \\
&\qquad d_1, d_3, d_4 \geq 0 \\
&\qquad \underline{x}^T \underline{V}\, \underline{x} = 1 \\
&\qquad \sum_{i=1}^{n} x_i = 1 - x_{n+1},\ \ x_i \geq 0, \quad i = 1, 2, \ldots, n\ ,
\end{aligned} \tag{5}$$

where d_1, d_3 and d_4 denote the deviations from the optimal policies derived in the first step. Hence, the objective function in the second step can be interpreted as minimisation of the deviations from each single optimal strategy. Thereby, each deviation is weighed accordingly to its preference parameter α, β and γ.

3. EMPIRICAL STUDY

3.1. Data

The data for the research were taken from the Standard and Poor's Emerging Market Database (S&P's EMDB). All in all, we relied on 600 series of equity returns from altogether 22 emerging markets of the world. Instead of local currency returns, US dollar returns have been used in each case. All variables in the cross-section regression models were calculated based on a time series of weekly returns on the above-mentioned equities. The time period comprised almost ten years, from 28 February 1997 until 31 December 2006. The emerging markets involved in the study were as follows (after the name of each country the number of equities taken from the particular stock market is presented in brackets).

Argentina (9), Brazil (35), Chile (27), China (97), Czech Republic (4), Egypt (11), Hungary (7), India (62), Indonesia (16), Israel (23), Korea (61), Malaysia (48), Mexico (23), Morocco (10), Peru (11), Philippine Islands (22), Poland (5), Russia (6), South-Africa (27), Taiwan (44), Thailand (30), Turkey (22).

For portfolio optimization the equity (price) index returns[7] of the 22 countries above were used. So, asset allocation decisions were simulated by creating emerging market portfolios from the viewpoint of US investors. This can be regarded as a usual decision making process of a hedge fund focusing on investments into emerging markets. In the out-of-sample analysis the weekly supervision of portfolio weights was allowed.

3.2. Research design

3.2.1. Testing the influence of higher moments on CVaR

The methodology for testing the influence of higher moments on CVaR was regression analysis. In the first regression model (original model) CVaR served as a resultant variable, and the expected return (E), the standard deviation of returns (σ), the skewness (s) and kurtosis (k) were used as explanatory variables.

The expected return[8], which can be estimated as the (arithmetic) average of returns in a given time period, is a typical measure of "location"[9]. In this case the location of observed values plays a crucial role in the magnitude of the measure. Based on the fact that an increase in the expected return means a decrease in expected loss (given that other conditions are unchanged), it is logical to expect that risk measured by CVaR will decrease in this case.

The standard deviation of returns is the square root of the variance. As it is well-known, the variance can be determined as the squared average of deviations of returns from the mean. As a "volatility" measure the variance or the standard deviation of the returns, respectively has been the traditional measure of risk[10]. As such, it belongs to the category of "location independent" measures, because its value is determined by the relative distance of each return observation from the mean and not by their absolute location.

The skewness[11] indicates the degree of asymmetry in the shape of the distribution function (in our case in the return distribution). If the skewness is positive the distribution function has a longer tail extending to the right (to the direction of large (positive) values) than to the other direction.[12] In the case of negative skewness just the opposite holds, i.e. the distribution function has a longer tail the left, namely to the direction of small (negative) values. Negative skewness suggests the occurrence of extreme negative returns (usually with a small probability, however). Considering that negative return can be interpreted as loss, an increase in the value of skewness – keeping other conditions unchanged – might cause a decrease in the value of risk measured by CVaR.

The kurtosis[13] intuitively refers to the fact how the different "scores" are distributed at the different parts of the distribution: namely in the center, at the tails, and between the center and the tails (in the "shoulders"). If we take the bell-shaped normal distribution function as a starting point, and we remove scores from the part between the center and the tails to the center as well as to the tails, the result is a so-called leptokurtic distribution which is thinner in the centre and thicker at the tails than the normal distribution. Based on the above-mentioned considerations, with an increase in the kurtosis – other conditions unchanged – we can expect intuitively an increase in the CVaR.

The first step was to calculate the expected return, the standard deviation of returns, the skewness, the kurtosis and the CVaR values for each stock. For these calculations we utilized the 600 time series, with 513 weekly return data in each time series. Then we run a linear regression on the cross-section data. The regression model applied can be written as follows:

$$CVaR_{\alpha} = c_0 + c_E \cdot E + c_{\sigma} \cdot \sigma + c_s \cdot s + c_k \cdot k + \varepsilon \qquad (6)$$

where c_E, c_σ, c_s , c_k are the regression parameters expressing the influence of the particular explanatory variables, the expected return (E), the standard deviation of returns (σ), the skewness (s) and the kurtosis (k) on CVaR, respectively. c_0 is the regression constant and ε denotes the error term. In addition, α serves as a notation for the confidence level chosen in calculating CVaR.

As it will turn out from the results presented in the next section, there is a significant degree of multi-collinearity in the model above (see formula 6). It is well-known that it restricts the analytical interpretation of the results, namely the regression coefficients given by the model. In particular, the main problem with multi-collinearity is that one is unable to separate the effects of the different explanatory variables in explaining the resultant variable.

In order to eliminate multi-collinearity, factor analysis was applied. However, in carrying out the factor analysis, we decided not to reduce the number of variables. Instead of doing that our intention was to be able to express the influence of the original explanatory variables on CVaR in terms of "independent dimensions". As it can be seen later, the positive impact of this endeavor has proved to be to keep the high explanatory power of original model in a new one.

The new model was built on the factors we got in the factor analysis. They were used as new explanatory variables and CVaR was kept as a resultant variable. The linear regression model containing the variables mentioned above takes the form as follows:

$$CVaR_\alpha = c_0^* + c_{F_1} \cdot F_1 + c_{F_2} \cdot F_2 + c_{F_3} \cdot F_3 + c_{F4} \cdot F_4 + \varepsilon^* \qquad (7)$$

In both models, i.e. in model (6) and (7) as well, we used a cross-sectional sample of 600 as sample size was equal to the number of equities considered.

3.2.2. Portfolio optimization

In our analysis we compare different approaches that take higher moments into account with the standard mean-variance framework. We consider the minimum variance portfolio (MVP) and the tangency portfolio (TP) [14] as well as their counterparts in the mean-CVaR framework (MCVaR, TP-CVaR)[15], each at different confidence levels (95%, 99%). Furthermore, we solve in the presence of conflicting higher moment preferences the multi-objective portfolio problem by polynomial goal programming.

Our research design is in line of previous work on international portfolio diversification. First we analyze the different portfolio strategies in an ex post setting utilizing all data in our sample for parameter estimation. This procedure abstracts from any effects that may arise from estimation risk and therefore allows us to focus on important differences or similarities among the portfolio strategies. Subsequently we take estimation risk into account and backtest our set of portfolio strategies. We use a sliding estimation window of 250 observations and allow for weekly rebalancing of portfolio weights. More precisely, at each time step we calculate from our estimation period the required parameters, then derive the optimal strategies and finally move our estimation window one time step ahead. We iterate over these steps and obtain in total 263 out-of-sample portfolio returns.

3.3. Results

3.3.1. Results of regression analysis

The results given by model (6) are summarized in Table 1. It can be observed from the table that, despite the fact that at 99 percent confidence level for CVaR the explanatory power is somewhat lower (but still around 90 percent), at both confidence levels for CVaR the explanatory power is high. In addition, all the regression coefficients are significantly different from zero at 5 percent significance level. Moreover, with the exception of the regression constant at 99 percent confidence level for CVaR, they are also significant at 1 percent level. It is worth mentioning that the signs of the regression coefficients are in agreement with the intuitive expectations outlined in the previous section. In particular, the positive sign of the regression coefficient for the standard deviation and the kurtosis indicate that an increase in the value in the respective variable − given that other conditions are unchanged − results in an increase in the value of CVaR. At the same time, the negative sign of the coefficient for the expected return as well as for the skewness refer to the fact that an increase in the value of the above-mentioned variables goes together with a decrease in the value of CVaR. This interpretation, however, has only a limited value because of the presence of multi-collinearity.

There is a high degree of multi-collinearity in model (6)[16]. The presence of multi-collinearity is already suggested by the pair-wise correlation terms between the different explanatory variables.

Table 1 Results of the multi-linear regression analysis
(at 95 percent and 99 percent confidence level for *CVaR*, respectively)

	c_0	c_E	c_σ	c_s	c_k	R^2
$CVaR_{95\%} = f(E,\sigma,s,k)$	0.011^{**}	-0.998^{**}	1.989^{**}	-0.014^{**}	0.001^{**}	**96.8%**
$CVaR_{99\%} = f(E,\sigma,s,k)$	0.006^{*}	-1.210^{**}	3.053^{**}	-0.044^{**}	0.003^{**}	**89.9%**

Note: ** Significant at 1% level
* Significant at 5% level

The correlation matrix is presented in Table 2. The high correlation between the skewness and kurtosis is the most conspicuous with the value which is above 0.8. At the same time, the correlation terms between the mean and the standard deviation, the standard deviation and the skewness as well as that one between standard deviation and the kurtosis are also not negligible (the values are approximately 0.45, 0.46 and 0.33, respectively).

Table 2 The correlation matrix of the explanatory variables

	Mean (E)	*S.D.* (σ)	*Skewness* (s)	*Kurtosis* (k)
Mean (E)	1.000	0.445	0.068	0.000
S.D. (σ)		1.000	0.464	0.329
Skewness (s)			1.000	0.831
Kurtosis (k)				1.000

The most important results of the factor analysis, which was carried out with the purpose to eliminate multi-collinearity, are presented in Table 3 and Table 4. For extracting the relevant factors Principal Component Analysis (PCA) was applied, and as a method for rotation varimax with Kaiser-normalization was used.[17]

Table 3 Total variance explained

Factor	*Eigenvalue*	*Variance explained (%)*	*Cumulative variance explained (%)*
F_1	1.689	42.217	42.217
F_2	1.024	25.591	67.808
F_3	1.014	25.355	93.163
F_4	0.273	6.837	100.000

As it can be seen from Table 3, in the four-dimensional space determined by the explanatory variables of model (6), 42% of the total variance is due to the first, 26% to the second, 25% to the third and 7% to the fourth factor, respectively. Despite the fact that only relatively low proportion of the total variance can be explained by the fourth factor, we decided to keep it with the intention to build a new regression model. As it was already emphasized earlier, this was motivated by the purpose to keep the high explanatory power of the original model.

The factors can be identified based on the rotated component matrix (see Table 4). The correlation terms in the matrix suggests that the first factor embodies the joint effect of the skewness and kurtosis. The standard deviation is predominantly represented by the second, while the mean is by the third factor, respectively. The fourth factor shows a noteworthy correlation with the skewness only (0.516), so it seems obvious to identify it as a factor expressing the effect of the skewness.

The results given by the new regression model, which was built on the factors provided by the factor analysis, are summarized in Table 5 (see model (7)).[18] In fact, similarly to model (6), two versions were considered, one with 95 percent and a second one with 99 percent confidence level for CVaR, respectively. The table shows not only the regression coefficients but also the components of the explanatory power which are attributable to the different factors. The decomposition of the explanatory power in the new model is possible only because of the (linear) independence of factors. The mathematical consequence of this, on one side, is that the value of the regression coefficient belonging to a specific factor is independent from those ones of the other factors, and, on the other side, the explanatory power component of each factor does not change by the inclusion or exclusion of different factors into and from the model.

Table 4 Rotated component matrix

	F_1	F_2	F_3	F_4
Mean (E)	-0.014	0.218	0.976	0.009
S.D. (σ)	0.209	0.943	0.248	0.080
Skewness (s)	0.815	0.263	0.018	0.516
Kurtosis (k)	0.990	0.135	-0.016	-0.021

It is observable from Table 5 that all the regression coefficients of both versions of model (7) are significant at 1 percent level. In addition, in coincidence with our purpose declared earlier, it can also be recognized that the explanatory power is the same as that of the versions of model (6).[19]

Table 5 Results of the factor model
(at 95 percent and 99 percent confidence level for *CVaR*, respectively)

	c_0^*	c_{F_1}	c_{F_2}	c_{F_3}	c_{F_4}	R^2
$CVaR_{95\%} = f(F_1, F_2, F_3, F_4)$	0.131**	0.004**	0.035**	0.007**	-0.003**	**96.8%**
Explanatory power component due to the given factor	-	**1%**	**91.3%**	**3.8%**	**0.7%**	
$CVaR_{99\%} = f(F_1, F_2, F_3, F_4)$	0.197**	0.011**	0.052**	0.011**	-0.015**	**89.9%**
Explanatory power component due to the given factor	-	**3.4%**	**76.9%**	**3.5%**	**6.1%**	

Note: ** The respective parameter is significant at 1% level.

Based on the results presented in Table 5, we can draw the conclusion that predominantly the second factor, which represents the effect of the standard deviation, is responsible for the volatility in the value of CVaR (in particular at 95 percent confidence level for CVaR the explanatory power component attributable to it is 91.3%, while at 99 percent level the respective value is 77%). It is not so striking given that the standard deviation is also a risk measure. Therefore, it is understandable that the factor dominated by the standard deviation is highly correlates with the risk measured by CVaR. At 95 percent confidence level for CVaR in the magnitude of the explanatory power component the second factor is followed by the third factor which is dominated by the mean, with a much lower contribution than that of the second factor, however (3.8%). At 99 percent confidence level the fourth factor, in which the effect of the skewness is "condensed", has the second highest contribution (6.1%). The first factor, which embodies the joint effect of the skewness and kurtosis, together with the skewness dominated fourth factor have contributed to the explanatory power with only about 2% at 95 percent confidence level for CVaR, but with almost 10% at 99 percent confidence level!

Based on the results shown above it can be concluded that with an increase in the confidence level for calculating CVaR, the explanatory power component of those factors dominated by the skewness and kurtosis increases. As it is well-known the magnitude of skewness and kurtosis is related to the non-normality characteristics of the distribution.

3.3.2. Ex-post portfolio analysis

Table 6 presents the higher moment- and CVaR-metrics of 10 different ex post portfolio optimization strategies. Among the mean-variance portfolio allocation strategies, the minimum-variance (MVP) as well as the mean-variance tangency portfolio (TP) was considered. The CVaR counterparts of these portfolios, namely the minimum-CVaR (MCVaR) and the one with the same mean as the mean-variance tangency has (TP-CVaR) were also examined both at 95 and 99 percent confidence level, respectively. For the sake of comparability, among the higher moment portfolios special attention was given to different combinations of zeros and ones for the preference parameters of the mean (α), skewness (ß) and kurtosis (γ), respectively.

Table 7 shows the portfolio weights for the 10 different strategies considered.

It is remarkable that the CVaR values presented in the sixth and seventh rows of Table 6 for each strategy were calculated based on the empirical distributions of the relevant equity index returns, while those shown in the eighth and ninth rows were computed as if the returns had been normally distributed. The confidence level applied for calculating CVaR is indicated in the sub-index of CVaR. Please note that both in the case of MCVaR and TP-CVaR strategies the sub-index refers to the confidence level which was applied in optimizing CVaR.

Table 6 Higher moment- and CVaR-parameters of 10 different portfolio allocation strategies

	MVP	$MCVaR_{95\%}$	$MCVaR_{99\%}$	TP	TP – $CVaR_{95\%}$	TP – $CVaR_{99\%}$	[1 1 1]	[1 0 0]	[0 1 0]	[0 0 1]
Mean (%)	0.27	0.27	0.29	0.35	0.35	0.35	0.32	0.35	0.29	0.27
S.D (%)	1.66	1.71	1.80	1.89	1.95	1.98	2.47	1.88	2.08	2.29
Skewness	-0.34	-0.09	-0.13	-0.38	-0.09	-0.21	-0.17	-0.40	0.13	-0.27
Kurtosis	4.43	4.59	4.00	4.88	4.59	4.40	3.57	4.89	5.11	3.21
$CVaR_{95\%}$ (%)	3.57	3.43	3.81	4.05	3.44	3.73	5.14	4.03	4.11	4.86
$CVaR_{99\%}$ (%)	5.88	5.81	5.34	6.11	5.57	5.89	7.19	6.09	7.11	6.29
$CVaR^*_{95\%}$ (%)	3.16	3.25	3.43	3.54	3.68	3.74	4.78	3.53	4.00	4.46
$CVaR^*_{99\%}$ (%)	4.16	4.28	4.52	4.68	4.85	4.93	6.27	4.66	5.25	5.84

Note: * denotes the CVaR for normally distributed returns.

It is observable in Table 6 that the parameters of the mean-variance tangency portfolio (TP) and those of the [1, 0, 0] are almost identical. However, considering that in the later case there is neither preference for a higher skewness nor a lower kurtosis in model (5) was set up, higher moment optimization is expected to provide the portfolio with the highest excess return over the risk-free rate to unit of variance. It is exactly the TP in the mean-variance context. Therefore, the slight difference between TP and [1, 0, 0] is due to rounding errors in numerical portfolio optimization (see Table 6 and also the portfolio weights presented in Table 7).

It seems logical to compare and contrast those strategies which result in the same mean return because the mean return indicates the average profitability of a given strategy. The MVP has 0.27 percent weekly return on average, the same as the $MCVaR_{95\%}$ as well as [0, 0, 1]. In the later case the only preference in model (5) was given to reduce the kurtosis. Indeed, [0, 0, 1] possesses the lowest observed kurtosis value (3.21) among the three strategies. Its skewness (-0.27), however, is between that of the MVP (-0.34) and $MCVaR_{95\%}$ (-0.09). The standard deviation is the smallest for the MVP (1.66 percent weekly), as it must be the case ex post, considering that it is the portfolio with the lowest possible variance. The standard deviation of the $MCVaR_{95\%}$ is only slightly higher (1.71 percent weekly). The standard deviation is the highest for [0, 0, 1] with the value of 2.29 percent. The risk measured by CVaR is also the highest in the latter case (4.86 at 95 percent, and 6.29 at 99 percent confidence level, respectively). The portfolio weights given by MVP are much more similar to those of $MCVaR_{95\%}$ than to those of [0, 0, 1] (see Table 7). The explanation of this fact might be that at the same return level there is only a small difference in the standard deviation as well as the kurtosis values of these two strategies. By applying $MCVaR_{95\%}$, however, proved to increase the skewness compared to the MVP.

$MCVaR_{99\%}$ and a pure-skewness-preference [0,1,0] portfolio have 0.29 percent mean return. At the same time, in the earlier case both the standard deviation (1.8 percent) and the kurtosis (4.00) are lower than the respective parameters of the latter one (2.08 percent and 5.11).

Those “worsening” values are compensated by the positive skewness value of [0,1,0]. It is 0.13 while the skewness of the $MCVaR_{99\%}$ is negative (-0.13).

There are four strategies in Table 6 possessing the mean return 0.35 percent per week. As it has already been mentioned [1,0,0] converges to TP. Therefore, it is enough to compare TP, TP-$CVaR_{95\%}$ and TP-$CVaR_{99\%}$[20]. Both TP-$CVaR_{95\%}$ and TP-$CVaR_{99\%}$ have higher standard deviation and lower kurtosis than TP. It is noteworthy that they also have higher (but still negative) skewness than the TP. In sum, CVaR optimization seems to increase the skewness and decrease the kurtosis compared to mean-variance optimization.

It is also remarkable that [1,1,1] portfolio has a slightly lower mean return (0.32 percent) than the above-mentioned four strategies.[21] However, the standard deviation of return (2.47 percent) is much higher in this case than that of the TP-$CVaR_{95\%}$ portfolio (1.95 percent). This goes along with lower skewness (-0.17) as well as with a lower kurtosis value (3.57). The skewness of the TP-$CVaR_{95\%}$ is -0.09 while its kurtosis is 4.59.

Table 7 Portfolio weights provided by 10 different portfolio allocation strategies

	MVP	$MCVaR_{95\%}$	$MCVaR_{99\%}$	TP	TP – $CVaR_{95\%}$	TP - $CVaR_{99\%}$	[1, 1, 1]	[1, 0, 0]	[0, 1, 0]	[0, 0, 1]
ARG	0	1.98	0	0	0	0	0	0	4.03	0
BRA	0	0	0	0	1.17	1.17	0	0	0	0
CHI	15.29	9.78	0	0	0	0	0	0	0	0
CHN	0	0	0	0	0	0	0	0	0	0
CZE	1.45	1.10	0	10.93	9.69	9.69	0	9.88	0	0
EGY	8.04	8.55	0	11.43	9.74	9.74	42.81	10.17	0	30.69
HUN	0	0	0	0	0	0	0	0	0	0
IND	5.44	0	0	6.50	0	0	0	6.74	0	0
IDN	0	0	0	0	0	0	0	0	0	0
ISR	13.02	10.00	24.47	0	0	0	0	0	12.55	0
KOR	0	0	0	2.71	5.10	5.10	0	1.20	0	8.18
MAL	4.28	9.10	3.95	0	0	0	0	0	0	6.57
MEX	0	0	0	9.00	0.28	0.28	0	10.15	0	0
MOR	40.02	48.76	45.95	40.13	51.04	51.04	9.59	41.54	82.49	9.67
PER	9.33	10.72	24.49	14.66	16.71	16.71	47.60	15.02	0	22.11
PHI	1.64	0	0	0	0	0	0	0	0	0
POL	0	0	0	0	0	0	0	0	0	0
RUS	0	0	0	3.92	6.29	6.29	0	2.86	0	0
SAF	0	0	0	0	0	0	0	0	0	0
TAI	1.48	0	0	0	0	0	0	0	0	22.79
THA	0	0	0	0	0	0	0	0	0	0
TUR	0	0	1.14	0.73	0	0	0	2.44	0.93	0

By observing the portfolio weights presented in Table 7 one can draw the conclusion that [1,1,1] is the least diversified portfolio among all of the 10, including only three countries to invest in.

All in all, based on the results presented above, CVaR optimization seems to offer a compromise between mean-variance and higher moment optimization in the sense that some implicit preference is given to create a portfolio with a higher skewness and a lower kurtosis than in the case of its mean-variance counterpart.[22]

In the following section we are going to sketch the pair-wise relationship of the first four higher moment metrics by tracking out the mean-variance as well as mean-CVaR efficient portfolios in the coordinate systems determined by the different pairs of higher moments.

Figure 1 shows the pair-wise combinations of the different higher moment metrics for the mean-variance and the mean-CVaR efficient portfolios. In the later case three different confidence levels, 90, 95 and 99 percent, were applied, respectively.

As it can be seen in part (a), the mean-standard deviation space does not present striking results, considering that it is the traditional space of metrics applied by Markowitz in his mean-variance efficiency analysis. In fact, as it should be, at the same return level the mean-variance efficient portfolios have always a lower standard deviation than those of their mean-CVaR efficient counterparts. At the same time, despite the fact that there is a tendency for the standard deviation to increase with higher confidence level, there is not a clear-cut order which can be observed between the standard deviation values of the mean-CVaR efficient portfolios belonging to different confidence levels.

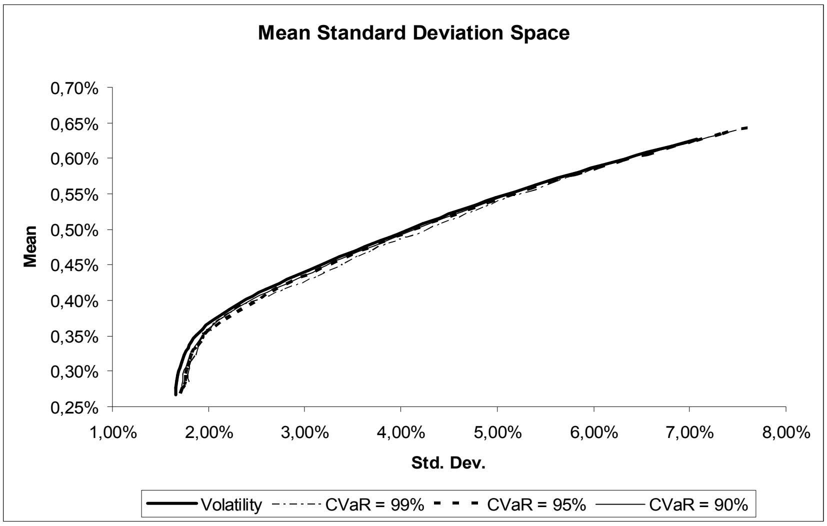
Based on the graphs shown in part (b), one can draw the conclusion that at the same return level minimizing the risk measured by CVaR goes along with creating a portfolio possessing a higher skewness than that of the one minimizing the risk measured by the standard deviation.[23] To put it another way, optimizing CVaR seems to support the investors' preference for higher skewness.

However, the graphs presented in part (c) do not reveal the general ability of CVaR optimization to reduce kurtosis, i.e. to facilitate the preference of the investors for a lower kurtosis. To be more exact, in this respect the results are somehow mixed. At 99 percent confidence level, at the same return level, the kurtosis of the CVaR optimum portfolio tends to be lower than that of its mean-variance efficient counterpart. At the same time, just the opposite holds at 95 as well as at 90 percent level. The results regarding the relationship between the kurtosis and standard deviation for mean-variance and mean-CVaR efficient combinations are mixed alike (see part (e) of Figure 1). Indeed, keeping the value of the standard deviation constant, we can only report on a lower kurtosis value compared to that of the corresponding mean-variance efficient portfolio at 99 confidence level. At 90 and 95 percent levels the kurtosis of the CVaR optimal portfolio is tending to be higher than that of the mean-variance efficient portfolio with the same standard deviation of returns.

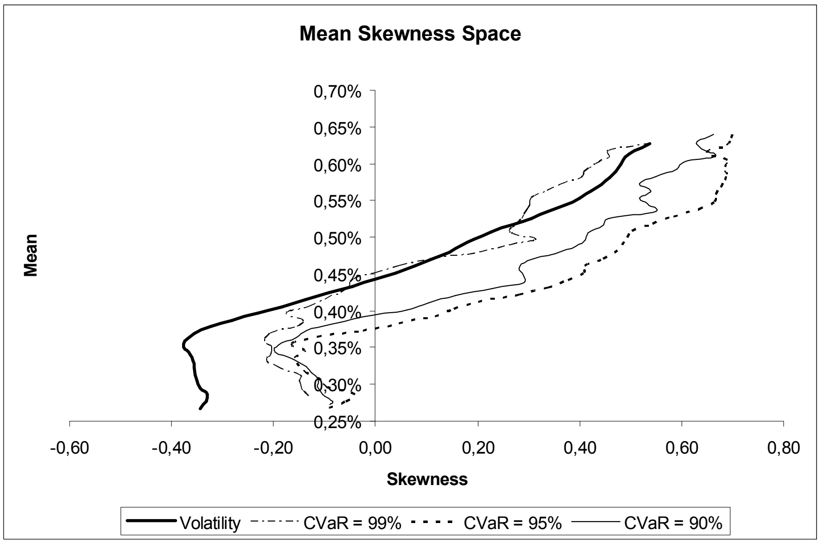
By looking at part (d) of Figure 1, we can observe that at the same level of the standard deviation the CVaR optimal portfolios tend to possess higher skewness than that of their mean-variance efficient counterparts. However, the above mentioned property does not hold for the full range of standard deviations in case of CVaR optimum portfolios at 99 percent confidence percent level.

Finally, by observing part (f) of Figure 1, it is remarkable that optimizing CVaR has a tendency to decrease the kurtosis of the respective portfolio as compared to its mean-variance counterpart while keeping the skewness constant. This has proved to be true within the whole range of skewness values in the case of CVaR portfolios at 99 percent confidence level. At 90 and 95 percent level, however, for higher than (about) 0.4 skewness level the kurtosis of the CVaR optimum portfolio is higher than that of the correspondent mean-variance efficient one

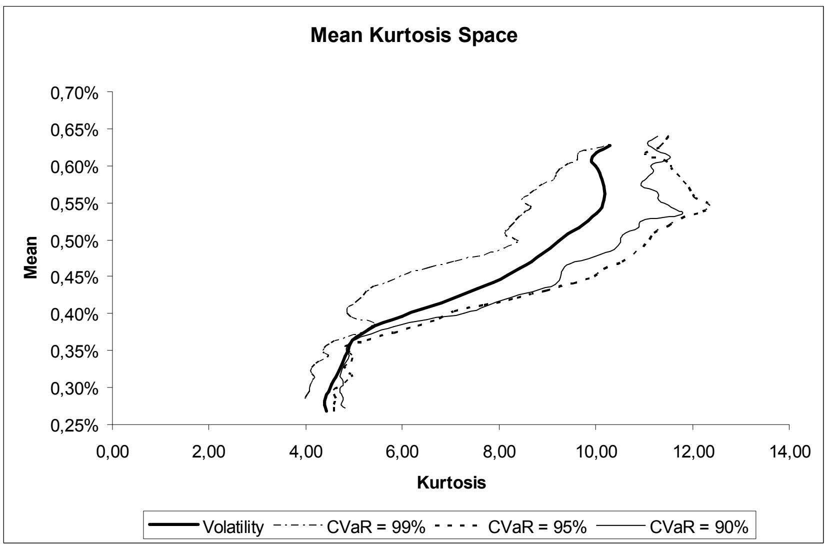
Figure 1 Mean-Variance and Mean-CVaR efficient portfolios plotted in the 6 different spaces determined by the pair-wise combinations of higher moment metrics



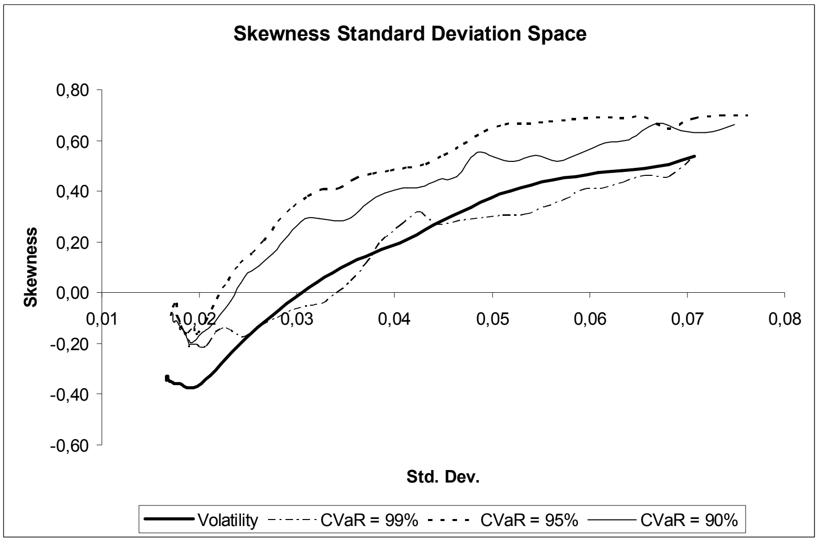
(a)



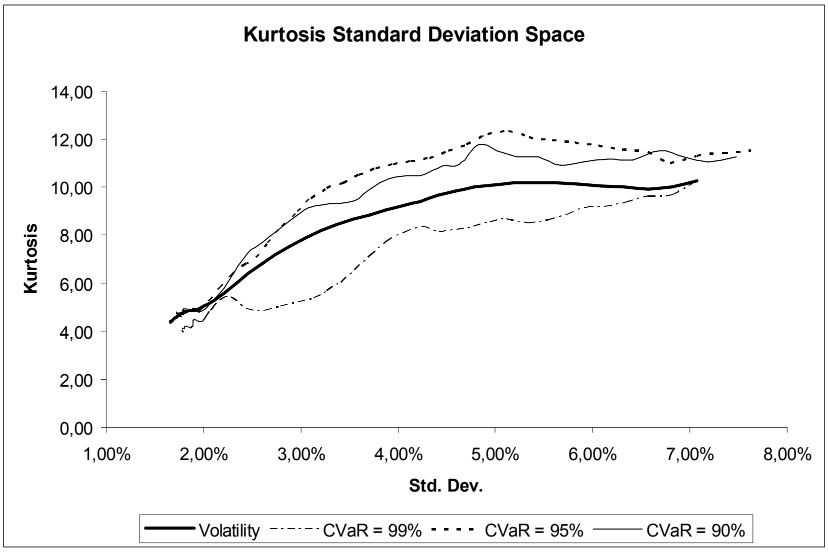
(b)



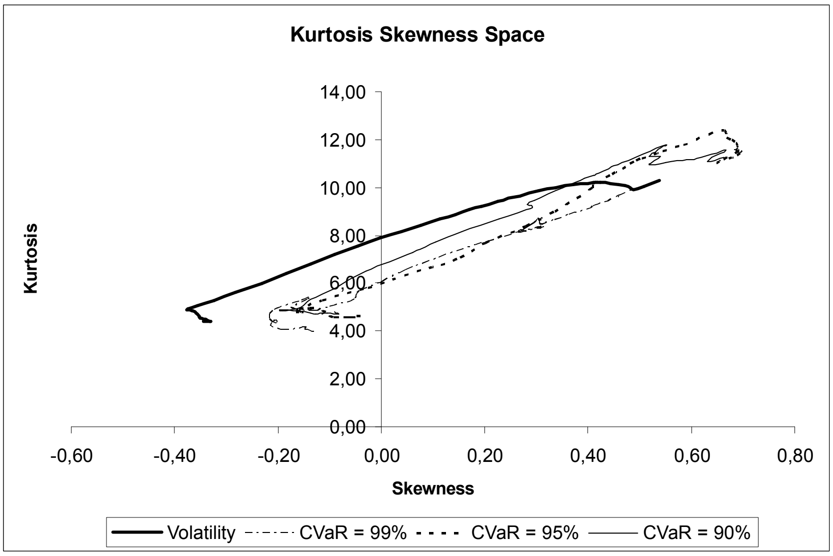
(c)



(d)



(e)



(f)

Note: CVaR efficient portfolios were calculated at three different confidence levels, 90, 95 and 99 percent, respectively.

The effects analyzed above are summarized in Table 8. A "+" / "– " sign in a particular cell of the table indicates an increase / a decrease in the respective variable of the CVaR optimal portfolio at the particular confidence level depicted in the appropriate column as compared to that of the same parameter of mean-variance efficient portfolio under the condition of keeping the variable shown in the appropriate row constant. A "0" refers to the cases when there was not a clear-cut tendency in the order of the respective values, i.e. for certain values of the higher moment parameter presented in the row the particular parameter shown in the column was lower in the case of the CVaR portfolio than in the mean-variance case, while for other values it was higher.

Table 8 Comparison of CVaR optimum portfolios and their mean-variance counterparts in terms of different pairs of higher moment metrics[24]

	Standard deviation			Skewness			Kurtosis		
Keeping constant	$CVaR_{90\%}$	$CVaR_{95\%}$	$CVaR_{99\%}$	$CVaR_{90\%}$	$CVaR_{95\%}$	$CVaR_{99\%}$	$CVaR_{90\%}$	$CVaR_{95\%}$	$CVaR_{99\%}$
Mean	+	+	+	+	+	0	+	+	–
Standard Deviation				+	+	0	+	+	–
Skewness							0	0	–

Note: In the comparison mean-variance efficient portfolios were used as benchmarks.

3.3.3. Out-of-sample portfolio analysis

The results of the out-of-sample analysis are shown in Table 9 and 10. All in all, two different mean-CVaR optimization strategies were compared to that of their mean-variance efficient counterparts. In fact, here the same strategies considered in the ex post part were examined, namely the MVP, TP, MCVaR as well as TP-CVaR. The optimization of CVaR at 99 percent confidence level, however, was not possible because of the number data has not proved to be enough for this purpose.

Table 9 presents the average in-sample values of the mean, the standard deviation, the skewness, the kurtosis and CVaR at 95 as well as at 99 percent confidence level for each of the four strategies. The respective in-sample values were calculated for all of the 263 optimum portfolios created by each strategy, and then the average value of each parameter was computed.

As it should happen, the in-sample standard deviation is lower for the mean-variance strategies than for their mean-CVaR counterparts. At the same time, as it also should be, the CVaR is lower for the latter than for the earlier strategies. These parameters go along with a higher level of skewness and a lower kurtosis in the case of CVaR optimization. Indeed, the average in-sample skewness value is 0.18 for the $MCVaR_{95\%}$ and it is -0.08 for the MVP, while the kurtosis values are 3.57 and 3.72, respectively. The respective skewness value is -0.04 for the $TP\text{-}CVaR_{95\%}$ and -0.15 for the TG, while the kurtosis is 3.88 for the earlier one and 4.05 for the latter one.

Table 9 Comparison of the average in-sample values of four different portfolio optimization strategies

	MVP	$MCVaR_{95\%}$	TP	TP - $CVaR_{95\%}$
Mean (%)	0.16	0.19	0.57	0.57
S.D (%)	1.46	1.54	3.20	3.25
Skewness	-0.08	0.18	-0.15	-0.04
Kurtosis	3.72	3.57	4.05	3.88
$CVaR_{95\%}$ (%)	3.06	2.85	6.62	6.50
$CVaR_{99\%}$ (%)	4.33	3.95	9.52	9.25

Table 10 summarizes the out-of-sample values of the four higher moment parameters as well as those of CVaR at 95 and 99 confidence levels for the different strategies studied. The values shown in the table were calculated based on the series of realized (out-of-sample) returns[25] given by the four portfolio optimization strategies.

Table 10 The out-of-sample values of four different portfolio optimization strategies

	MVP	$MCVaR_{95\%}$	TP	TP - $CVaR_{95\%}$
Mean (%)	0.54	0.54	0.53	0.59
S.D (%)	1.60	1.79	3.04	3.06
Skewness	-0.44	-0.70	-0.25	-0.24
Kurtosis	4.50	5.11	5.08	4.86
$CVaR_{95\%}$ (%)	3.33	3.96	6.51	6.42
$CVaR_{99\%}$ (%)	5.00	6.11	9.56	9,38

It is remarkable that the out-of-sample mean return on the MVP and $MCVaR_{95\%}$ (0.54 percent per week) has proved to be about three times as high as those of the average in-sample values (0.16 and 0.19 percent weekly). It has coupled with comparatively the same standard deviation but much lower negative skewness (-0.44 and -0.70, respectively) than those of the in-sample counterparts (-0.08 and 0.18). At the same time, higher kurtosis values could also be registered (4.5 and 5.11 versus 3.72 and 3.57). In case of the TP as well as TP-$CVaR_{95\%}$ the out-of-sample mean return and standard deviation values were about at the same level as the respective in-sample averages. In both cases the skewness for the distribution of the realized returns, however, turned out to be negative and lower than the in-sample average values (-0.25 and -0.24 versus -0.15 and -0.04, respectively). In addition, the kurtosis values are also higher (from 4.05 and 3.88 to 5.08 and 4.86, respectively).

By analyzing the results presented in Table 10 further, it is worth to mention that $MCVaR_{95\%}$ has resulted in a more negatively skewed distribution of the realized returns than MVP, while possessing the same level of mean return and a slightly higher standard deviation of return. The CVaR values (i.e. value of the CVaR both at 95 as well as 99 percent confidence level) were also higher for the $MCVaR_{95\%}$ strategy than those of the MVP. It indicates the superiority of the MVP over the $MCVaR_{95\%}$ with respect to all parameters. However, the mean return for TP-$CVaR_{95\%}$ (0.59 percent per week) was higher than that of the TP (0.53 percent weekly) and the standard deviation was almost the same for the two strategies (3.06 versus 3.04 percent per week). Furthermore, the skewness was also about the same in the case of TP-$CVaR_{95\%}$ (-0.24) as it was in the case of the TP (-0.25), while the kurtosis and the CVaR values were more favorable in the earlier case than in the latter one. Therefore, regarding the out-of-sample performance TP-$CVaR_{95\%}$ seems to be more appealing than the TP.

All in all, if one had to choose among the four strategies based on their out-of-sample performance, MVP would appear to be the best strategy to apply.

4. CONCLUSIONS

In the paper we intended to reveal some characteristics of a prosperous risk measure, the conditional value at risk (CVaR), which can be utilized in portfolio optimization. In particular, the main aim was to study in which extent the CVaR is determined by the moments of the return distribution and which consequences this relationship has in portfolio allocation.

First, the relationship between the conditional value at risk (CVaR) and the first two central moments of return distribution, namely the mean and the standard deviation as well as the skewness and kurtosis, which can be generated from the third and the fourth moment, was studied empirically. We relied on a cross-section database including 600 equities from 22 emerging markets of the world. The method applied was linear regression combined with factor analysis. Eventually, a factor model was constructed in order to eliminate multi-collinearity from the original model.

Then portfolio optimization was performed. On an ex post basis different approaches that take higher moments into account were compared with the standard mean-variance framework. We considered the minimum variance portfolio (MVP) and the tangency portfolio (TP) as well as their counterparts in the mean-CVaR framework (MCVaR, TP-CVaR), each at different confidence levels (95%, 99%). In addition, we solved in the presence of conflicting higher moment preferences the multi-objective portfolio optimization problem for different sets of preferences. As a part of the ex post analysis, the pairwise comparison of the different higher moment metrics of the mean-variance and the mean-CVaR efficient portfolios were also made.

Finally, the out-of sample performance of two different mean-CVaR optimization strategies were compared to that of their mean-variance efficient counterparts. In fact, here the same strategies considered in the ex post part were examined, namely the MVP, TP, MCVaR as well as TP-CVaR.

For portfolio optimization the equity (price) index returns of the 22 emerging stock markets above were used. Asset allocation decisions were simulated by creating emerging market portfolios from the viewpoint of US investors. This can be regarded as a usual decision making process of a hedge fund focusing on investments into emerging markets. In the out-of-sample analysis the weekly supervision of portfolio weights was allowed.

The conclusions of the study can be summarized as follows:

- The explanatory power of the factor model built on the factors given by principal component analysis as explanatory variables and CVaR as a resultant variable proved to be very high for both confidence levels for CVaR. Furthermore, all the regression coefficients were significant at 1 percent level. However, the explanatory power of the above-mentioned factor model has decreased with an increase in the confidence level for calculating CVaR.

- For the volatility in the value of CVaR the factor conveying the effect of the standard deviation has predominantly proved to be responsible. At the same time, it is remarkable that the strength of the influence of this factor has decreased as the confidence level for CVaR has increased. In addition, parallel to the decrease in the effect of the factor dominated by the standard deviation, the effect of the factors dominated by the skewness and kurtosis, i.e. those factors representing the non-normality characteristics of the distribution has increased as a result of an increase in the confidence level for CVaR.

- It has been shown in the ex post case that minimizing CVaR can be regarded as a substitute for higher moment portfolio optimization. It can be explained by the implied preference for a higher skewness (and mean) and a lower kurtosis (and standard deviation). Indeed, it became obvious from our empirical analysis that the portfolios in the mean-CVaR framework clearly trade mean variance efficiency for more skewness and less kurtosis. In other words, optimizing CVaR seems to support the investors' preference for higher skewness and lower kurtosis.

- In an ex ante setting, however, the results are less appealing. In fact, the minimum variance portfolio seems to be the most robust portfolio optimization strategy. Considering and comparing the values of the first four higher moments, except for skewness, it dominates all other strategies. We assume that the main reason for such disturbances in ex ante settings is parameter uncertainty.

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ENDNOTES

[1] Within this framework the standard deviation can be regarded as an equivalent risk measure with the variance.

[2] See Joe (1997) and also Embrechts-McNeil-Straumann (2002).

[3] Since the loss is the negative return, the loss density function is the mirror image of return density function on the y-axis.

[4] See Jorion (1997) and Frey-McNeil (2002) for details.

[5] $CVaR_{\alpha}^{+} = E\{L | L > VaR_{\alpha}\}$

[6] The number of elements in the sample set equals the number of the return observations in the time series of returns, while the dimension of the vectors is equal to the number of assets in the portfolio.

[7] The price index of each country comprised the same individual equities which were used in the cross-section analysis.

[8] Empirically it is often referred to as the mean return.

[9] This expression is used by Pflug (1999, p.1) as he differentiates measures of “dispersion” (such as the variance) and measures of “location” (such as the expected value).

[10] Cf. Eftekhari-Pedersen-Satchell (2000).

[11] $s = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^3}{\sigma^3}$

[12] In this case the mean is higher than the median.

[13] $k = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^4}{\sigma^4}$

[14] For a detailed description of these strategies see e.g. Eun-Resnick (1994).

[15] Actually, the TP-CVaR is not the tangency portfolio in the mean-CVaR space, but the portfolio that provides at the same expected return level as the tangency portfolio the lowest CVaR.

[16] The multi-collinearity was tested by χ^2 test. The value of the test statistics has proved to be $\chi^2 = 436.326$ while the critical value at 5 percent significance level is $\chi^2_{0,05;6} = 12.592$ (the degree of freedom is 6).

[17] The factor analysis was performed by SPSS.

[18] There is an option in the SPSS which makes possible to save the factor loadings as new variables for further analysis. Relying on this we got 600 values for each of the 4 factors.

[19] For the sake of comparison see Table 1.

[20] By comparing TP-CVaR$_{95\%}$ and TP-CVaR$_{99\%}$, we can report on a slightly higher standard deviation, lower skewness and kurtosis values in the case of TP-CVaR$_{99\%}$.

[21] The difference between the mean returns is not so big even in annual terms, as the respective yearly returns are around 16.6 and 18.2 percent.

[22] Our intention was to determine the values of the preference parameters given the weights of a certain portfolio, such as the MCVaR and TP-CVaR. The values of the preference parameters turned out to be very sensitive to small changes in the portfolio weights, so we have not found stable solutions.

[23]Please note that the above-mentioned property does not completely hold for the CVaR optimal portfolios at 99 percent level of confidence.

[24] See also Figure 1 and the interpretation above.

[25] Each series consisted of 263 return data, formulating a particular empirical distribution of realized returns.