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MODERN PORTFOLIO THEORY - SOME ACTUARIAL PROBLEMS

PAR / BY

Alasdair D. WILKIE

Grande - Bretagne / United Kingdom

THEORIE MODERNE DU PORTEFEUILLE - QUELQUES PROBLEMES ACTUARIELS

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A. D. WILKIE, R. WATSON & SONS, ACTUARIES CONSULTANTS, REIGATE (ANGLETERRE)

RESUME

L'article conteste certaines des hypothèses et méthodes de versions courantes de la théorie moderne du portefeuille. Il traite d'abord de l'application du critère de variance moyenne à des distributions lognormales. On montre que, dans une perspective d'espérance normale et logncxmale, et en envisageant les mêmes moyennes et variances, le choix fondé sur l'utilité maximale espérée dépend de la forme particulière de la fonction d'utilité, et donc que le critère de variance moyenne ne peut pas être utilisé à la place du critère d'utilité maximale espérée.

Lorsque Son compare des espérances lognormales différentes, des résultats de Lévy montrent que le critère adéquat n'est pas la variance moyenne, mais le coefficient moyen de variation. Ce résultat peut être décrit géométriquement dans le plan E - .

Dans la seconde partie de l'article, on examine le problème des devises multiples, et on suggère que le modèle de fixation des prix d'actifs immobilisés n'est pas applicable. Les devises peuvent suivre différentes lignes du marché financier, et il est possible que tous les titres ne reposent pas sur la même ligne de ce march6

Bien que la discussion soit conduite en termes de devises multiples, les mêmes résultats s'appliquent dans un temtoire unique, si différents investisseurs ont différents types d'engagements, par exemple des engagements financiers et "réels". Il est possible, voire probable, que pour certains investisseurs, il n'existe pas de placement sans risque.

Modern Portfolio Theory - Some Actuarial Problems

BY A. D. WILKIE

SUMMARY

The paper questions certain of the assumptions and methods of the popular versions of modem portfolio theory. It first discusses the application of the mean - variance criterion to lognormal distributions. It is shown that, when a normal and a lognormal prospect, with the same mean and variance, are considered, the choice based on maximum expected utility depends on the particular form of utility function, and hence the mean-variance criterion is not a substitute for the maximum expected utility criterion.

When different lognormal prospects are compared, results of Levy show that the appropriate criterion is not mean-variance, but mean - coefficient of variation. This result can be described geometrically in the E - σ plane.

In the second pan of the paper the problem of multiple currencies is considered, and it is suggested that the capital asset pricing model cannot apply. Different capital market lines may exist for each currency, and it is possible that not all securities lie on one capital market line.

Altough the discussion is in terms of multiple currencies, the same results apply within one territory, if different investors have different types of liability, for example money and "real". It is possible, indeed likely, that no risk - free investment exists for certain investors.

1 - INTRODUCTION

Modern portfolio theory has become the classical paradigm of academic financial economists - particularly in the United States. Many well written text books put forward the theoretical and practical background for interested students, and there are very many articles in many journals that have explored many parts of the theory. The concepts of "efficient portfolios", "beta factors" and "risk - adjusted returns" have entered into the conversation of many investment managers.

Although the methods of modern financial economics greatly resemble actuarial methods, actuaries, at least in Britain, have been reluctant to take them into their intellectual portfolio. This may be because of misunderstanding or of sheer conservatism ; but it may also be because of an inherent feeling that some of the conclusions of modern financial economists do not ring true. The mean-variance criterion is perhaps not thought to be a satisfactory method for selecting efficient portfolios and the capital asset pricing model is not thought to be a satisfactory way of pricing individual securities.

My purpose in this paper is to draw attention to two aspects of modern portfolio theory, to indicate why I feel that the classical model is not wholly satisfactory, and to suggest areas for further research. I do not come to conclusions, but suggest topics for discussion, and while the points I shall raise do not not seem to be covered in the usual textbooks, they may well have been discussed in articles of which I am unaware, in addition to those to which I refer. I have not attempted a comprehensive review of the literature to seek such articles out.

I first discuss, in Section 2, certain aspects of normal and lognormal distributions in relation to portfolio selection and the mean - variance criterion. Some of this has been selected from the references cited, and I am comfortable with the conclusions. In Section 3 I raise a problem in relation to the Capital Asset Pricing Model, which I describe as a problem of "multiple currencies", but which is really a problem of multiple types of liability. This has particular application in the actuarial investment field. In combination the results of Sections 2 and 3 suggest that a new - actuarial - look at Modem Portfolio Theory is desirable.

2 - NORMAL AND LOGNORMAL DISTRIBUTIONS

2. 1 The mean - variance criterion for portfolio selection is justified in the usual textbooks on the basis of one of two possible assumptions : returns on various securities over the single time - period considered are jointly normally distributed, and investors have increasing and concave utility functions ; or investors have quadratic utility functions, and the returns on possible investments have a finite upper limit not greater than the peak of the utility function. The utility function $u(x) = - (x-b)^2$ is only increasing for $x < b$, so it is unrealistic if the total return on the portfolio could exceed b.

Provided the time - period considered is sufficiently short, either of these assumptions may be satisfactory. Over short periods return distributions may be approximately normal, and portfolios formed as the weighted sum of such securities will have a distribution that approaches the normal. This is true even if the return distributions of individual securities are not normal, provided that there is a reasonably large number of securities included, and their variances are limite. Alternatively, it is reasonable to assume that over any reasonably short period return distributions have a finite upper limit.

Such a model may suit short - sighted investment managers, but they are not satisfactory for far-sighted actuaries, for whom the time horizon may be the lifetime of existing policies of an insurance company or the lifetime of existing members of the pension fund. Even if the actuary is prepared to limit his outlook to somewhat less than the final demise of the last survivor, he is unlikely to be satisfied with a time horizon less than 10 or 20 years away. Over such a period retum distributions are much better represented by a lognormal than by a normal distribution.

Over a very short period - a day, a week - it may not be possible to distinguish between normal and lognormal distributions of returns on investments. Over any long period - a year or m a e - the empirical evidence is that return distributions are positively skewed, and the distribution of the logarithm of total return is approximately normal. There is a clear rationale f a this : the return on an investment wer any long period is the product of the returns over the shorter periods comprising that long period. Just as the sum of a sufficiently large number of random variables (whether they are normally distributed or not) is itself normally distributed, so the product of a large number of random variables.

(whatever their distribution) tends towards the lognormal ; provided always that we are dealing with distributions with finite variances.

It looks as if we may be heading towards a paradox: whatever the short-period returns of individual securities, their long - period returns tend to be distributed lognormally. The short - period return on a portfolio consisting of the weighted sum of a large number of individual securities should be approximately normal. The long - period return on the portfolio can be approached in two ways : if it is seen as the weighted sum of the long - period returns on the individual securities forming it, it should tend towards normality, if the number of securities is sufficiently large ; however, seen as the product of the short-period returns on the portfolio, it should tend towards lognormality, if the number of periods is large enough.

A resolution of this apparent paradox has been suggested by Levy (1977). If the portfolio is re - balanced at the end of each short period, then it is appropriate to consider its long - period return only as the product of its successive short - period returns. If, further, the return distributions for each security are the same in each period, and the weightings of each security in the portfolio are kept the same at each rebalancing, then the distribution of returns on the portfolio is the same in each period However, the general conclusion does not seem to rely on this identity.

Some such result is necessary in order to keep us within the practicable mean-variance framework. Otherwise the problem is this : a random variable formed from the sum of a small number of normally distributed variables is itself normally distributed. However, if the constituent parts are lognormally distributed, the resulting sum is distributed neither normally nor lognormally, but as the convolution of a number of lognormals.

As will be discussed in Section 2.2 below, an investor's choice between a prospect whose return is normally distributed and another prospect with identical mean and variance whose return is lognormally distributed depends on the investor's utility function, and no generalisations can be made. The mean-variance criterion is insufficient. Similarly. if we are comparing several portfolios , whose returns are each distributed as the convolution of lognormals, then the investor's choice is dependent on the form of his utility function, and again no generalisations can be made. It is only if we can assume that portfolio returns are also lognormally distributed, like their constituent securities, that we can make progress. The choice between different lognormal securities and lognormal portfolios formed from them is discussed in Section 2.3

2.2 We now consider how a utility - maximising investor approaches the investment decision when faced with two prospects, one of which offers a normal distribution of returns. the other of which offers a lognormal distribution of returns, both with the same mean and variance.

Consider two investments, or prospects, called N and L. The proceeds of N are a random variable, X, which is normally distributed with mean μ and standard deviation σ. Thus $E(x) = \mu$ and $Var(x) = \sigma^2$. Let the proceeds of L be the random variable Y, which is lognormally distributed, with the same mean and standard deviation as X.

Let Z = log Y. Then Z is normally distributed, and we shall denote its mean by M and standard deviation by S. It is well known that, given the values of M and S, then, for the lognormal distribution:

$$\mu = e^{M+S^2/2}$$
$$\sigma = \mu\,(e^{S^2} - 1)^{1/2}$$

It is convenient to defme the "coefficient of variation", C, given by

$$C = \sigma/\mu = \{e^{S^2} - 1\}^{1/2}$$

from this we &rive

$$S = \{\log(1 + C^2)\}^{1/2}$$
$$M = \log\mu - \log(1 + C^2)/2$$

Let the investor have a utility function, u(x), which consists of two upwards - sloping straight line segments which meet at a point. Since utility functions which are linear transforms of each other have the same effect, we can, without loss of generality, take:

$$u(x) = \begin{cases} x & x > c \\ x - d(c - x) = (1 + d)x - dc & x < c, \text{ with } d > 0 \end{cases}$$

Such a utility function is continuous for all values of x, is upward sloping, and is concave. It therefore represents a possible utility function for a risk - averse investor, even though the derivative, $u'(x)$, is not defined at $x = c$.

(Incidentally, the popular class of hyperbolic absolute risk aversion (HARA) utility functions *(see,* for example, Miiller, 1988) has the inconvenience that most of its members are not defined over the whole real line. The only one that is: $u(x) = -\exp(-ax)$, which seems less mathematically tractable for the present purpose than my "kinked linear" utility function).

It is convenient to introduce the following notation. Let N (.) be the distribution function of the unit normal distribution.

Let

$$F(x) = N((x - \mu)/\sigma)$$

be the distribution function of the normally distributed random variable X.

Let

$$f(x) = (2\pi)^{-1/2}\sigma^{-1}\exp(-(x - \mu)^2/2\sigma^2)$$

be the density function of X.

Let

$$G(y) = N((\log y - M)/S)$$

be the distribution function of the lognormally distributed random variable Y.

Further, let

$$H(y) = N((\log y - M - S^2)/S)$$

Then the expected utility for N can be shown to be given by

$$EN = E_N(u(x)) = \mu + d(\mu - c)F(c) - d\sigma^2 f(c)$$

and the expected utility for L can be shown to be given by

$$EL = E_L(u(x)) = \mu + d\mu H(c) - dcG(c)$$

If $d = 0$ so that the utility function everywhere is $u(x) = x$ then the expected utilities are equal, $EN = EL = \mu$, and the investor is indifferent between them.

For $d > 0$, both expected utilities are diminished (unless $c \leq 0$, in which case the value of EL is unchanged).

However, the effect on the normal and lognormal prospects is different and depends on the values of μ, σ, d and c. In every case, if c is small relative to μ, then EL > EN, and the lognormal prospect is preferred. This is most obviously seen if we put $c = 0$, since then EL is unaffected, and EN is diminished. For such an investor the prospect of ending up with a liability instead of an asset is particularly undesirable, and the lognormal prospect gives no possibility of this.

In every case, if c is large relative to μ, then EL < EN. This too can be seen intuitively: if the kink is at a very high value. then utility function over most of its range is given by

$$u(x) = (1 + d)x - dc,$$

ie mostly a single straight line. For this investor, however, particularly high returns are relatively uninteresting, and these are more likely to happen with the lognormal than with the normal prospect.

If d, μ and σ are held constant, then there is a critical value of c, say c*, such that

EL > EN	$c < c^*$
EL < EN	$c > c^*$

The value of c* is given by the solution of

$$(\mu - c)F(c) - \sigma^2 f(c) = \mu H(c) - cG(c)$$

which does not depend on d. A little calculation shows that c* is proportionate top, so it can be expressed as

$$c^* = \mu . r(C)$$

where r depends only on the coefficient of variation,

$$C = \sigma / \mu.$$

Numerical solution of this equation gives the following values of r for given values of C

C	r
0,01	1,0001
0,02	1,0002
0.05	1,0015
0,1	1,0058
0,2	1,0232
0,5	1,1397
1,0	1,5062
2,0	2,6203
3,0	4,0038
4,0	55438
5,0	7.1024

The preceding analysis demonsuates that the choice between a normal and lognormal prospect, which have the same mean and standard deviation, depends upon the utility function of the investor. It would be unsafe to draw conclusions about utility functions other than the kinked linear one considered here, but the results of this investigation suggest that, if the investor emphasises to himself the disadvantages of returns well below the mean, then he will prefer the lognormal prospect, whereas if he is relatively unattracted by returns well above the mean, then he will prefer the normal prospect.

2.3 In this section we discuss comparisons between normal and lognormal prospects with different means and variances, and we compare two lognormally distributed prospects. We are now able to generalise for the class of utility functions, U, that can represent the preferences of the risk - averse investor, that is, they are increasing, $u'(x) \geq 0$, and concave, $u''(x) \leq 0$, where both inequalities are sharp for at least one value of x each. (This excludes linear or constant utility funcrions).

We say that prospect A dominates prospect B if the expected utility under A is not less than the expected utility under B, ie

$$E_A\, u(x) \geq E_B\, u(x),$$

for all utility functions in the class U, with at least one inequality being sharp.

It has often been shown hat, if the distribution of returns for prospects A and B are both normal, then A dominates B if

$$E_A(x) \geq E_B(x),$$
$$Var_A(x) \leq Var_B(x),$$

with at least one of the inequalities being sharp. In words : if the means are equal, A is preferred if it has the lower variance ; if the variances are equal, A is preferred if it has the greater mean ; and A is preferred if it has both a greater mean and a smaller variance than B.

A similar result for the lognormal distribution has been proved by Levy (1973). He shows that, if the distributions of two prospects, A and B, are both lognormally distributed, then A dominates B for utility functions in U if

$$E_A(x) \geq E_B(x)$$
$$C_A(x) \leq C_B(x)$$

with at least one of the inequalities being sharp, where $C_A(x)$ is the coefficient of variation of A, ie

$$C_A(x) = E_A(x) / \{Var_A(x)\}^{1/2}$$

The effect of this is most easily denionsuarcd by using a diagram in the E - a plane. In Figure 1 (a), which applies to normally distributed prospects, prospect A dominates prospects in the shaded region marked B_N, including the vertical and horizontal edges. In Figure 1 (b), which applies to lognormal prospects, prospect A dominates prospects in the larger shaded region B_L, including the left - hand sloping edge OA, and the horizontal line to the right of A.

(Figure 1 about here)

The effect of this on the "efficient frontier", the locus of portfolios which are not dominated by any other portfolio, has been discussed by Levy (1977) and can be demonstrated as shown in Figure 2. We now assume that portfolios have been formed in the following way. If security returns are normally distributed, we construct portfolios as the weighted sum of individual securities, held over the whole time - period. Thus portfolio returns are also normally distributed. Where the portfolio returns are lognormally distributed, we assume that the portfolio of securities is re - balanced sufficiently frequently during the total time - period, as discussed in 2.1, for the portfolio distributions to be lognormal.

(Figure 2 about here)

In each case we consider the region of feasible portfolios in the E - σ plane. In the normal case this region is bounded by a hyperbola, as shown in Figure 2 (a). The upper boundary of this region gives the locus of efficient portfolios, from the "nose" of the hyperbola, A, upwards and to the right.

The lognormal case is shown in Figure 2 (b). The feasible region has a similar shape, but the locus of efficient portfolios is given by the upper boundary of the region upwards and to the right of point B, where B is the point at which a straight line from the origin is tangential to the boundary of the feasible region.

The consequence of this result is simply that the mean - variance criterion for selecting efficient portfolios can be used in the lognormal case in almost the same way as in the normal case. The only difference is that, when we come to select optimal portfolios for any particular investor, we should eliminate from the efficient set those portfolios that lie on the segment AB of the boundary. The analysis is even simpler when a risk - free security exists, ie a security for which $\sigma = 0$. In this case the boundary of the feasible region becomes two straight lines, and the "nose" becomes a point on the axis, $\sigma = 0$, at the point where $E = r$, the return on the risk - free security.

In the normal caw the point A has moved to represent the risk - free security, and the locus of efficient portfolios is the straight line upwards and to the right of A. In the lognormal case the points A and B coincide, and the boundary is the same in the two cases.

Although in the classical presentation of the portfolio selection problem, the existence of a risk - free security is assumed, it is my contention that such an assumption is unwarranted for most long - term investors, because they are interested in real rather than nominal returns. I shall return to this point at the end of Section 3.

3 - MULTIPLE CURRENCIES

3.1 In the classical presentation of modern portfolio theory the first step is usually the presentation of the portfolio selection decision for the individual investor. In a development of this, it is assumed that the investor has a clear set of securities to choose from, is prepared to work with a single time period, and can attribute an expected value and variance of return to each security over that time period, along with a matrix of covariance terms. It is assumed that he uses the mean - variance criterion for portfolio selection, and a feasible region and efficient frontier in the E - σ plane is developed.

It is then usually assumed that one of the securities has zero variance, so is described as "risk - free", and it is shown that the efficient frontier collapses to a straight line in the E - σ plane.

In applying such a model, the individual investor can take account of his own situation and his own beliefs. He can allow for his own tax position, allow for transaction costs in the light of his existing holdings, choose his own time - period, make his own estimates of the expected returns, variances and covariances of the securities in which he is interested, and do the calculations in his own currency.

In the development of the Capital Asset Pricing Model (CAPM) the differences between investors are generally assumed not to exist. It is commonly assumed in the presentation of the CAPM that there are no taxes (or at least that all investors are subject to the same taxes) and no transaction costs, that all investors have the same time period, have the same beliefs about expected returns and variances, use the same mean-variance criterion, have no restrictions on holdings of particular assets, all of which are supposed to be infinitely divisible (unlike single large properties). and that there exists one risk-free security, which is the same for all investors. All this implies that all investors work in the same currency.

On the basis of all these assumptions, and perhaps more, the CAPM is developed, along with concepts such as the market portfolio, and the capital market line. It is shown that in equilibrium the expected returns on all securities are a linear function of the "beta factor", the regression factor which connects the return on the security with the return on the market portfolio.

It is my contention that, although the portfolio selection paradigm is extremely useful, the CAPM is not a satisfactory representation of the real world, and the concept of one capital marker line is an illusion. One could enumerate many objections to the assumptions of the CAPM : the existence of differential taxes on investors , restrictions on asset ownership, non - divisibility of certain assets, etc. A telling criticism would be the existence of different beliefs by different investors about the values of expected returns, variances and covariances. After all. stock markets do not flourish on equilibrium, but precisely because active investors hold different beliefs, one thinking that a security is too expensive and worth selling and another thinking that it is too cheap and worth buying. However, the existence of different beliefs is difficult to prove, and it could be argued that the state of the market at any time represents some sort of "average market belief", even if it does not represent the views of all investors.

The point I want to discuss in this section, however, is that of multiple currencies. I discuss it in these terms, because I believe that this makes it easily understood. But the same principles apply because investors have different liabilities that they need, or may wish, to take into account.

3.2 Consider two countries, A and B. Assume that the currencies in use in these countries are called dollars and pounds respectively. At any time, t. the exchange rate between the currencies is given by $x(t)$, the number of dollars exchangeable for one pound. An alternative exchange rate could be given by $1/x$, the number of pounds per dollar.

We shall consider a single time - period. At thc bcginning of thc time period the currencies arc at par, ie x (0) = 1. The value of thc cxchangc rate at the end of the time period is a random variable, lognormally distributcd, such that log (x (t) / x (0)) is normally distributcd with zero mean and standard dcviation o . Putting it in this way has the advantage that the change in the logarithm of the rcciprocal exchange rate has the same valuc with changed sign, and hence is also lognormally distributed with the same parameters (but changed sign).

The propcrtics of thc lognormal distribution givc us thc cxpcctcd value and variance of the exchangc rate at the end of the period :

$E(x) = e^{\sigma^2/2}$

$Var(x) = e^{\sigma^2}(e^{\sigma^2}-1)$

The expecte4value and variance of the rcciprocal exchange rarc arc idcntical and are

$E(1/x) = e^{\sigma^2/2}$

$var(1/x) = e^{\sigma^2}(e^{\sigma^2}-1)$

Thcy arc equal bccausc we have takcn thc mcan as zcro. If thcrc were an expected "driit" in the exchange rate, so that the expcctcd value of log x was μ, thcn there would bc additional tcrms in e^{μ} and $e^{-\mu}$ in thc above cxprcssions.

Now imagine that a risk - free security, such as Trcasury bills, for thc appropriate period, exists in each currency, and that each carries thc samc intcrest rate (strictly force of intcrcst) 6. The expcctcd retum at thc end of thc pcriod (for example o r year, but we nccd not bc so prccise) on dollar Treasury bills, mcasurcd in dollars, is e^{δ} with variance zcro. Similarly, thc cxpected return in pounds on pound Treasury bills is e^{δ} with variance zero.

But, bccausc of thc possiblc changc in thc cxchangc ratc, thc expcctcd rcturn in dollars on pound Trcasury bills is :

$e^{\delta + o^2/2}$

with variancc :

$e^{\sigma^2}(e^{\sigma^2}-1)$

Likcwisc, thc mcan and variancc of thc rcturn, nicasurcd in pounds, on dollar Treasury bills arc thc samc as mcan and variancc of thc rcturn, nicasurcd in dollars, on pound Trcasury bills. This symmctry relics on thc cquality of thc intcrcst rate, and the assumption that thc expcctcd drift of thc cxchangc ratc is zcro. Suitablc adjustments would be nccdcd if thcsc assumptions wcrc not appropriatc.

Now imaginc that thcrc is a markct in ordinary sharcs in cach couny, and, continuing our symmctrical approach, imaginc that thc rcturn on sharcs in cach country, in its own currcncy, is lognormally distributcd with the mcan and standard deviation of the log rctum bcing M and S. Let us furthcr assumc, pcrhaps unrealistically, that the returns on the share portfolios are not correlated, and that ncithcr is correlated with changes in the exchangc rate. In practice such corrclations are likcly to exist. The expcctcd retum on sharcs in each country in thc domcstic currency is thcrcfore

$e^{M + S^2/2}$

and thc variancc is

$e^{2M + S^2}(e^{S^2}-1)$

However h e expected rctum and variance of shares in each country, as measured in the currency of the other are given by

$$e^{M + S^2/2 + \sigma^2/2}$$

and

$$e^{2M + S^2 + \sigma^2}(e^{S^2 + \sigma^2} - 1)$$

If we label the possible investments as follows

P dollar Trcasury bills
Q pound Treasury bills
R dollar shares
S pound shares

We may find a pattern show such as shown in Figures 3 (a) and 3 (b) which show the sccuritics in the E - o diagram for each currcncy. Note that although the position of the points is the samc in cach diagram, the labelling is diffcrcnt. Trcasury bills (P and Q) are rcvcrscd, and ordinary shares (R & S) are rcvcrscd.

(Figure 3 about here)

It would now bc possible to continue this line of rcscarch by using specimen numerical values, dcvcloping the region of fcasible portfolios and the boundary line of efficient portfolios, for each currency separately. But it is clear that different results will be obtained (except perhaps in very special cases) in cach of the two different currencies. Two diffcrcnt capital market lines will be produced, and it is quite possible that the "foreign" investments will not appear on the capital market line of a particular currency, but lie below it.

The theory of the CAPM suggests that, in equilibrium, prices of sccuritics should move so that they arc all taken up in equal proponions by all invcstors. But it is quite possible that share prices in the two currcncics cannot move so as to become equally attractive to investors in the other currency. It is possible that an equilibrium situation can be found in which dollar invcstors do not include pound Treasury bills in their portfolios at all, and similarly pound invcsrors do not include dollar Trcasury bills in their portfolios at all, even if rhcy may include somc ordinary sharcs from the other currency. Alternatively, it is possible that the exchange risks sccni sufficiently large that ordinary sharcs in the opposing currcncy arc not atrracdvc.

In effect, market segmentation can exist. This is similar to the markct segmentation that can appear when diffcrcnt invcsrors have different tax positions. For example, British Government securities with a high coupon rate (nominal percentage rate of interest), are relatively attractive to "gross" investors (those who pay no tax), whereas similar sccuritics with a low coupon rare are rclativcly attractive to those invcstors who have a high tax rate ("net" investors). The markct is actively traded and arbitrage opportunities are difficult to find. When expected returns arc calculated on a gross tax basis, then high coupon stocks appear rclativcly attractive, and low coupon stocks unattractive. When calculated on a net tax basis, the position of these stocks is rcvcrscd.

Although I have talked in terms of diffcrcnt "currcncics", the samc argument can apply within the markets of one country, provided that some investors measure in units of nominal currcncy, and other invcsrors measure in units of "real" currency (ie adjusted to maintain constant purchasing power, or measured in terms of a retail prices index).

In the United Kingdom the British Government has issued two types of Government stock : "conventional" ones, where interest payments and redemption amounts are fixed in terms of a specific number of pounds ; and "index - linked" stocks where the interest and redemption amounts are adjusted proportionately to changes in the official Retail Prices Index for an appropriate period.

For investors who are interested in money returns, index - linked stocks, whose value may depend on the uncertain changes in the Retail Prices Index, are relatively risky, whereas for those who are interested in a particular real return, index - linked stocks form a relatively risk - free investment, and nominal stocks are relatively risky. Each of these classes of investor also has the opportunity to invest in other classes of investment, such as ordinary shares. But the nominal investor will measure returns on shares in nominal terms, and the real investor will measure in real terms. Their assessment of the relative attractiveness of different investments may therefore be different.

These remarks indicate that the conventional Capital Asset Pricing Model, as described in the usual textbooks at least, is an unsatisfactory description of securities markets where different investors have different approaches to what constitutes risk - free - ness. In order to find equilibrium solutions, it may be necessary to take into account the weight of money for each class of investor.

This brief review of the problem does not constitute completed research. It only shows that the CAPM is an unsatisfactory description of markets. It is possible that work on these lines has already been carried out and published, in which case it would be useful for this research to be brought to the notice of the actuarial profession. If not, it gives an opportunity for someone to carry out original investigations.

4 - CONCLUSION

The combination of the results of the two investigation in the paper, both of which are of relevance to actuaries, suggest that a new and actuarial look at modern portfolio theory is desirable. In Section 2.2 we have shown that the mean - variance criterion is not equivalent to the maximum expected utility criterion when we are comparing two prospects, if the return on one of them is distributed normally, and on the other lognormally. However, this may not matter if we can restrict ourselves to the situation described in Section 2.3, where we are only comparing lognormal prospects.

Over any reasonably long period, such as actuaries are typically interested in, lognormal returns prevail. Levy's results, discussed in 2.3, showed that, provided the portfolio was actively managed, lognormal returns over a long period could be assumed, even when there was a mix of securities. The criteria for selecting between different lognormal prospects was discussed, and it was shown that a mean - coefficient of variation criterion should replace the mean - variance criterion.

It was then shown that, provided a risk - free security existed, the mean -coefficient of variation results were identical to the mean - variance results, so that it might be thought that the mean - variance criterion were still on a sound basis. However, the conclusion of Section 3, that for investors who are interested in "real" return.. a risk - free security seldom exists. shows that the mean - coefficient of variation criterion is of practical importance.

It was further discussed in Section 3 that, where different currencies exist, or different investors have different types of liability, eg money and real liabilities, a single capital asset pricing model for all securities will not normally cxist, and different capital market lines for different currencies are rcquircd. This result too has substantial actuarial implications.

I suggested in the introduction that actuaries wcrc not enthusiastic ahout modem portfolio theory, perhaps because of certain difficulties with the theoretical foundation. I hope the ideas cxprcssed in this paper, though not carried to a full conclusion, may stimulate actuarial interest and further rcscarch in a satisfactory actuarial version of MPT.

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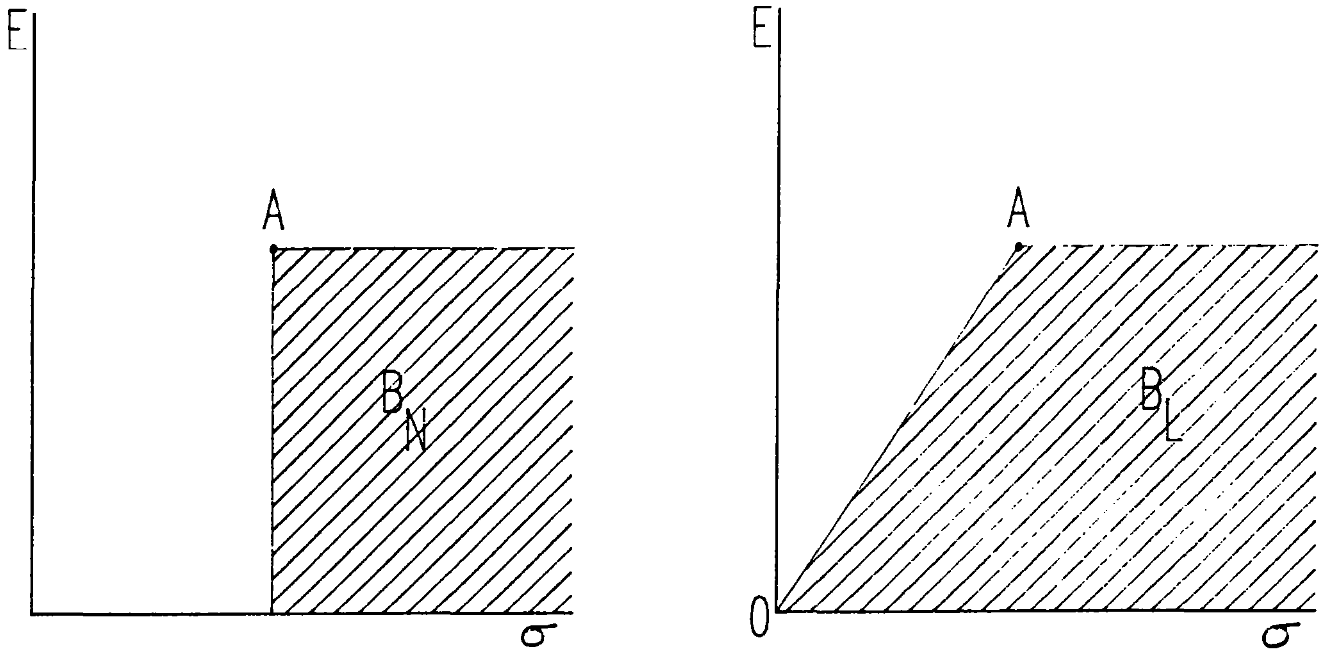


(a) Normal (b) Lognormal

Figure 1.
In each case **A** dominates portfolios in the shaded region.

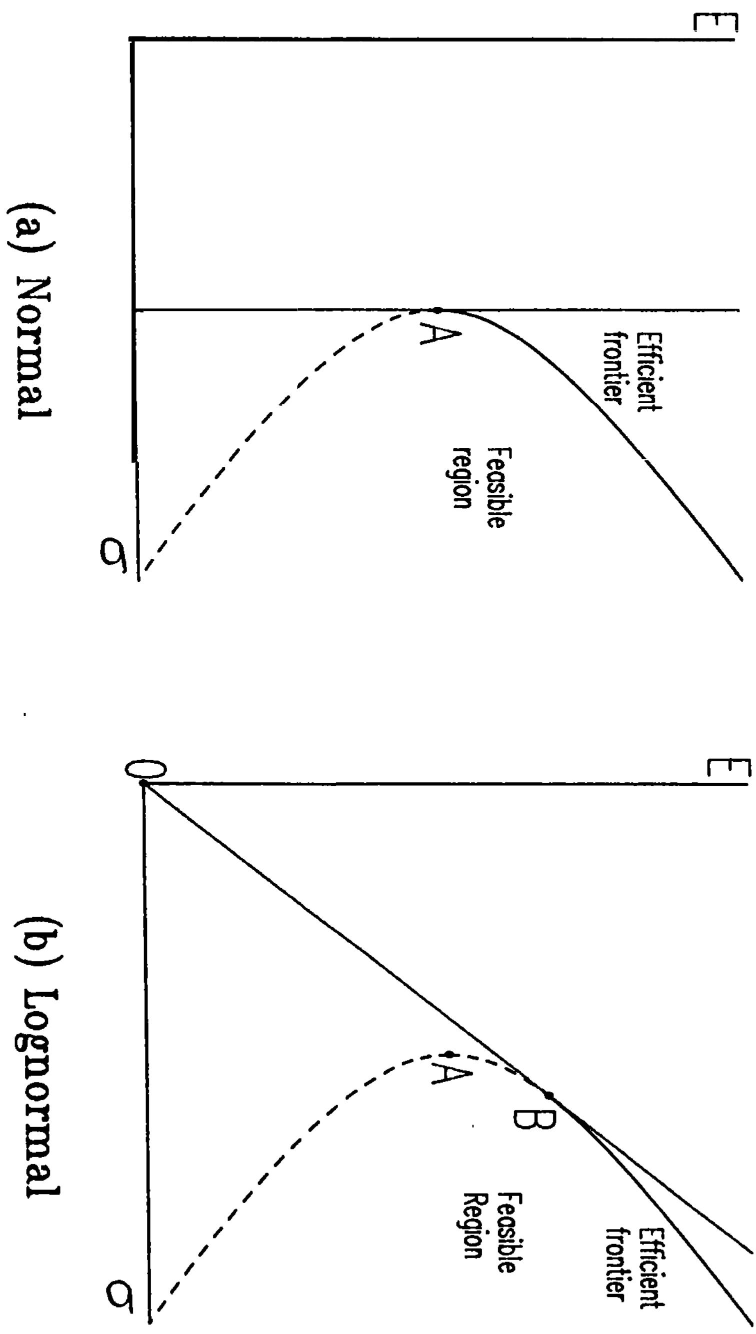
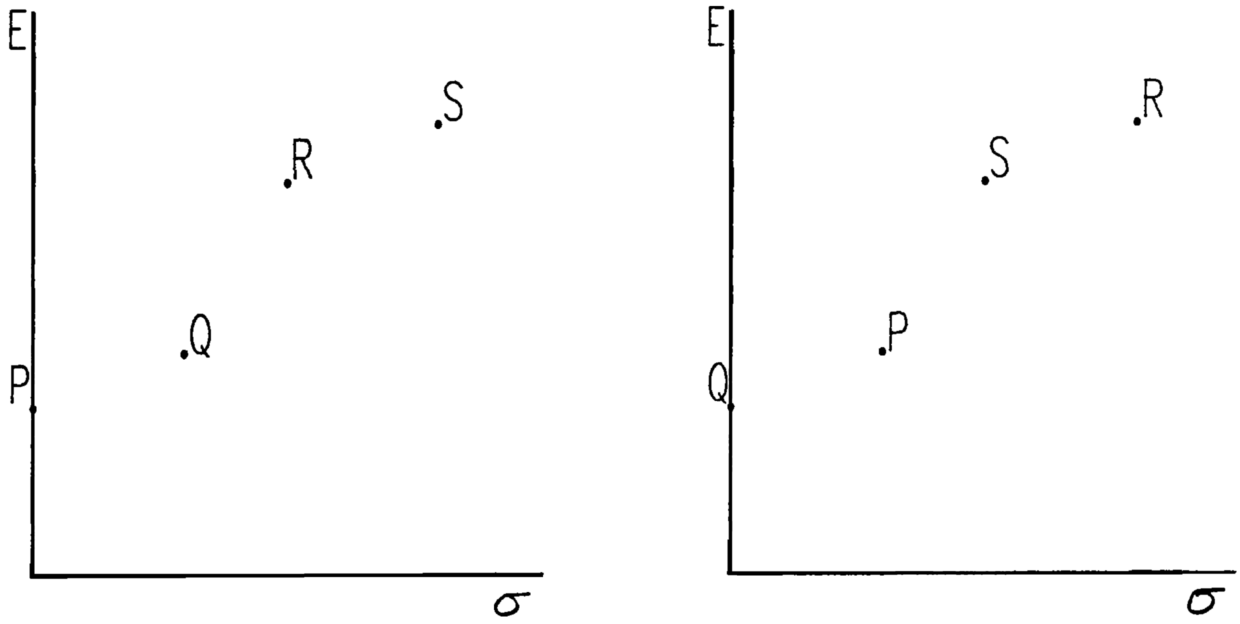


Figure 2.
Feasible region and efficient frontier



(a) Dollars (b) Pounds

Figure **3.**
E–σ diagrams in different currencies