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THE RATE OF RETURN FOR DISCOUNTING NON - LIFE INSURANCE LOSS RESERVES

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TAUX DE RENDEMENT APPLICABLE A L'ACTUALISATION DES RESERVES POUR PERTES DES ASSURANCES NON - VIE

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RESUME

L'article traite des taux de rendement applicables à l'actualisation des réserves pour pertes. Cette question a déjà fait l'objet d'un article (Taylor, 1984), rapidement analysé dans la Section 2.

La Section 3 passe en revue la théorie moderne de la structure des termes des taux d'intérêts d'obligatians. Des facteurs d'actualisation d'engagements de payement unique à terme fme, sont développés dans le cas où ces engagement reposent sur un portefeuille d'obligatians, et ces facteurs sont interprétés dans le cadre de la théorie de la structure des termes.

La Section 4 applique les facteurs d'actualisation de la Section 3 à l'évaluation d'un portefeuille d'engagements reposant sur un portefeuille d'obligations. On trouve que la valeur actualisée des engagement est égale à la valeur du march6 du portefeuille d'obligations qui génère les cash - flows correspondants, à condition que ce portefeuille puisse être constitué, indépendamment du fait qu'il soit soit ou non détenu. Le cas où il serait impossible de constituer un tel portefeuille est également envisage.

La Section 5 envisage les implications fmancikres d'une non - correspondance entre les actifs et les engagements, lorsque les premiers ne comprennent que des obligations. La non - correspondance n'affecte pas la valeur des engagements, mais affecte le risque associé. Ce risque peut être couvert par une provision pour déviation adverse (PAD : Provision for Adverse Deviation), a ajouter à la valeur des engagements. La PAD est calculée en sorte que les actifs totaux requis résultants soient suffisants, avec une probabilité prescrite, pour remplir les engagements. La courbe des taux à terme en vigueur à la date de l'évaluation est la cheville ouvrière de cette évaluation de risque.

La Section 5 mentionne en outre l'évaluation des titres à intérêt fixe comportant un risque de défaut.

La Section 6 concerne l'évaluation des engagements supportés par un portefeuille form6 uniquement d'actifs à rendement variable, c'est à dire autres que les actifs à intérêt fixe. Dans ce cas, on constate que les engagements doivent être actualisés à des taux de rendement essentiellement égaux aux taux espérés (au sens statistique) qui seraient générés par les actifs à rendement variable. La CAPM est utilisée pour relier ces taux à ceux impliqués par la courbe des taux des obligations à terme en vigueur à la date d'évaluation. Le calcul d'une PAD associée est également examiné.

Ces procédures conduisent à des résultats qui pourront sembler étranges à certains. Ils impliquent, en particulier, que la valeur attribuée à un même portefeuille d'engagements sera en général plus faible s'il est supporté par des actions que s'il est supporté par des investissements à intérêt fme. Les aspects paradoxaux de ce résultat sont résolus par la PAD. Ce point est traité au paragraphe 6.4.

La Section 7 revient aux concepts de fonds ouvert et de fonds fermé, concernant l'évaluation des engagements supportés par un portefeuille d'actifs généraux, c'est à dire formé dune combinaison d'actifs à intérêts fixes et à rendement variable. On trouve que le concept de fonds fermé est suffisant, à deux conditions :

a) la composition du portefeuille d'actifs, par secteur et par volatilité dans chaque secteur, doit Ctre maintenue constante ;

b) les rendements espérés générés par le portefeuille doivent être seuls concernés.

Dans les autres cas, et en particulier lorsque l'on doit examiner le risque associé aux rendements espérés, le concept de fonds ouvert est requis.

Jusqu'à la fin de la Section 7, toute l'incertitude concerne le risque d'investissement, les cash-flows des engagements étant considérés comme fixes et connus. La Section 8 introduit I'incertitude associée à ces cash-flows et l'intègre à I'incertitude dérivant du risque d'investissement.

Il est suggéré à plusieurs reprises dans cet article que la plupart des calculs pratiques doivent être effectués au moyen de simulations. En cette matière, il est fait mention des simulations présentées dans de récentes séries de publications anglaises et finlandaises.

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SUMMARY

The paper considers the rates of return appropriate for use in the discounting of loss reserves.

This question was discussed in an earlier paper (Taylor, 1984) which is briefly reviewed in Section 2. That paper made extensive use of the concepts of open and closed funds.

Section 3 reviews some of the modern theory of term structure of bond interest rates. Discount factors for fixed term single payment future liabilities are developed for the case in which those liabilities are supported by a bond portfolio, and the factors are interpreted within the term structure theory.

Section 4 applies the discount factors of Section 3 to the valuation of a portfolio of liabilities supported by a bond portfolio. It is found that the discounted value of the liabilities is equal to the market value of the bond portfolio which provides matching cash flows, provided that such a matching portfolio **can be** constructed, whether or not such a portfolio is in fact held. The alternative case, in which it cannot be constructed, is also considered.

Section 5 considers the financial implications of mismatching between assets and liabilities, when the former comprises only bonds. The mismatch does not affect the **value** of the liabililties, but does affect the **associated risk.** This risk may be covered by a provision for adverse deviation (PAD), to be added to the value of liabilities. The PAD is calculated in such a way that the resulting total asset requirement will be sufficient to meet liabilities with prescribed probability. The forward yield curve in existence at the valuation date is central to this evaluation of risk.

Section 5 also makes brief reference to valuation of fixed interest securities which involve default risk

Section 6 concerns the valuation of liabilities supported by an asset portfolio consisting of only **variable return assets. i.e.** those other than fixed interest assets. In this case, it is found that liabilities are to be discounted at rates of return essentially equal to the **expected** (in the statistical sense) rates to be generated by the variable return assets. The CAPM is used to relate these rates to the bond rates implicit in the forward bond yield curve existing at valuation date. The computation of an associated PAD is also discussed.

These procedures lead to results which may seem foreign to some. In particular, they imply that the same portfolio of liabilities will usually be assigned a **lower** value when supported by shares than when supported by fixed interest investments. Any paradox perceived in this result is resolved by the PAD. The question is discussed in Section 6.4.

Section 7 returns to the concepts of open and closed funds in dealing with the valuation of liabilities supported by a general asset portfolio, **i.e.** a mixture of fixed interest and variable return assets. It is found that the closed fund concept is sufficient provided that:

(a) the composition of the asset portfolio by sector, and by volatility within sector, is to be maintained constant at **all** times; and

(b) one is concerned only with **expected** returns to be generated by the portfolio.

Otherwise, and particularly when the risk associated with the expected returns is to be examined, the open fund concept is required.

All uncertainty up to the end of Section 7 relates to investment risk, the liability cash flows being **treated** as fixed and known. Section 8 introduces the uncertainty associated with these cash flows and integrates it with the uncertainty deriving from the investment risk.

It is suggested throughout the paper that most practical computations need to be carried out by means of simulation. Section 8 also relates such simulations to those which have been carried out in recent series of English and Finnish publications respectively.

1. INTRODUCTION AND BACKGROUND

1.1 Relevance

Loss reserves, at least in their actuarial forms, are very likely to be derived from a projection of future claim payments. These claim payments will normally be projected initially in money values current at the valuation date **(i.e.** excluding the effects of future claims escalation); and then including due allowance for such escalation.

It may or may not be possible or **desirable** to establish the loss reserve on a basis which is discounted, effectively anticipating the investment return earned by the assets which support that reserve. This is an issue over which conflict has raged many times in the past. The present paper is silent on the matter. Instead it deals with the situation in which it has been decided, for whatever reason, that the loss reserve is to be discounted. A question then arises regarding the appropriate rate of return to be used in **discounting** the liability.

In the English speaking world, certainly, there has been a considerable upsurge of interest in this topic over the past few year. Impetus was given to this by the US **tax** Reform Act of 1986 and the debate preceding it. Whereas many US insurers did not explicitly discount loss reserves in years past (note, however, the fact that some of these "undiscounted" reserves would have fallen short of independently assessed discounted reserves, suggesting am implicit discount in the "undiscounted" reserves), they are now obliged by the new legislation to do so. The basis for determining the discounting rate of **return** is prescribed.

Though no parallel legislation has been forthcoming in the UK, considerable interest in the concept of discounting has emerged over the past few years. This has been fuelled by the Inland Revenue's intention to challenge some insurers who do not discount their reserves., The situation in Australia is that discounting is reasonably widespread and has been so for some years. The choice of whether or not to discount loss reserves is one which is at the discretion of the insurer concerned. So is the rate of discount, though open to challenge by the Insurance Commissioner if the chosen rates exceed what he perceives as reasonable.

Thus, in the cases of both UK and Australia, there is ample scope for useful discussion of the precepts from which the discounting rates of return should be derived.

In the case of the US, the situation is not so simple. While the regulation of discounting rates might appear to render a discussion of the proper basis of those rates futile, there are other issues to be considered.

First, an assessment of the reasonableness of the regulated rates is a legitimate one. If these are unreasonable in some way, the result might be to impair the profitability of some or all insurers. Inequities could be created between insurers with different characteristics, or between the insurance industry and others.

Second, as will be seen in Sections 5 to 8, prudence may require a margin in the loss reserve to accommodate adverse fluctuation in investment return. The prudent loss reserve may exceed that prescribed by regulation. A proper assessment of this requires the application of principles of a type developed in the named sections.

1.2 Objectives

The questions raised in Section 1.1 were discussed in Taylor (1984). The first objective of the present paper will be to review that earlier one, suggesting changes where possible.

While techniques suggested in the earlier paper do not appear invalid, it is possible to simplify them to some degree. It is also possible to tighten the theoretical underpinning by making use of certain aspects of financial economics such as the CAPM and models of interest rate term structure.

Key concepts introduced in the earlier paper were those of open and closed funds.

The latter was defined as follows. (Taylor, 1984, p. 70):

"the insurer's fund is closed [if it may be regarded as] receiving no future income except that generated by what remains of the current investment portfolio (including income generated by reinvestment of that income)"

and

"the current investment portfolio is [regarded as] subject to gradual depletion over future years as the proceeds of sale and/or maturity of investments are required to pay clai'ms as they fall due".

A fund subject to other income, e.g. new premium income, is open.

The concept of a closed fund was used in the case in which an insurer had a stated investment policy of matching assets and liabilities by amount and term.

For reasons which will become clear in Section 2, if not already, the concept of an open fund was intimately related in the earlier paper to the need to project future rates of return on new investments.

A major objective of the present paper will be to review these two key concepts of the earlier paper:

(a) the distinction between open and closed funds;

(b) the projection of future rates of investment return.

Attention will be focused particularly on the necessity for these concepts, and the extent to which they fit with the somewhat more modem theory offered in the present paper.

First, however, Section 2 will provide a brief review of the key concepts of the earlier work in order to set a perspective against which the developments of the paper may be viewed.

2. REVIEW OF EARLIER WORK

The earlier paper dealt with a number of issues, for example the treatment of debtor and creditor items generally, the treatment,of brokers' premium balances as specific debtor items, etc. While all of these matters have a bearing on the rates of return at which loss reserves should be discounted, attention will be restricted here to the issue of open and closed funds, as defined in Section 1.

The earlier paper contrasted open and closed funds in the following terms (Taylor, 1984, p. 79), where the closed fund under consideration was assumed matched to liabilities:

"(i) in the case of a closed fund, a particular bloc of assets is identified with claims outstanding at a time t; these assets are held to maturity, and the cash flow which they provide (and hence the rate of investment return which they provide) and no other supports the liability in question;

(ii) in the case of an open fund, the bloc of assets to be identified with claims outstanding at time t changes ontinually over the period of run-off of those claims, the bloc of assets at time $s(>t)$ being simply a proportion of the entire asset portfolio associated with the total outstanding claims liability at that time."

It follows from (i) that future investment conditions will be totally irrelevant to a closed fund which is exactly matched. For closed funds generally, those conditions will be relevant only to the extent that any mismatch of assets to liabilities generates reinvestments or early realization of assets. Unless the mismatch is severe, future investment conditions will be only marginally relevant to such closed funds.

Proposition 4.3.1.1 of the earlier paper states that, in the case of a matched portfolio of assets, the "rates of investment return included in the discounting of outstanding claims should be those rates **earned** from time to time, by the remainder of those assets which, initially, matched liabilities exactly."

The contrary situation follows from (ii). Since the bloc of assets identified with loss reserves is continually changing in the case of an open fund, the terms on which those assets are added to by new investment in future years is highly relevant to the returns which they provide.

The earlier paper (Sections 5 and 7) suggested that in this situation the rates of return to be used in the discounting process should be determined by means of the following steps:

(a) project future movements in new money yields in the various investment sectors, e.g. fixed interest, equities, property, etc;

(b) project future investment policy in terms of the composition of the asset portfolio or of new investments by investment sector;

(c) on the basis of (a) and (b) project future:

- (i) market values of the investment portfolio;
- (ii) income from that portfolio;
- (iii) capital appreciation/depreciation in that portfolio;

(d) hence project the market value investment return provided period by period by the investment portfolio, and use these rates of return to discount loss reserves.

In the discussion of the paper at a sessional meeting of the Institute of Actuaries of Australia (Taylor, 1986, pp. 133-137) Dr. S. Bymes **criticised** the above procedure as follows:

"[If the] market expects [a return of x% p.a.] forever ... I find it hard to see how a rate other than [x% p.a.] could be used for the interest assumption",

and further, in relation to step (c);

"If one calculates the yield [over a year] using different future views at the beginning and end of the year, then one will **capitalise** these differences in expectations."

My own interpretation of this criticism is that one should not carry out step (a) in respect of fixed interest investments without reference to the current yield curve. This curve embodies in some way "the market's" view of future interest rate movements. To make an independent projection of these movements, as was done in Taylor (1984), amounts to out-guessing the market.

This criticism appears valid. Note, however, the scope of the criticism. According to the interpretation I have given, the concepts of open and closed funds and the procedure set out in steps (a) to (d) above would not be questioned; only the making of a projection in (a) **independent** of current market conditions would be criticised.

Presumably, this situation would be repaired if future movements in fixed interest rates of return were projected in (a) according to the current market view, as reflected in current market conditions.

This is the objective of Section 3. The implications of the procedure developed there for valuation of liabilities which are supported by fixed interest assets are worked out in Sections 4 and 5. The conclusions are extended in Section 6 and 7 to general portfolios of assets, including those containing assets, such as equities and property, which provide variable returns.

3. FUTURE RETURNS ON BONDS

3.1 Development of pricing model

In the following, the term bonds will be understood to refer to default free fixed interest assets. A reference to **fixed** interest assets will be to assets which may or may not be risk free.

Suppose that at a given point of time one is able to observe the market prices of bonds of all terms to maturity. These prices contain "the market's" expectations of future interest rate movements. The objective is to deduce these expectations from the schedule of prices.

In practice, observation of prices in respect of all terms to maturity will not be feasible. Suppose, however, that the following situation exists. Prices of securities are observed at time T. All subsequent transactions involving these securities (payment of interest, maturity of asset) take place at epochs T + A, T + 2A, ..., T + NA for some particular time interval $\Delta>0$ (e.g. day, month, quarter). Note that NA is the maximum term to maturity for which price is observed, though N will be infinite when there are perpetuities in the market.

Suppose that there is at most one type of security with any given maturity date. Then, at time T, securities may be labelled by their terms to maturity $n\Delta$(= A, 2Δ, ..., NA).

Let $p^{(n)}$ denote the price at time T of the security with term to maturity $n\Delta$. Henceforth in this sub-section A will be adopted as the unit of time and so $p^{(n)}$ will be referred to as the price of the security with term to maturity n (time units).

Now this security will be characterized by the cash flows which it provides at epochs 1, 2, ..., n. Let the cash flows at epoch t be $A_t^{(n)}$, t=1,2,...,n. In many cases:

$$A_t^{(n)} = \text{coupon rate}, \; t = 1, 2, ..., n-1;$$

$$A_n^{(n)} = 1 + \text{coupon rate};$$

but this restriction is not made here.

The price $p^{(n)}$ is the value assigned by the market to the flows $A_t^{(n)}$, t = 1, 2, ..., n. Let z_t denote the (possibly hypothetical) value which would be assigned to a zero coupon bond maturing for one unit at epoch t. Then it must be the case that:

$$p^{(n)} = \sum_{t=1}^{n} A_t^{(n)} z_t, \qquad (3.1.1)$$

if transaction costs can be ignored and the market is free of arbitrage opportunities.

Now as (3.1.1) applies to each $p^{(n)}$ observed, it generates a system of equations:

$$p^{(n_1)} = A_1^{(n_1)} z_1 + \ldots + A_{n_1}^{(n_1)} z_{n_1};$$

$$p^{(n_2)} = A_1^{(n_2)} z_1 + \ldots + A^{(n_2)}_{n_1} z_{n_1} + A_{n_1+1}^{(n_2)} z_{n_1+1} + \ldots + A_{n_2}^{(n_2)} z_{n_2}; \qquad (3.1.2)$$

etc.;

where $n_1, n_2, \ldots$ N denote the observed terms to maturity written in ascending order. The objective is to solve for the z_t.

For some purposes, the procedure of accepting (3.1.2) as a collection of identities, and so solving for the z_t, would be inappropriate. In practice, small price anomalies are likely to occur, whence the left side of (3.1.2) needs to be regarded as just an approximation to the right side. In this case, the right side may be expresible as a linear combination of a collection of basis functions (e.g. Vasicek & Fong). The fitting of this linear combination to observed prices could then be carried out by means of regression.

Literal interpretation of (3.1.2) in the presence of pricing anomalies may magnify the latter in the inversion process which leads to the solution in the z_t. However, this will not usually create particular difficulty in the use of the z_t for valuation of liabilities as occurs later in this subsection.

As will be demonstrated in Section 4.1, straightforward solution of (3.1.2) will give precisely the correct result in the case in which assets and liabilities are matched. It follows that the same procedure will give reasonable results more generally provided that any mismatch between assets and liabilities is not too great.

If, however, it appears in any particular case that pricing anomalies in the market are likely to'cause material inaccuracies in the valuation of liabilities, then the more sophisticated approach to estimation of the z_t should be used.

The solution to (3.1.2) will not be possible in general, since there will be more unknowns (the z_t) than observations (the $p^{(n)}$). An exception to this arises in the case $n_1 = 1, n_2 = 2$, etc. Then the system of equations (3.1.2) reduces to:

$$p^{(1)} = A_1^{(1)} z_1;$$

$$p^{(2)} = A_1^{(2)} z_1 + A_2^{(2)} z_2;$$

$$p^{(N)} = A_1^{(N)} z_1 + A_2^{(N)} z_2 + \ldots + A_N^{(N)} z_N. \quad (3.1.3)$$

This may conveniently be put in matrix form:

$$p = Az, \quad (3.1.4)$$

where p, z are the column N-vectors of $p^{(n)}$ and z_t respectively, and A is an NxN matrix whose elements are obvious from (3.1.3).

Now A is lower triangular and therefore invertible provided only that all diagonal elements are non-zero. This will be the case since the diagonal element $A_n^{(n)}$ denotes the maturity proceeds of the security which matures at epoch n.

Since A is invertible, (3.1.4) yields a solution for z:

$$z = A^{-1} p. \quad (3.1.5)$$

While it is convenient for some purposes to express the solution in this matrix form, one may also note that, because of the triangular form of A, numerical solution of (3.1.3) is achieved in straightforward manner by successive elimination of the z_t, thus:

$$z_1 = p^{(1)} / A_1^{(1)};$$

$$z_2 = [p^{(2)} - A_1^{(2)} z_1]/A_2^{(2)}$$

etc. (3.1.6)

All of the development from (3.1.3) to (3.1.6) is based on the assumption that $n_1 = 1, n_2 = 2$, etc., and a word needs to be said about the appropriateness of this assumption.

The meaning of the assumption is that, whatever the unit of time A chosen, there exist observations of prices of securities of all terms to maturity A. 2A, ..., NA. This is in fact unlikely to be precisely the case. If it is not, then the above analysis needs to be modified slightly. Appendix A discusses the sort of modification which might be made. The discussion there involves the "manufacture" of bond prices applicable to the regularly spaced terms to maturity needed for application of (3.1.3) to (3.1.6).

If, on the other hand, the regression approach to estimation of z_t mentioned above is being used, **no modification** to the theory is required. Equation (3.1.2) may then be applied directly with z_t expressed in terms of the chosen basis functions.

3.2 Interpretation of the pricing model

The pricing model of Section 3.1 leads to estimation of the vector $z = (z_1, z_2, ..., z_N)^T$, the T denoting transposition.

As they were introduced in Section 3.1, the z_t were values of zero coupon bonds. It is, however, possible to interpret them in terms of market expectations of future rates of return (RORs).

Before making this interpretation, however, it is of interest to note that use of the z_t is as if there will be a certain rate of return of

$$z_t/z_{t+1} - 1$$

in the period between epochs t and $t+1$, $t=1,2,...,N-1$.

It may be appropriate to remark at this point that little that is said in the present sub-section will be required in the development of later sections. The reader may therefore skip to the next section if so wishing. In other words, the interpretation of z_t as price of a zero coupon bond is sufficient for applications.

Nevertheless, it would seem less than desirable for there to be no enquiry into the manner in which fundamental market dynamics are translated into bond prices. The present sub-section, which is necessarily of a more technical nature, treats this issue.

Now return to the interpretation of the z_t in terms of market expectations.

If, for example, A is regarded as the basic unit of time in the fixed interest market, then

$$z_1 = 1/(1+i_1), \tag{3.2.1}$$

where i_1 is the market spot rate on term to maturity A. Investment of z_1 in a security of this term will yield one unit with certainty at maturity. The outcome is certain because, with a basic time unit of A, a security of term A will yield only one cash flow, and so there will be no uncertainty generated by reinvestment.

Consider now the value of z_2. This will need to reflect somehow the (certain) ROR i_1 and also the uncertain return i_2 obtainable at epoch T + A by investment then in a zero coupon bond of term to maturity A. This is because a pay-out at epoch T + 2Δ can be obtained by either:

(a) investment at epoch T in a zero coupon bond of *term* 2Δ;

or

(b) investment at epoch T in a zero coupon bond of term A, followed by reinvestment at epoch T + A in a further zero coupon bond of term A.

The values of z_1, z_2, together with market expectations of the future must be such as to equate in some way the outcome of investment of a unit of capital according to each of (a) and **(b)**. The literature contains various approaches to this question, a couple of which will be illustrated in this sub-section.

To **formalise** this argument, it will be necessary to **introduce** the random variable:

i_t = ROR obtainable at epoch T + (t-l)A

on a zero coupon bond with term to maturity A. As noted above. i_1 will in fact be deterministic, but the i_t, t > 1 **will** be genuinely stochastic.

It is evident from the sort of argument given immediately prior to the introduction of i_t that the values of z_1, z_2, **etc.** at epoch T + (s-l)A **(i.e.** all market prices at that epoch) are governed by the random variables i_t, $t \geq s$, as assessed by the market at epoch T + (s - 1)Δ.

The values of i_t will be supposed to evolve continuously over time but subject to perturbations which, viewed in prospect, may reasonably be regarded as random. Then the evolution of i_t may reasonably be represented **as**:

$$1 + i_t = \exp \int_{T+(t-1)\Delta}^{T+t\Delta} r(s)\, ds, \tag{3.2.2}$$

with

$$dr(s) = g(r(s);\theta)\, ds + de_s \tag{3.2.3}$$

for some function g, an external **parameter** θ chosen from a space Θ which may be multi-dimensional or even abstract, and a continuous random process de_s with zero expectation.

For example, θ may incorporate a long term value to which i_t is assumed to regress, and perhaps some measure of the strength of coupling of i_t to **earlier** values of that variable.

Specific examples of (3.2.3) occur in the literature. For example, Vasicek (1977) and Cox, Ingersoll and Ross (1977) suggest specific forms of this equation in continuous time (A $\to$ 0).

Vasicek suggests:

$$dr = \alpha(\gamma - r)\, ds + \rho dz, \tag{3.2.4}$$

with dz denoting a Wiener process with zero drift and instantaneous variance dt, and α, γ, ρ denoting constants.

Cox, Ingersoll and Ross (1977), quoted by Boyle (1978), suggest an alternative which ensures that r cannot go negative (as it can in the Vasicek model):

$$dr = \kappa(\mu - r)\, dt + \sigma \sqrt{r}\, dz, \qquad (3.2.5)$$

with κ, μ, σ constants.

For the moment it will not be necessary to specify the form off, but merely to note that the accumulator

$$(1 + i_{s+1})(1 + i_{s+2}) \ldots (1 + i_t) \qquad (3.2.6)$$

will be a random variable equal to

$$\exp \int_{T + s\Delta}^{T + t\Delta} r(s)\, ds \quad [\text{by } (3.2.2)] \qquad (3.2.7)$$

and so will have a **d.f.** $G(.,\phi_{s,t})$ say. characterized by a parameter set $\phi_{s,t}$ which depends on the form off, the process $\{e_s, s \le t\}$, the parameter θ, and the **initial** value **r(T).**

Let **I(s,t)** denote the accumulator (3.2.6). Then an investment at epoch T of 1 unit of capital in a zero coupon bond of term t to maturity, followed by reinvestment at unit intervals until epoch **T+N,** will yield a terminal payoff of $I(t,N)/z_t$.

Then relative magnitudes of the z_t are then given by the equivalence in some form of the N quantities $I(t,N)/z_t$, t = **1, 2,** ..., N, **i.e.** an assumption of no arbitrage over term to maturity. One approach to this equivalence interprets it in terms of the representative investor of Rubinstein (1976) and **Brennan** (1979). as described in Appendix B. This deals with the situation in which:

(a) de_s follows a Wiener process in **(3.2.3),** implying normality of log **I(s,t);**

and

(b) log **I(s,t)** and log W are jointly normally distributed, W denoting aggregate end period wealth in the economy.

It is demonstrated that, under these conditions,

$$z_t = \{1/E[I(0,t)]\} \times \exp[a\, C(0,t) - D(t,N)], \qquad (3.2.8)$$

with ,

$$C(0,t) = \text{Cov}[\log I(0,t), \log w], \qquad (3.2.9)$$

$$D(t,N) = \text{Cov}[\log I(0,t), \log I(t,N)], \qquad (3.2.10)$$

E denoting the expectation operator, and a a parameter characterizing the investor's **utililty** function **u(.)** thus:

$$u'(x) = Kw^{-a},\ a>0,\ K\ \text{const.} >0. \qquad (3.2.11)$$

Examples are worked out in Appendix B, demonstrating the following results:

(a) when interest rate movements are deterministic, (3.2.8) reduces to just the component in braces;

(b) the same reduction in (3.2.8) occurs when interest rate movements are stochastic **but:**

(i) there is no coupling between rates of disjoint periods;

and

(ii) either investors are risk neutral, or aggregate wealth is not correlated with bond rates;

(c) if:

(i) there is no coupling between interest rates of disjoint periods;

and

(ii) $C(t,t+1) = c$, **const. > 0** for each t, then the shape of the spot yield curve is just as for the deterministic **case;**

(d) under realistic assumptions, the effect of the second factor in (3.2.8) is to raise redemption **yields at** longer terms to maturity relative to those at shorter terms, **i.e.** to create an upward slope in the yield curve when the market has a neutral view of future interest rate movements, **i.e.** the braced factor of (3.2.8) taken alone produces a flat yield curve.

An alternative evaluation of the z_t **is** provided by Heath, **Jarrow** and Morton (1988). For their case let **P(s,t)** denote the **price** at time s of a zero coupon bond maturing at time t. They work in **continuous** time and define the forward rate at time for maturity date t **by**

$$f(s,t) = -\partial \log P(s,t)/\partial t, \qquad (3.2.12)$$

whence

$$P(s,t) = \exp \cdot \int_s^t f(t,v)\, dv. \qquad (3.2.13)$$

Note that $z_t = P(0,t)$.

The spot rate **r(s)** previously modelled in (3.2.3) is now defined as

$$r(s) = f(s,s). \qquad (3.2.14)$$

Whereas the suggestions of other authors quoted earlier in this subsection was to model the spot rate $r(s)$, Heath, Jarrow and Morton model the evolution of the entire curve of forward rates:

$$df(s,t) = \alpha(s,t)\,ds + \sigma(s,t,f(s,t))\,dW(s), \tag{3.2.15}$$

with a and σ real-valued functions, and $W(s)$ is a Wiener process with zero drift and unit instantaneous variance.

In fact, Heath, Jarrow and Morton allow any finite number of terms of the type involving $W(s)$ in (3.2.15). For simplicity, only the restricted model is discussed here. With this restriction, note the similarity of (3.2.15) to the earlier process (3.2.3).

Note that investor utility is not introduced. This is avoided by the use of continuous-time analysis and no-arbitrage arguments, which yield the result:

$$z_t = E\{[1/I(0,t)] \times \exp[\textstyle\int_0^t \phi(s)\,dW(s) - \frac{1}{2}\int_0^t \phi^2(s)\,ds]\}, \tag{3.2.16}$$

with

$$\phi(s) = \{-\textstyle\int_s^u \alpha(s,v)\,dv + \frac{1}{2}[\int_s^u \sigma(s,v,f(s,v))dv]^2/\int_s^u \sigma(s,v,f(s,v))dv, \tag{3.2.17}$$

which is shown to be independent of $u \epsilon (s,t)$.

The result (3.2.16) compares with (3.2.8) derived from the model of the spot rate process.

4. VALUATION OF LIABILITIES SUPPORTED BY BONDS

4.1 Bonds available at terms as long as liabilities

Consider a portfolio which generates liability cash flows y_1, y_2, etc. at epochs 1, 2, etc., these epochs being the same as discussed in Section 3 in connection with asset cash flows.

For the purpose of the present discussion. it will be supposed that the y_t are fixed and known. In practice, they will be random variables. Hence. uncertainty in their present value will arise from uncertainty in both;

(a) the values of the y_t themselves; and

(b) future rates of return, as reflected in the discounting of the y_t.

The issue of uncertainty due to (a) is dealt with elsewhere (e.g. Taylor, 1988; Taylor and Ashe, 1983). For the moment the objective is to isolate component (b) of uncertainty. The question of total uncertainty is considered further in Section 8.

Recall from Section 3.1 that N is the maximum term of available assets. It will be supposed in the present subsection that liability cash flows do not occur at greater delays than this, i.e. those cash flows are $y_1, y_2, \ldots, y_N$, possibly with some of these y_t equal to zero.

The value of these liabilities may reasonably be taken to be:

$$V = \sum_{t=1}^{N} z_t \, y_t \,. \tag{4.1.1}$$

In this case, the net value of a portfolio consisting of precisely matched assets and liabilities will be **zero,** as it must be since it yields only zero net cash flows.

Now write the y_t as a vector y:

$$y^T = (y_1, y_2, ..., y_N). \tag{4.1.2}$$

Then (4.1.1) may be put in the form:

$$V = y^T z \tag{4.1.3}$$

$$V = y^T A^{-1} p \quad \text{[by (3.1.5)]} \tag{4.1.4}$$

$$= m^T p, \tag{4.1.5}$$

say, with

$$m = (A^T)^{-1} y. \tag{4.1.6}$$

Equations (4.1.4) to (4.1.6) are interesting.

First, (4.1.4) indicates that the discounted value of liabilities is obtained by a simple matrix and vector multiplication if only the following are known:

(a) the liability cash flows;

(b) the cash flow profiles of bonds available in the market (this usually means just those coupon rates and terms to maturity);

(c) the market prices of those assets.

This discounted value encapsulates market expectations regarding future rates of return. The simplicity of (4.1.4) occurs despite the complexity of interpretation of the z_t discussed in Section 3.2.

Second, (4.1.5) shows that the discounted value of liabilities is equal to the market value of a notional bond portfolio, characterized by the vector $m_T = (m_1, m_2, ..., m_N)$, with m_t representing the magnitude of the holding of the security with term to maturity t.

This vector of bond holdings is given by (4.1.6) which is capable of a simple interpretation, most easily obtained by recalling the form of A from (3.1.3) and (3.1.4) and obtaining **m** as the explicit solution of:

$$A^T m = y, \qquad (4.1.7)$$

i.e.

$$A_1^{(1)} m_1 + A_1^{(2)} m_2 + \ldots + A_1^{(N)} m_N = y_1;$$

$$A_2^{(2)} m_2 + \ldots + A_2^{(N)} m_N = y_2;$$

$$\vdots$$

$$A_N^{(N)} m_N = y_N. \qquad (4.1.8)$$

This system of equations may be solved in reverse order

$$m_N = y_N / A_N^{(N)}; \qquad (4.1.9a)$$

$$m_{N-1} = [y_{n-1} - A_{N-1}^{(N)} m_N] / A_{N-1}^{(N-1)}; \qquad (4.1.9b)$$

$$m_{N-2} = [y_{n-2} - A_{N-2}^{(N)} m_N - A_{N-2}^{(N-1)}] / A_{N-2}^{(N-2)}; \qquad (4.1.9c)$$

etc.

The quantity m_N is the holding of security of term N required to meet liability cash flow st epoch **N**. The square bracketed term in **(4.1.9c)**, for example, is net liability cash flow at epoch N-2 after allowance is made for flows deriving from subsequently maturing investments (terms **N** and **N-1)**. Hence, $\mathbf{m_{N-2}}$ is the holding of security of term N-2 required to meet this net liability cash flow.

Similarly for all the $\mathbf{m_t}$. Since this algorithm arranges asset cash flows to meet liability cash flows precisely, **m** denotes a matched portfolio of assets.

Thus, (4.1.4) gives the discounted value of liabilities as the market value of the notional matching portfolio of assets. This remains the value of liabilities, whether **the** portfolio of assets in fact matches them or not.

The above results are summarized in the following proposition.

PROPOSITION 4.1.1. When liabilities are supported by a bond portfolio, the present value of those liabilities is equal to the market value of the bond portfolio whose cash flows precisely match the liability cash flows. This value is given in various algebraic forms by (4.1.3) to (4.1.6).

□

It should be noted in connection with the matching portfolio discussed here that cases are possible in which strict matching will not be possible. This will occur whenever one or more components of m are negative. This would required the investor to adopt a short position in the assets concerned, and this may not be possible.

Despite this the valuation formulas (4.1.3) to (4.1.6) remain valid. In any event, negative components of m are comparatively rare in practice. Indeed, (4.1.9) indicates that m_t will be positive for t = s, s+1, ..., N provided that the sequence y_s, y_{s+1}, ..., y_N decreases sufficiently rapidly.

4.2 Bonds not available at terms as long as liabilities

The valuation formula (4.1.4) cannot be applied when the term of liabilities extends beyond the longest term of the available bonds. In this case, the matching bond portfolio required by Proposition 4.1.1 is not available. From an alternative viewpoint, the factors z_t required by (4.1.1) are not available (at least not directly from bond market prices) for the larger values of t.

To obtain these z_t in a rigorous fashion, one requires an evolutionary interest model such as (3.2.3). The parameters θ of this model would be estimated, and hence expected values of discount factors.

For example, if a normal model of instantaneous interest rates is used as in (3.2.8) to (3.2.10), and if utility function (3.2.11) is also used, then the parameters of the interest rate model are:

(a) the function $\mu(s,t) = E[\log I(s,t)]$;

(b) instantaneous variance and auto-covariance of the Wiener process followed by de_s (these parameters contributing to C(0,t) and D(t,N) in (3.2.8) in the manner indicated by Appendix B2.

These parameters govern the values of z_t, as in (3.2.8), and so may be estimated from market data just as the z_t may be. Once they are available for all values of s,t, one can use (3.2.8) to extrapolate the z_t to values of t outside the range of market data.

In practice, it may well be possible to abbreviate this whole process by means of a simple expedient. As pointed out early in Section 3.2, the quantity $z_t/z_{t+1}-1$ may, for discounting purposes, be treated as the certainty equivalent of the stochastic interest rate i_t. It may therefore be possible to carry out a rough empirical extrapolation of the quantities $z_t/z_{t+1}-1$ (and hence of z_t). They may, for example, be approximately constant for $t > N$.

This sort of approximation is all the more justified when one considers the roughness in the sequence of these quantities for $t \leq N$, likely to arise from small market anomalies in the price vector p.

5. MISMATCHING OF ASSETS AND LIABILITIES

5.1 Financial implications of mismatched bond portfolio

According to Proposition **4.1.1,** the discounted value of liabilities is calculated as if the supporting bond portfolio matches them precisely. Any mismatch between assets and liabilities therefore has no effect on the discounted value of the latter.

It is clear, however, that the financial risk to the insurer increases with the **degree** of any mismatch.

These two facts are reconciled in the remainder of the present sub-section, which examines the issue of mismatching generally.

Consider the present value of the net gain which will accrue to an insurer in the settlement of **known** liability cash flows $y_1, ..., y_N$ from the asset cash flows $x_1, ..., x_N$. These asset cash flows are those generated **directly** by the bonds currently held, **i.e.** without any allowance for early sale or reinvestment.

This gain is

$$G = \sum_{t=1}^{N} (x_t - y_t) z_t \tag{5.1.1}$$

$$= \sum_{t=1}^{N} g_t z_t , \tag{5.1.2}$$

with

$g_t = x_t - y_t$ = surplus cash flow at epoch t. **(5.1.3)**

This equation may conveniently be written in vector form:

$$G = g^T z, \tag{5.1.4}$$

with g^T denoting the vector $(g_1, ... g_N)$.

The case of a precisely matched portfolio is characterized by:

$$x = y, \tag{5.1.5}$$

i.e. direct cash flows x_t are precisely sufficient to meet liability cash flows y_t.

By (5.1.3) and **(5.1.6),**

$$g = 0, \tag{5.1.6}$$

and so (5.1.4) gives:

$$G = 0. \tag{5.1.7}$$

If the portfolio is not precisely matched, some of the g_t will be non-zero. Equation (5.1.7) will continue to hold for the appropriate amount of total assets. However, by (5.1.4), this equation is:

$$g^T z = 0, \tag{5.1.8}$$

which is a constraint on g rather than an identity as in the case g = 0.

Now if g is a solution of (5.1.8) for given z, and z undergoes an infinitesimal change, then g may or may not remain a solution of (5.1.8). This merely expresses the familiar fact that, if there is a mismatch between assets and liabilities, and if the yield curve shifts, then net assets, commencing at zero, will shift to G which may be positive, negative or zero.

The chance of adequacy of a particular bond portfolio to meet the associated liabilities depends on the probability distribution of G, now regarded as the random variable:

$$G = \sum_{t=1}^{N} (x_t - y_t)\, Z_t, \tag{5.1.9}$$

with Z_t the random variable underlying z_t, i.e. E $Z_t = z_t$.

The actual form of Z_t will depend on the interest rate dynamics being assumed. For example, under the assumptions which led to (3.2.8) as a representation of z_t, (B1.4) shows that:

$$Z_t = I(t,N)\, E[y(w) \mid \{r_s, t \le s \le N\}] / E[I(0,N)\, E[y(w) \mid \{r_s, 0 \le s \le N\}]]. \tag{5.1.10}$$

Alternatively, under the assumptions leading to representation (3.2.16),

$$Z_t = [1/I(0,t)] \times \exp[\textstyle\int_0^t \phi(s)\, dW(s) - ½ \int_0^t \phi^2(s)\, ds], \tag{5.1.11}$$

with $\phi(s)$ defined by (3.2.17).

There are two great difficulties in attempting an analytic treatment of the distribution of G.

Fist, the form of the distribution is not tractable. This is essentially due to the fact that, as the assumption of a Wiener process of spot rates implies lognormal Z_t (in both (5.1.10) and (5.1.11)), G is a linear combination of lognormal variates. This is algebraically inconvenient. For small values of N. G will be approximately lognormal; for large N. G and be approximately normal. For intermediate values of N, it will not have a convenient form.

Second, the Z_t are not stochastically independent. Computation of the **covariances** is laborious. For example, when Z_t is given by (5.1.10), then

$$\text{Cov}[Z_s, Z_t] = K\ \text{Cov}[I(s,N), I(t,N)]$$

$$= K\ \text{Cov}[\exp \textstyle\int_N^s r(u)\,du, \exp \int_t^N r(u)\,du], \qquad (5.1.12)$$

with K the constant given by the square of the ratio of expectations appearing in (5.1.10). Now, in general, all the $r(u)$ for different u are correlated because of (3.2.3) (see **e.g.** special cases (3.2.4) and **(3.2.5)**, rendering computation of (5.1.12) particularly awkward. Even if obtainable, the covariances (5.1.12) need to be **further** manipulated to give the variance of G:

$$V[G] = \sum_{s,t=1}^{N} (x_s - y_s)(x_t - y_t)\ \text{Cov}[Z_s, Z_t]. \qquad (5.1.13)$$

Despite these computational difficulties, the distribution of G is well-defined when the underlying interest rate dynamics are properly defined. Let $\Psi(.)$ be the **d.f.** of G. Then the probability of adequacy of assets held to meet corresponding liabilities is:

$$\text{Prob}[G \geq 0] = 1 - \Psi(0-). \qquad (5.1.14)$$

The above computational difficulties involved in evaluation of the distribution of G suggest that the most effective approach to this question in practice would be simulation. There is no particular difficulty in this. Once the basic interest rate dynamics have been specified, as in (3.2.3) or **(3.2.15)**, the trajectory of future spot rates (and forward rates in the case of **(3.2.15)**) may be simulated, leading to a simulated collection of Z_t for values of t up to the largest required. The distribution of G is then simulated via **(5.1.9)**.

Emphasis should be given here to the assumption made at the start of this sub-section, namely that asset cash flows are those generated directly by the assets held at valuation date. Any rearrangement of the composition of the portfolio, either at the valuation date or subsequently, will change these cash flows.

Such a change would **not** change the value of assets held at the time of the change, and therefore would cause no change to the value of assets held at the valuation date if anticipated then.

The reason for this is that, at time t when purchase or sale of bonds is assumed to occur, the model of market prices involves their encapsulation of all available information relevant to future rate (and price) movements. Likewise, the distribution of these prices at time t is incorporated in the value of Z_t. As a result, the inclusion of explicit reinvestment and early sale in the simulation will lead to the same value of G **provided** that simulated prices move in a manner consistent with the relation between the Z_t and the **stochastic** interest rate dynamics.

In noting the constraint on price movements imposed by past rate movements and the dynamics of future rate movements, it is perhaps opportune to recall and emphasise a further constraint in the model (3.2.15) of Heath, Jarrow and Morton (1988). Specifically, the independence of $\phi(s)$ of u in (3.2.17) implies a coupling between the forward drift and volatility terms, α and σ respectively. This coupling needs to be incorporated in any simulation of the interest rate model (3.2.15).

Although variations in composition of the bond portfolio have no influence on the value of that portfolio, they do affect the risk to which the insurer is exposed. For example, if the insurer were to move from an exactly matched to a dead short position in the instant immediately after valuation, the investment risk would be changed from nil to something positive. Thus, any assessment of the financial effects of mismatching, as considered in Section 5.2 and later sections of this paper, will need to take future investment policy into account.

5.2 Mismatching reserve

Section 5.1 considers the situation in which the market values of assets and liabilities were equal at the valuation date, i.e.

$$E[G] = 0. \tag{5.2.1}$$

Then (5.1.14) gave the probability of ultimate adequacy of assets, taking into account any mismatch with liabilities. The probability depended on the d.f. Ψ of G.

An alternative approach, and one which is desirable in practice, is to set the probability of adequacy of assets to a predefined level and, for a given profile of maturity dates, calculate the value of assets (i.e. calculate the value of $E[G]$) required to achieve this probability.

Treatment of this question will require a small extension of notation to recognise the volume of assets as a variable in the problem. As in Section 5.1, let $y_1, ..., y_N$ denote the (supposedly known) liability cash flows, and $x_1, ..., x_N$ the asset cash flows when the total volume of assets is such that (5.2.1) holds.

The definition of the asset cash flows will need to be broadened at this point, for the reason given at the end of Section 5.1. The cash flows $x_1, ..., x_N$ are those generated directly by assets held at the valuation date, and are known with certainty.

In the case of any future changes in bond portfolio composition, the prices at which the changes take place will contain a component of uncertainty. Hence, so will the asset cash flows subsequent to the transactions concerned. Thus, asset cash flows should be represented as random variables $X_1, ..., X_N$, which depend on:

(a) future changes in composition of bond portfolio in accordance with investment policy; and

(b) the simulated investment conditions at the times of these changes.

Let, $X = (X_1, ..., X_N)^T.$

It will be supposed that the control variable by which the probability of asset adequacy is varied is a multiplier, λ say, of the vector X. Thus one envisages a bond portfolio of particular term structure, greater in value than liabilities ($E[G] > 0$) and providing a cash flow vector kX say with $k>1$. This portfolio is to be divided into two parts:

(a) a proportion λ/k (generating cash flow XX) assigned to cover liabilities with prescribed probability of adequacy; and

(b) a proportion $1-\lambda/k$ regarded as surplus to the service of liabilities.

The net surplus cash flow vector g deriving from the proportion λ/k will now depend on λ:

$$g(\lambda) = \lambda X - y, \tag{5.2.2}$$

and G will also depend on X, so that (5.1.4) becomes:

$$G(\lambda) = g^T(\lambda) Z, \tag{5.2.3}$$

with Z denoting the random vector $(Z_1, Z_2, ... Z_N)^T$.

Setting $\lambda=1$ equates values of assets and liabilities, i.e.

$$E[G(1)] = g^T(1) E[Z] = 0 \tag{5.2.4}$$

The distribution of G will now also depend on X, and so its d.f. will now be written as $\Psi_\lambda(.)$.

As in (5.1.14). the probability of adequacy of assets to meet liabilities is:

$$\text{Prob}[G(\lambda) \geq 0] = 1 - \Psi_\lambda (0-). \tag{5.2.5}$$

Section 5.1 suggested the computation of $\Psi_1(.)$ (just $\Psi(.)$ in Section 5.1) by means of simulation. The performance of simulations for various values of λ in order to solve (5.2.5) would be laborious.

In fact, such a procedure is in any case unnecessary for the following reason. Substitution of (5.2.2) in (5.2.3) yields:

$$G(\lambda) = (XX - y)^T Z$$

$$= \lambda G_X - G_y, \tag{5.2.6}$$

where'

$$G_X = X^T Z, G_y = y^T Z. \tag{5.2.7}$$

Now suppose that p is the required probability of adequacy of assets, i.e. by (5.2.5), the asset multiplier λ must be such that:

$$p = 1 - \Psi_\lambda(0-). \qquad (5.2.8)$$

But

$$\Psi_\lambda(a) = \text{Rob}\,[G(\lambda) \le a]$$
$$= \text{hob}\,[\lambda G_X - G_y \le a] \quad [\text{by } (5.2.6)]$$
$$= \text{hob}\,[(a + G_y)/G_X \ge \lambda]. \qquad (5.2.9)$$

Substitution of (5.2.9) in (5.2.8) yields

$$\text{Prob}\,[G_y/G_X \le \lambda] = p. \qquad (5.2.10)$$

Thus, the steps required in the procedure to calculate λ satisfying (5.2.8) are as follows:

(a) simulate values of the random vector **Z** as described in Section 5.1;

(b) for each simulated value of Z, compute G_X, G_y defined by (5.2.7);

(c) hence obtain the simulated distribution of G_y/G_X (remembering that numerator and denominator of this ratio are not independent);

(d) the p-percentile of this distribution is the required value of λ.

A value of $\lambda > 1$ implies an asset holding of greater value (at the valuation date) than the value of liabilities. Indeed, the value of the net asset holding is $G(\lambda)$. This quantity is therefore the value of the mismatching reserve required to achieve probability p of adequacy of assets.

It is noteworthy that, as long as one is concerned with a central estimate of loss reserve, this mismatching reserve should play no part. Any recognition of mismatch is in the nature of a provision for adverse deviation of experience. This conclusion is summarized in the following proposition.

Proposition 5.2.1. When assets supporting a loss reserve comprise only bonds, the central estimate of that loss reserve is obtained by discounting the centrally estimated liability cash flows at market rates of return, i.e. using the discount factors z_t. This central estimate of loss reserve thus takes no account of any mismatch between assets and liabilities. Such mismatch may be recognised in a provision for adverse deviation, calculated as described in the earlier part of this sub-section. This particular provision would be incorporated in a total provision for adverse deviation which also deals with unfavourable experience in the liability cash flows.

The conclusion will seem foreign to those who are accustomed to discount at conseivative rates of interest in order to recognise the possibility of unfavourable future development of interest rates. It appears nevertheless to be the only approach logically consistent with the basic concepts of central estimate and provision for adverse deviation (PAD) respectively.

One further matter deserves a little discussion before this sub-section is closed. This is the assignment of a **cross-section** of the **total** asset portfolio (via the parameter λ as discussed earlier in this sub-section) to the support of liabilities.

This is not the only way of apportioning the asset portfolio between liabilities and surplus assets. For example, one might decide that:

(a) liabilities are supported by that part of the bond portfolio which provides cash flows matching liability cash flows, provided that total cash flow generated by the portfolio is sufficient at all durations to do this; and

(b) the balance of the bond portfolio is surplus.

There seems to be no particular logic for choosing between these two possible apportionments of assets (nor from all the intermediate apportionments). A similar question was discussed by Taylor (1984) in connection with assets which do not earn income (e.g. brokers' balances). When these exist, should the assets regarded as supporting liabilities be a cross-section of the total asset portfolio, including a component not generating income? Or might one assign only income earning assets (even the highest yielding) to liabilities?

In that earlier paper, as in this one, the cross-sectional approach was preferred, since the alternatives seem rather contrived. However, the absence of hard logic from this justification is to be emphasised.

5.3 Fixed interest default risk

Up to this point, 'the only fixed interest assets considered have been (default free) bonds. More generally, it is necessary to cbnsider fixed interest holdings which include securities subject to risk of default.

Such securities carry higher yields than bonds in compensation for the default risk, possible illiquidity, and sometimes other factors as well such as tax or stamp duty considerations. It will anticipate Section 6 slightly to suggest that:

(a) the ultimate expected rate of return on risky fixed interest securities exceeds that on risk free, after due allowance is made for expected defaults;

(b) as a consequence, the expected discounted value of liabilities supported by a risky fixed interest portfolio will be somewhat less than if supported by a bond portfolio of the same term structure;

(c) as a consequence of the default risk, an additional to the PAD is needed.

A rigorous treatment of these matters would require a stochastic model of the occurrence of defaults. While there is no reason in principle why this could not be done, it would involve the establishment of quite elaborate structure to deal with what is usually, in financial terms, a relatively small issue.

Unless the fixed interest assets include high risk items, such as junk bonds, the interest rate differentials are usually quite small. A practical expedient might be:

(a) to value assets at market value, i.e. in accordance with market redemption yields where these include a risk premium in cases where default risk exists;

(b) to value liabilities at default free rates of discount, i.e. as if they were supported by default free bonds of the same term structure as the fixed interest assets actually held.

If a portfolio of (matched) assets so valued were equal in value to liabilities, the expected value of net assets at the completion of payment of liabilities would be slightly positive.

6. VALUATION OF LIABILITIES SUPPORTED BY VARIABLE RETURN ASSETS

6.1 General

Sections 3 to 5 have considered the valuation of liabilities only in the case in which supporting assets are entirely of the fixed interest type. Rather similar considerations arise in connection with other types of assets, such as shares, property, etc. These are generically labelled variable return assets hereafter.

As was the case with fixed interest assets, a central estimate and provision for (asset) adverse deviation can be established only on the basis of a model of the relevant asset prices. The purpose of Section 6.2 will be to provide some of the details of such a model, and to deal with the central estimate of liability. Section 6.3 will then deal with the question of PAD.

Throughout Sections 6.2 and 6.3, the relevant asset portfolio will be assumed to consist entirely of variable return assets. Portfolios containing a mixture of fixed interest and variable return assets will be the subject of Section 7.

6.2 Central estimate of liability

For the sake of brevity, the present sub-section and also Section 6.3 will be phrased in terms of shares. All that is said is intended, however, to apply equally to all forms of variable return asset.

In attempting to assign a value to a shareholding as a support for a liability portfolio, and more particularly in calculating a reasonable PAD, one may return to the structure set up in (5.1.1) to (5.1.4). Those equations apply equally to a shareholding if two of the components are reinterpreted:

(a) the asset cash flows are now those generated by the shares, and are therefore regarded as stochastic;

(b) the factors z_t are those appropriate to discount of the shares involved rather than the bonds considered earlier.

Because of the stochastic nature of the asset cash flows, these are denoted by X_t instead of x_t. Further, the discount factors will also be regarded as random, as in (5.1.9), and represented by Z_t. Then (5.1.1) (or (5.1.9)) is replaced by:

$$G = \sum_{t=1}^{N} (X_t - y_t) Z_t . \tag{6.2.1}$$

Now as long as the income generated by a shareholding is considered to comprise only dividends, N must be infinite. However, the value of the shareholding may alternatively be regarded as consisting of:

(a) the value of dividends payable at epochs 1.2, ..., N; and

(b) the market value of the shareholding at epoch N (since that cash flow could be generated by sale at that point).

These two viewpoints will be assumed to lead to the same total value of the shareholding. This amounts to an assumption of no arbitrage over time in terms of date of sale. This is essentially the same as the assumption of no arbitrage over term made in relation to the valuation of fixed interest assets, an assumption made in the preamble to (3.2.8) and also made explicitly by Heath. Jarrow and Morton in the development of (3.2.16).

When the asset cash flows are interpreted in this way, it is perhaps more informative to revise the notation slightly, as follows:

D_t = dividend income at epoch t = 1, 2, ..., N;

S_t = market value at epoch t = 1, 2, ..., N.

Then X_t may be replaced in (6.2.1) as follows:

$$G = \sum_{t=1}^{N} (D_t - y_t) Z_t + (D_N + S_N - y_N) Z_N . \tag{6.2.2}$$

It follows from this that the shareholding is adequate (in expected value) to meet liabilities if:

$$E[G] = \sum_{t=1}^{N-1} E[D_t Z_t] + E[(D_N + S_N) Z_N] - \sum_{t=1}^{N} y_t E[Z_t] = 0. \tag{6.2.3}$$

Now, by the no arbitrage assumption shortly preceding (6.2.2), the last result may be put in the form:

$$S_0 - \sum_{t=1}^{N} y_t z_t = 0, \tag{6.2.4}$$

where S_0 = market value of shareholding at valuation date, and

$$z_t = E[Z_t].$$

This last relation is notationally consistent with Section 5 (see just after (5.1.9)), but note that the discount factors Z_t now relate to shares rather than fixed interest investments.

Finally, (6.2.4) may be written as:

$$\sum_{t=1}^{N} y_t z_t = S_0, \tag{6.2.5}$$

i.e.,

$$\text{discounted value of liabilities} = \text{market value of, shareholding} \tag{6.2.6}$$

in order for that shareholding to be just adequate in terms of expected value.

The discount factors z_t in (6.2.5) must be consistent with those of Section 5, but recognise the different return generating process affecting shares. For example, if z_t is given by (5.1.10) for fixed interest assets, the same formula holds for shares, but with rate of return r_s now that produced by the relevant shareholding.

Computation of the z_t in this case would require specification of the process by which share prices evolve and their joint distribution with aggregate wealth in the economy. The process of price evolution would need in turn to be consistent with the CAPM (Sharpe, 1964; Lintner, 1965) which gives the relation between risk and return:

$$r_s = r_{fs} + \beta_s(r_{Ms} - r_{fs}) + e_s, \tag{6.2.7}$$

where

r_{fs}	=	the risk free rate of return at epoch s;
r_{Ms}	=	the share market index rate of return at epoch s;
e_s	=	a stochastic term with $Ee_s = 0$;
β_s	=	$\text{cov}[r_s, r_{Ms} - r_{fs}]/V[r_{Ms} - r_{fs}]$.

Then

$$E[r_s] = E[r_{fs}] + \beta_s \{E[r_{Ms}] - E[r_{fs}]\} \quad (6.2.8)$$

$$V[r_s] = V[r_{fs}] + \beta_s^2 V[r_{Ms} - r_{fs}] + V[e_s], \quad (6.2.9)$$

provided that r_{fs}, $r_{Ms} - r_{fs}$, and e_s are mutually stochastically independent.

6.3 Provision for adverse deviation

Let X be as in Section 5.2, **i.e.** the multiplier of the portfolio of Section 6.2 required to achieve some predefined probability of adequacy of the share portfolio. The reasoning follows that of Section 5.2. In parallel with **(5.2.2), g(λ)** denotes:

$$g(\lambda) = X(D+S) \cdot y, \quad (6.3.1)$$

where $D^T = (D_1, ..., D_N)$ and $S^T = (0, ..., 0, S_N)$.

Then (5.2.3) to **(5.2.10)** continue to hold with X replaced by D+S. Thus, (5.2.6) may be represented as:

$$G(\lambda) = [\lambda(D+S) - y]^T Z = \lambda G_X - G_y, \quad (6.3.2)$$

now with

$$G_X = (D+S)^T Z, G_y = y^T Z. \quad (6.3.3)$$

Then the reasoning leading to (5.2.10) continues to hold for the present case, and **(5.2.10)** holds, **i.e.**

$$\text{Prob}[G_y/G_X \leq \lambda] = p, \quad (6.3.4)$$

with G_X, G_y now defined by (6.3.3).

Note that in the case in which Z_t is given by (5.1.10).

$$G_X = [\sum_{t=1}^{N-1} D_t \; I(t,N) + (D_N + S_N)] \times R, \quad (6.3.5)$$

where R denotes the ratio of expected values appearing in (5.1.10). The square bracketed term of (6.3.5) may be interpreted simply as the **accumulation** to epoch **N** of the shareholding with dividends reinvested in a cross-section of the shareholding. A similar interpretation applies to G_y.

Note also that (6.3.4) may be put in the alternative form:

$$\mathrm{Prob}[(G_y/R)/(G_X/R) \leq \lambda] = p. \qquad (6.3.6)$$

Although this appears a less convenient form than **(6.3.4),** it in fact has a simple interpretation: the subject of the square bracketed inequality is just the ratio of the accumulation of liability cash flows to the accumulation of asset cash **flows,** accumulation always being in a cross-section of the relevant **shareholding.**

6.4 Summary and discussion of the results

The first major conclusion, given by **(6.2.6),** is summarized as follows.

Proposition **6.4.1.** A share portfolio will be just adequate to meet a particular fixed liability portfolio in terms of expected values provided that the market value of the shares is equal to the discounted value of the liabilities on the basis of discount factors appropriate to the return generation process of the shares (the meaning of this last qualification **being** discussed in Section **6.2).**

The result is quite consistent with Proposition 4.1.1 on fixed interest asset portfolios, having been obtained by the same reasoning.

The question of adverse deviation is dealt with as follows.

Proposition 6.4.2. The magnitude of the share portfolio required in order to be adequate to meet a particular fixed liability portfolio with a predefined probability is given by **(6.3.6).**

In particular circumstances, **e.g.** when Z_t is given by (5.1.10), this involves an examination of the ratio of accumulated **liability** cash flows to accumulated asset cash flows, with accumulation always being in a cross-section of the relevant shareholding.

There is one major implication of Propositions 4.1.1 and 6.4.1 which may be a little disconcerting to actuaries. It arises if the expected future rate of return from shares exceeds that from fixed interest investments, as most historical studies suggest is usually the case.

Now in very rough terms, the discounting suggested in Propositions 4.1.1 and 6.4.1 consists of taking present values at approximately the expected future rate of return. This implies that the same portfolio of liabilities will usually be assigned a lower value when supported by shares than when supported by fixed interest investments. Many actuaries would, I suspect, reverse the relationship on the ground of the higher risk involved in share investment.

There is, in fact, no paradox here. The conclusion drawn above applies only if the amounts of assets required to support a particular portfolio of liabilities can be calculated so as to be adequate in terms of statistical expectation. In this case, higher expected return implies lower volume of required assets.

If, however, one requires greater certainty of adequacy of assets by the establishment of a provision for adverse deviation (in investment returns), then the greater risk of share investment becomes relevant. While the greater expected return on shares decreases the value of liability, relative to that supported by fixed interest investments, the greater volatility of shares would generate an increase in the PAD, again relative to the fixed interest case.

As the confidence level required of the assets is increased, the total value of liabilities including PAD increases more rapidly under share than under fixed interest investment. At a certain confidence level, the total value of liabilities under share investment will exceed that under fixed interest investment.

Note, however, the distinction between the value of liabilities in terms of statistical expectation - call this the "central estimate" of liabilities - and the PAD. The equating of statistical expectation with "central estimate" terminology is in accordance with conventional actuarial usage.

It is common (though not universal) actuarial practice in loss reserving to provide separate opinions on:

(a) the central estimate of liability;

(b) the PAD in the form of:

either (i) a single recommendation regarding the quantum of this provision;

or (ii) just the identification of the levels of provision needed to achieve various probabilities of adequacy.

It would appear to follow from this that the central estimate of liability should comprise the statistical expectation of unpaid claims, discounted in terms of statistical expectations. The PAD would then be the additional amount required to achieve the prescribed probability of adequacy taking into account both:

(a) fluctuation in the claims process itself; and

(b) the investment risk involved in the selection of assets held.

Such a division of the total loss reserve would preserve the situation discussed above in which the central estimate would be lower when supported by share investment than by fixed interest investment.

While this division is logical, it may create practical difficulties for actuaries to the extent that management in receipt of the actuarial advice is inclined to take liberties with the PAD. It nevertheless seems preferable that the logic be followed and that management, perhaps with some coercion from auditors, come properly to grips with the nature and necessity of the PAD.

Just as considerable computational difficulties were noted in Section 5.1 in connection with fixed interest asset portfolios, no less difficulty occurs in connection with the share portfolios discussed in the present section. Just as in Section 5, the only practical solution would appear to involve simulation.

7. OPEN AND CLOSED FUNDS

7.1 Fixed interest investments

The concepts of open and closed funds were discussed in the earlier paper by Taylor (1984). Their relevance was reviewed in Section 2.

The concept of an open fund was applicable only to the case in which the assets to be identified over future years with the liabilities under valuation would be changing in composition; the liabilities at any point of time being supported by a slice taken across all assets held at that time.

Effectively, an open fund involves continual purchase and sale of assets between separate generations of policyholders. Only thus does the portfolio of assets held at the valuation date, and so identified with liabilities in existence at that date, change to the portfolio held at some later date instead of to just the remnants of the original holding unconsumed by claims expenditure in the interim.

These notional purchases and sales take place at market value since only market values have been considered throughout this paper.

The approach to this question adopted in the earlier paper (Taylor, 1984) was, as discussed in Section 2 here, to project future rates of return and so the capital profits and losses made on the notional purchases and sales.

Again as noted in Section 2, there seems' nothing objectionable in principle in this procedure. It is necessary, however, to ensure that the projected future rates of return are consistent with the information about market expectations contained in current prices.

This last requirement is encompassed in the asset values computed in Sections 3 and 4, leading to Proposition 4.1.1. Those values assume no arbitrage over different terms to maturity. The effect of this is that, in expected values, the value derived from investing a unit in a particular zero coupon bond at market price will be the same as continuously rolling the investment of that unit over at the spot rate until the maturity date of the bond.

In other words, the discounted value of the bond in unaffected by whether it is held to maturity or sold early. Current market information anticipates the capital profit or loss generated by an early sale, and this is incorporated in the market value.

This market value is therefore the discounted value of the bond whatever its owner's intentions with respect to sale.

Once the need to relate projected future rates of return to current yield curve information is properly recognised, the whole procedure of bond valuation explicitly incorporating future rates of return and capital movements becomes identical with the adoption of market values.

It follows that, while the open fund concept reflects the process of transfer of bond holdings between generations of policyholders, it is not a tool needed for the valuation of bonds, whose central value should always be taken as market value.

The open fund concept is required, however, in dealing with questions of risk of, as opposed to return on, the fixed interest portfolio, and particularly in the computation of a suitable PAD (see Section 5.2).

7.2 Variable return investments

Much the same arguments as in Section 7.1 apply to a portfolio of variable return investments (for brevity, they will be referred to again as "shares"), provided that the composition of that portfolio is maintained constant. The case in which it is allowed to change is considered in Section 7.3.

As in Section 7.1, notional sales of parts of the portfolio between generations of policyholders can be contemplated. These sales take place at market price.

As for bonds, it is assumed that there is no opportunity for arbitrage across future dates of sale of a holding of shares. This is discussed in the passage following (6.2.1). It is then shown in Section 6.2 that the discounted value of shares is equal to the market value, regardless of the owner's intention with respect to future sale.

Just as for bonds, while the open fund concept reflects the process of transfer of variable return assets between generations of policyholders, it is not required for valuation purposes. The open fund concept is still not required for evaluation of a PAD as long as the assumption of constant composition of the variable return portfolio continues to hold.

7.3 Mixture of fixed interest and variable return assets

The arguments of Section 7.1 require some modification when the asset portfolio is a mixture of fixed interest and variable return assets, or involves a changing composition in either sector. In this case, the concept of an open fund becomes relevant to asset valuation, as discussed below.

For expository purposes, the case considered here initially will involve asset holdings in two sectors, say:

(a) a holding of bonds (various terms to maturity permitted); and

(b) a holding of shares whose precise composition matches some market index.

The specific quality of this portfolio which distinguishes it from those considered in Sections 7.1 and 7.2 is that its expected rate of return depends on the relative proportions held in the two sectors. Thus, for example, the following two cases yield different expected returns:

Case I: Assets are initially split 50150 between to the two sectors. As liability cash flows occur in excess of income generated by assets, assets are sold down in **parcels** which are **50/50** by value.

Case **II**: Assets **are** initially split **50/50** between to the two sectors. As liability cash flows occur in excess of income generated by assets, assets are sold down in such a way as to maintain the **50/50** split of the portfolio.

In Case I, the expected proportion of the portfolio (by market value) represented by shares would increase in the absence of any asset sales, and with income of each sector reinvested in that sector, since the shares will carry the greater expected rate of return. If the selling strategy of Case I is adopted, therefore, the expected proportion in shares will rise steadily from its initial value of 50%. This contrasts with Case **II** in which the proportion is specifically maintained at 50%.

It follows that Case I would produce a higher expected rate of return than Case **II.** Hence, the discounted (central) value of liabilities would **be** less in Case I than in Case **II.**

It is evident from this argument that the concept of an open fund becomes relevant to the determination of the rates of discount of loss reserves when the investment portfolio contains more than one sector.

Indeed, the same arguments apply to any changes in the composition of assets, even within sector, which shift the portfolio's expected rate of return. For example, in an asset portfolio consisting only of shares, there might **be** a planned shift **from** lower to higher **beta** shares, with the accompanying increase in expected rate of return.

Thus, in any case in which the composition of the asset portfolio, with respect to expected rate of return, will undergo shifts in future years. the valuation of discounted loss reserves will need to employ the open fund concept. The projected changes in composition will be reflected in changes in expected yield and hence in the discounted value of liabilities.

A more precise statement of the valuation procedure is as follows.

As in Section 5, consider a cash flow vector **kX** deriving from the fixed interest investments held at valuation date, taking into account an future sales of those investments (and corresponding purchases of new investments) in accordance with investment policy, and where k is a constant defined below. As in Section 6, let kD represent the (stochastic) **vector of** dividend payments generated by the variable return assets and let $S = (0, \ldots, 0, S_N)^T$ with kS_N the market value of those assets at epoch N. Now expand the notation slightly by decomposing D and S as:

$$D = \sum_{i=1}^{m} D^{(i)}, \quad S = \sum_{i=1}^{m} S^{(i)}, \tag{7.3.1}$$

where the suffix **i** labels the different segments of variable return assets which provide different expected rates of return.

It will not be necessary to retain a separate value of **i** for every beta. For example, if the only variable return asset were an indexed equity fund, then the average beta would always be unity, and only one value of **i** would need to be considered.

Alternatively, if investment were always in **units of** a group of high beta stocks and in units of a group of low beta **stocks,** then two values of i would need to be considered.

As before, let y denote the vector of liability cash flows; k the number of times liabilities are covered by **assets, i.e.** the discounted **central** value of liabilities is **equal** to **1/k** times the market value of **assets**; and λ is a multiplier such that a proportion λ/k of all asset holdings are of sufficient value to cover liabilities with prescribed probability p.

Then the basic equations (5.2.3) and (6.3.2) are combined in the following form:

$$G(\lambda) = \{\lambda[X + \sum_{i=1}^{m} (D^{(i)} + S^{(i)})] - y\}^T Z$$

$$= \lambda [G_X + \sum_{i=1}^{m} G_{(i)}] - G_y , \quad (7.3.2)$$

with

$$G_X = X^T Z, \quad G_y = y^T Z, \quad (7.3.3)$$

$$G_{(i)} = (D^{(i)} + S^{(i)})^T Z, \quad (7.3.4)$$

and with the discount factors Z_t appropriate to the mixed portfolio under consideration.

For example, in the case in which Z_t is given by **(5.1.10),**

$$G_X = \sum_{t=1}^{N} X_t \, I(t,N) \times R, \quad (7.3.5)$$

where R denotes the ratio of expected values appearing in **(5.1.10).** As in (6.3.5), the factor other than R on the right side of (7.3.5) may be interpreted as the accumulation to epoch N of fixed interest cash flows (both income and maturing capital) at the rates of return generated from time to time by the total portfolio.

This rate of return of the tote! portfolio, over a particular time interval, would bring to account all income and all movement in capital values over that **interval.** It would also reflect the planned composition of the portfolio over the interval.

The other terms G_y, $G_{(i)}$ appearing in (7.3.2) are subject to interpretation similar to that applying to G_X.

The value of λ required to produce the prescribed probability p of adequacy of assets is now obtainable just as in **(5.2.10)** and (6.3.4):

$$\text{Prob } \{G_Y / [G_X + \sum_{i=1}^{m} G_{(i)}] \leq \lambda\} = p. \tag{7.3.6}$$

The discounted central value of liabilities is

$$E[G_y] = y^T E[Z], \tag{7.3.7}$$

which is the present value of liability cash flows calculated at expected rates of return generated by the mixed asset portfolio under consideration.

The issues involved in the division between a central value of liability and a PAD are exactly as discussed in Sections 6.3 and 6.4.

While it may be possible to compute the central value (7.3.7) analytically, simulation will almost certainly be required to compute the magnitude of the PAD in accordance with (7.3.6). The steps required are:

(a) simulate investment conditions over the next N periods, including:

(i) bond spot yields (consistent with the forward yield curve at time 0);

(ii) for each sector of variable return assets, the dividend yields and total market value rates of return (again consistent with the fixed interest forward yield curve if there is any hypothesized linkage with **fixed** interest returns);

(b) from these simulated investment conditions, compute the total rate of return on the mixed portfolio over each interval between epochs 0 and N, taking due account of the planned composition of the portfolio, and allowing for all income and movement in capital values;

(c) **from** these rates of return compute the random vector Z according to whatever model (e.g. (5.1.10)) is in use for converting 'random rates of return into discount factors;

(d) for each simulated value of Z, compute $G_X, G_{(1)}, \ldots, G_{(m)}, G_y$ defined by (7.3.3) and (7.3.4).

(e) hence obtain the simulated distribution of the ratio

$$G_y \,/\, [G_X + \sum_{i=1}^{m} G_{(i)}]$$

(remembering that numerator and denominator are not independent);

(f) the p-percentile of this distribution is the required value of λ.

Note that step (b) allows for the insurance portfolio to be treated as a going concern. The investment of new money generated by future business may be reflected in the portfolio composition anticipated in (b). As explained in Sections 7.1 and 7.2, this may involve notional sales of assets between different generations of policyholders.

Note also, however, that despite the bringing of new money to account in this way, there is **no need to bring new money explicitly into computations.** Since sales between generations of policyholders are assumed to take place at market value, any change in the future composition of portfolio attributable to loss reserves at the valuation date can be viewed as achieved by on-market transactions.

It may be noted here that the above simulation procedure for determining a PAD is essentially the same as that suggested by Brooks, Condon and McLeod (1989) in connection with life insurance capital guarantees.

They point out that the investment risks undertaken by life offices in this field amount to the underwriting of put options. they also point out that, while it is possible to construct a riskless hedge against such an option, life offices tend not to do so in practice. The resulting exposed position requires the establishment of a mismatching reserve. The magnitude of this reserve is such as to produce adequacy of total reserves with prescribed probability.

The results obtained in Sections 7.1 and 7.2 are in fact special cases of the present sub-section. All of these results may be summarized as follows.

Proposition 7.3.1. **A** general portfolio of assets will be just adequate to meet a particular fixed liability portfolio in terms of expected values provided that the market value of the assets is equal to the discounted value of the liabilities. This discounted value is calculated on the basis of discount factors which reflect the planned future composition of the asset portfolio by sector and the return generating process of each sector, as discussed in the present sub-section.

□

Proposition 73.2. The magnitude of a general portfolio of assets required in order to be adequate to meet a particular fixed liability portfolio with a predefined probability is given by **(7.3.6).** In particular circumstances, **e.g.** when Z_t is given by (5.1.10), this involves an examination of the ratio of accumulated liability cash flows to accumulated **asset** cash flows, with accumulation always being at the rate of return earned by **whole** portfolio from time to time, taking due account of the planned composition of the portfolio, and allowing for all income and movement in capital values.

8. UNCERTAINTY IN LIABILITY CASH FLOWS

All computations up to this point have been based on an assumption that **liability** cash flows y_t **are** known with certainty (see **e.g.** (7.3.6)). In practice, only estimates $\hat{Y}(t)$ of these **will** be available. These estimates are random variables containing uncertainty additional to that which resides in the investment return process.

The extent of this additional uncertainty may be calculated as follows. Let

$$A = \sum_{t=1}^{N} \hat{Y}(t)\ I(t,N) \tag{8.1}$$

= accumulation of liability cash flows to epoch N.
Then

$$V[A] = \sum_{s,t=1}^{N} C[\hat{Y}(s)\ I(s,N),\ \hat{Y}(t)\ I(t,N)]. \tag{8.2}$$

Now, if $\hat{Y}(s)$ and I(t,N) are stochastically independent, it is routine to check that

$$\begin{aligned} C[\hat{Y}(s)\, I(s,N),\ \hat{Y}(t)\, I(t,N)] = {} & C[\hat{Y}(s),\ \hat{Y}(t)]\ C[I(s,N),\ I(t,N)] \\ & + C[\hat{Y}(s),\ \hat{Y}(t)]\ E[I(s,N)]\ E[I,(t,N)] \\ & + C[I(s,N),\ I(t,N)]\ E[\hat{Y}(s)]\ E[\hat{Y}(t)]. \end{aligned} \tag{8.3}$$

Substitution of (8.3) in (8.2) yields:

$$V[A] = \sum_{s,t=1}^{N} C[\hat{Y}(s),\ \hat{Y}(t)]\ \ C[I(s,N),\ I(t,N)]$$

$$+ \sum_{s,t=1}^{N} C[\hat{Y}(s),\ \hat{Y}(t)]\ E[I(s,N)]\ E[I(t,N)]$$

$$+ \sum_{s,t=1}^{N} C[I(s,N),\ I(t,N)]\ E[\hat{Y}(s)]\ E[\hat{Y}(t)]. \tag{8.4}$$

Note that, if there is uncertainty in the investment return process, but liability cash flows are known with certainty, then (8.4) reduces to:

$$V[A] = \sum_{s,t=1}^{N} C[I(s,N),\ I(t,N)]\ y_s y_t. \tag{8.4a}$$

If, on the other hand, investment returns were known with certainty but only estimates of liability cash flows were available, then (8.4) would reduce to:

$$V[A] = \sum_{s,t=1}^{N} C[\hat{Y}(s),\ \hat{Y}(t)]\ I(s,N)\ I(t,N). \tag{8.4b}$$

Comparison of (8.4) with **(8.4a)** and **(8.4b)** indicates that:
total uncertainty in accumulated (or discounted) liabilities

=

expected accumulation (or discounted value of) uncertainty in liabilities

+

uncertainty in accumulation (or discounting) of expected liabilities

+

interaction of two sources of uncertainty. **(8.5)**

While (8.4) and (8.5) are informative as regards the total uncertainty due to errors of prediction in both liability cash flows and investment returns, in practice **simulation will** be required.

Because of the interaction of these two variables (see (8.1)) it will be necessary to simulate them simultaneously. Simulation of future liability cash flows implies bootstrapping the loss reserving model, as described by **Ashe (1986) and** Taylor (1988). Each bootstrap replication provides a realization of the vector $[\hat{Y}(1), ..., \hat{Y}(N)]$. At each of these replications, it is necessary to simulate investment return between epochs 0 an N and so obtain the accumulated or discounted value of the liability cash flows.

Simulations of the type described above, incorporating random fluctuations of both asset and liability cash flows, represent a special case of the situation considered in a recent series of English papers (Coutts, Devitt and Ross, 1984; Coutts and Devitt, 1986; Daykin and Bernstein, 1985; Daykin et al, 1986,1987; Daykin an Hey, 1988) and another series of Finnish publications (Pentikainen and Rantala, 1982; Pentikäinen, 1988; Pentikäinen and Pesonen, 1988; Pentikäinen et al, 1989).

Generally, these papers were concerned with the use of simulation to assess the solvency of an insurer in a dynamic setting in which new premiums continued to be received, although the English papers do give special attention to the case of an insurer closed to new business.

Those earlier papers differ substantially, however, from the present one in emphasis. The present paper uses simulation for three separate purposes:

(a) to obtain the discounted value (i.e. central value) of expected liability, i.e. ignoring variability of the liabilities themselves (Section 7.3);

(b) to estimate the PAD required in respect of investment risk, still ignoring liability risk (also Section 7.3);

(c) to estimate the total PAD required in respect of both investment and liability risks (Section 8).

Because of the emphasis here on investment issues, the present paper contains more comprehensive modelling of investment return. For example, none of the earlier quoted papers incorporates any model of interest rate term structure.

9. NUMERICAL EXAMPLE

There has been insufficient time to produce a numerical example illustrating the very comprehensive simulation procedure outlined in Section 8. However, it is hoped that such an example will be produced for presentation at the Colloquium.

10. CONCLUSIONS

The main results of the paper are contained in Section 7 dealing with an asset portfolio which is a mixture of fixed interest and variable return assets.

The discounted value of liabilities supported by this portfolio is the present value of those liabilities calculated essentially at the expected (in the statistical sense) rates of return to be generated by the assets over future years.

The risk involved in holding an asset portfolio which does not precisely match liability cash flows is provided for by a PAD whose magnitude is determined as sufficient to ensure adequacy of total assets (= discounted value of liabilities + PAD) with prescribed probability. This determination is made by simulation of future investment conditions.

The simulation of future investment conditions must recognise a number of constraints imposed by theoretical considerations and current market conditions. First, future movements in bond yields must reflect the forward yield curve as it exists at the date of valuation. Second, the treatment of fixed interest assets generally must reflect the relation between their yields and those of bonds. Third, any projection of yields on variable return assets must reflect their relation with bond yields through the CAPM.

The simulation of future investment conditions must also recognise management plans as regards the future distribution of the asset portfolio over the various sectors. In general, this requires the open fund approach discussed by Taylor (1984), in order that due allowance be made for changes to the portfolio composition over future years.

However, a couple of special cases provide exceptions to the need for the open fund approach. One is the case of a portfolio of variable return assets of constant composition in terms of beta coefficients. A second exception is only partial. The open fund approach is not required in connection with a bond portfolio, provided that only the value (i.e. questions of return, as opposed to risk) of the portfolio is considered. The open fund approach is required for assessment of the portfolio risk, and particularly any associated PAD.

As noted in Section 7.3, the required open fund approach does not involve any need to bring new money explicitly into computations. What is needed is to track any change in portfolio composition, which may of course be partially due to new money, rather than the volume of new money *per se*.

The **total uncertainty** of the discounted value of liabilities will depend not only on investment conditions but also on uncertainty in the liability cash flows themselves. The assessment of this total uncertainty is discussed in Section 8.

Most of the computations involved in practical evaluation of the central discounted value of loss reserve and an associated PAD will be feasible only by means of simulation. The simulations involved can be seen in the context of the extensive simulation work carried out by unglish and Finnish authors over recent years (Section 8).

APPENDIX A

ESTIMATION OF DISCOUNT FACTORS WITH IRREGULARLY SPACED TERMS TO MATURITY

Section 3.1 dealt with estimation of $z_1, z_2, ..., z_N$ from equations (3.1.3) when bonds are available in the market at terms to maturity $\Delta, 2\Delta, ..., N\Delta$. The same subsection also raised the question of how to deal with the situation in which bonds are not available at these regularly spaced terms to maturity.

Consider, for example, the case in which prices are observed on 31/12/88. Suppose that interest payments are made half-yearly (synchronized with maturity) on all securities, and that there exist securities (inter alia) with maturity dates 15/6/91 and 30/6/92. It is necessary to choose $\Delta = \frac{1}{2}$ month, in order that multiples of Δ capture all the transaction dates, 15/6/89, 30/6/89, 15/12/89, 31/12/89, ..., 30/6/91, 31/12/91, 30/6/92, associated with these two securities. It is unlikely. however (at least in the Australian financial markets), that there will be a market in securities of outstanding terms to maturity equal to all multiples of ½ month (up to a limit of ½N months).

Nevertheless, it may be possible in such a case to choose a larger value of Δ and regard the assumption $n_1 = 1, n_2 = 2$, etc. as holding approximately. For example, in the case dealt with above, it might be possible to choose $\Delta = 6$ months, and take the security maturing on 15/6/91 as a proxy for a 30/6/91 maturity (if such maturity does not exist in its own right).

If this approximation were adopted, it would be necessary to adjust the price of the 15/6/91 maturity to a price appropriate to 30/6/91 maturity. Initially at least, this adjustment could be made simply by computing the net present value of the security as if it maturity date were 30/6/91, and 'with discounting carried out at the security's market redemption yield.

This adjustment wiil introduce a small error, which is most easily illustrated in the case in which the security in question provides a coupon rate of j per half-year and return of unit capital at maturity. Then, reverting to a ½ month time unit, the actual value of the security is:

$$p^{(59)} = j(z_{11} + z_{23} + z_{35} + z_{47}) + (1 + j)\, z_{59}. \tag{A.1}$$

The value of the "as if" security would be:

$$p^{(60)} = j(z_{12} + z_{24} + z_{36} + z_{48}) + (1 + j)\, z_{60}. \tag{A.2}$$

Evidently, the value of the "as if" security is obtained from that of the actual security by modifying the successive terms of (A.1) by factors of $z_{12}/z_{11}, z_{24}/z_{23}$, etc.

Now in terms of redemption yield, i per half-month say:

$$p^{(59)} = j(v^{11} + v^{23} + v^{35} + v^{47}) + (1 + j)\, v^{59}, \tag{A.3}$$

with $v = 1/(1+i)$. The suggested proxy value for $p^{(60)}$ was:

$$j(v^{12} + v^{24} + v^{36} + v^{48}) + (1 + j)\, v^{60} = vp^{(59)}.$$

That is each term of (A.3) has been modified by the factor v.

In general, this will not yield the same result as the above modification by factors of z^{12}/z^{11}, z^{24}/z^{23}, etc.

The error introduced by the adjustment of maturity date will be small if the adjustment to the date itself is small. This is particularly so when one considers that prices are likely to be varying continually on market, so that observation of the whole vector p at any particular instant will not be possible. Nevertheless, it would be possible, if considered necessary, to refine the above hypothetical price of the "as if' security.

In the example given above, $p^{(60)}$ could be re-estimated using the initial estimates of z_6, z_{12}, z_{18}, etc.

This would involve:

(a) interpolation of these values to obtain z_{11}, z_{23}, etc.;

(b) application of (A.1) to calculate $p^{(59)}$ on the basis of these interpolated values:

(c) "adjustment" of the interpolated values of z_t in order that this computed values of $p^{(59)}$ agree with the observed value;

(d) inversion of interpolation to obtain "adjusted" values of z_6, z_{12}, etc. and hence an "adjusted" value of $p^{(60)}$;

(e) re-estimation of the z_6, z_{12}, etc. on the basis of the "adjusted" values of $p^{(60)}$ and similar hypothetical prices;

(f) iteration of steps (a) to (e) if necessary.

The precise nature of the "adjustment" introduced in (c) is left unspecified. Any reasonable adjustment will do provided that it leads in step *(f)* to convergence to the correct solution.

APPENDIX B

INTERPRETATION OF DISCOUNT FACTORS z_t IN TERMS OF THE REPRESENTATIVE INVESTOR

B1. General development and examples

A single period economy is considered, in which the aggregation problem has been solved, i.e. security prices are independent of the allocation of wealth across investors. It is assumed that security prices are determined as if there existed only identical representative investors. Let $u(.)$ denote the utility function of such an investor defined over end period wealth.

The market prices of various securities are then taken to be those which maximize the utility of a representative investor taking into account:

(a) consumption during the period;

(b) end period wealth, including that deriving from holdings in the various available securities;

(c) that the totals of (a) and (b) over all investors must be equal to the corresponding exogenously determined aggregates in the total economy, i.e. aggregate consumption and aggregate wealth,

(d) an assumption that aggregate wealth and security prices are random variables subject to particular distributions.

It may be shown under these conditions (e.g. Brennan, 1979, pp. 56-57) that the price p_0 of any particular security at the beginning of the period is given by:

$$p_0 = r_f^{-1} E[p_1 E[y(w)|p_1]], \tag{B1.1}$$

where

r_f = 1 + risk free rate of return over the period concerned,

w = representative investor's end period wealth = W/m where W is aggregate end period wealth and m is number of investors;

p_1 = end period price of the security;

E = statistical expectation operator

$y(w)$ = $u'(w)/Eu'(w)$ = relative marginal utility of wealth,

and w, p_1 (equivalently W, p_1) are subject to some joint distribution in accordance with (d) above.

Applying (B1.1) to the investment of $p_0 = 1$ at epoch T for a term of t, followed by reinvestment at unit intervals until epoch $T + N$ (taken to be end of period), giving $p_1 = I(t,N)/z_t$ according to the above, yields:

$$1 = r_f^{-1} E[I(t,N) z_t^{-1} E[y(w) | I(t,N)/z_t]].$$

i.e. $$z_t = r_f^{-1} E[I(t,N) E[y(w) | \{r_s, t \le s \le N\}]], \qquad (B1.2)$$

If one allows the term of zero coupon bond to approach zero with z_0 denoting the value of such a bond. Then $z_0=1$, and so (B1.2) yields

$$r_f = E[I(0,N) E[y(w) | r_s, 0 \le s \le N\}]] \qquad (B1.3)$$

Then substitution of (B1.3) in (B1.2) yields:

$$z_t = E[I(t,N) E[y(w) | \{r_s, t \le s \le N\}]] / E[I(0,N) E[y(w) | \{r_s, 0 \le s \le N\}]]. \qquad (B1.4)$$

This is the fundamental expression for the discount factors z_t. Simplification requires specific assumptions about the u(.) and the joint distribution of w and $\{r_s, 0 \le s \le N]$. As is evident in Rubinstein (1976) and **Brennan** (1979). simplification occurs for conjugate pairs of this utility function and joint distribution, e.g.

(a) exponential utility and joint normal distribution of y and $(\exp r_s, 0 \le s \le N\}$;

(b) power utility function and joint normal distribution of log w and $\{r_s, 0 \le s \le N\}$.

Alternative (b) is of greatest interest in the present case. It arises in the case in which de, follows a Wiener process in (3.2.3). Normality of de_s implies normality of dr(s) by (3.2.3), which in turn implies normality of log I (s,t) by (3.2.6) and (3.2.7). That is, log w and log I(s,t) are jointly normally distributed.

Since normality is being assumed, the distributions involved will be completely specified by first and second moments. Thus, let

$$\mu(s,t) = E[\log I(s,t)], \qquad (B1.5)$$

$$\sigma^2(s,t) = V[\log I(s,t)], \qquad (B1.6)$$

$$\nu = E[\log w], \qquad (B1.7)$$

$$\tau^2 = V[\log w], \qquad (B1.8)$$

$$\rho(s,t) = \text{Corr}[\log I(s,t), \log w]. \qquad (B1.9)$$

Note that $\mu(s,t)$, $u^2(s,t)$ and $\rho(s,t)$ will be expressible in terms of the basic parameters of the stochastic process $\{r_s$, OSsSN). For the time being, however, this will not be a matter of concern.

Now let the utility function, assumed to be a power function, have derivative

$$u'(w) = Kw^{-a}, a>0, K \text{ const.} > 0. \qquad (B1.10)$$

Then, as shown in Appendix B2,

$$z_t = \exp[-\mu(0,t) - \tfrac{1}{2}\sigma^2(0,t)] \exp[a\, C(0,t) - D(t,N)], \tag{B1.11}$$

where

$$C(0,t) = \mathrm{Cov}[\log I(0,t), \log w] = \rho(0,t)\,\sigma(0,t)\,\tau(0,t), \tag{B1.12}$$

and

$$D(t,N) = \mathrm{Cov}[\log I(0,t), \log I(t,N)] \tag{B1.13}$$

Note that the first exponential factor in (B1.ll) is, by the log normal assumption, equal to $1/E(I(0,t)]$. Thus,

$$z_t = \{1/E[I(0,t)]\} \times \exp[a\, C(0,t) - D(t,N)]. \tag{B1.14}$$

It is possible to identify some special cases of (B1.14) which are of some interest.

Special case 1. $0^2(0,t) = 0$

This is the case of **deterministic** future rates. The fact that $0^2(0,t) = 0$ implies that $C(0,t) = D(t,N) = 0$, by (B1.12) and (B1.13). Then (B1.11) gives:

$$z_t = \exp - \mu(0,t), \tag{B1.14a}$$

which is the usual discount factor in the deterministic case.

Special case 2. $a\, C(0,t) = D(t,N) = 0$.

This case arises if:

(a) there is no coupling between bond rates of disjoint periods $[D(t,N) = 0]$;

AND

(b) either

(i) investors are risk neutral $(a=0)$;

or

(ii) aggregate wealth is not **correlated** with bond rates $[C(0,t) = 0]$.

Then (B1.14) reduces to

$$z_t = 1/E[I(0,t)], \tag{B1.14b}$$

i.e. a discount factor is merely the reciprocal of the expected corresponding accumulation factor.

Special case 3. $D(t,N) = 0$, $C(0,t) = ct > 0$.

This case arises if

(a) there is no coupling between bond rates of disjoint periods, [$D(t,N) = 0$];

and

(b) $C(t,t+1) = \text{Cov}[\log I(t,t+1), \log w] = c$, const. > 0. **(B1.15)**

Then (B1.14) gives

$$z_t = \{1/E[I(0,t)]\} \times \exp act \qquad \textbf{(B1.14c)}$$

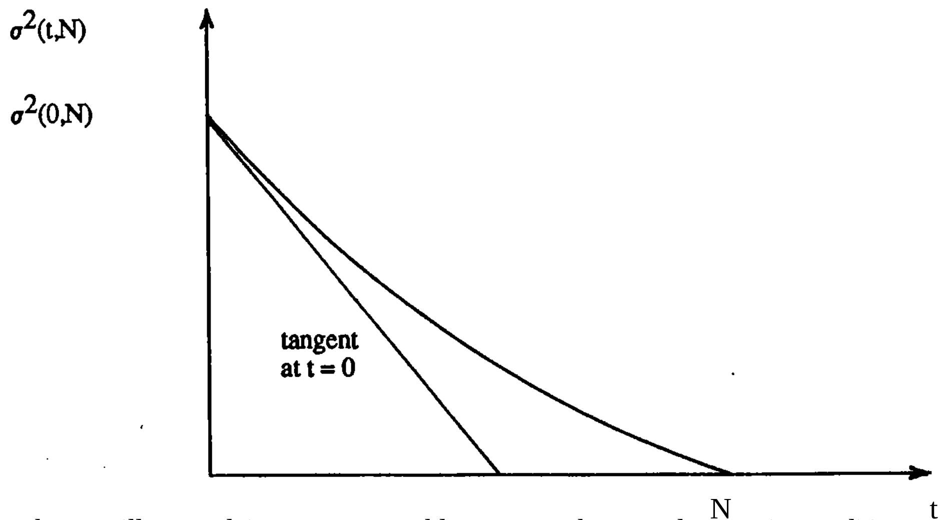
Note that the second factor here merely shifts the (continuous) redemption yield by ac at all terms to maturity. Thus, when assumption (B1.15) holds, the term $C(0,t)$ in (B1.14) has no effect on the shape of the yield curve.

Special case 4. $C(0,t) = ct > 0$, $(d^2/dt^2)\ \sigma^2(t,N) > .$

As in special case 3, the first condition arises if (B1.15) holds.

The condition on $\sigma^2(t,N)$ appears to represent the realistic case. If there were no coupling between interest rates of disjoint periods, then $\sigma^2(t,s) = (s-t)\sigma^2$ for some const. $\sigma^2 \geq 0$. In fact, correlations between disjoint periods would usually be assumed negative, indicating a long term tendency for rates to regress to a mean.

This would imply that $\sigma^2(t,N)$ would decrease less rapidly than linearly as t increases from 0 to N, as illustrated in the following diagram.



In the case illustrated, it seems reasonable to expect that stated convexity condition to hold. If it does, then

$$\sigma^2(t,N) = \sigma^2(0,N) + t(d/dt)\, \sigma^2(t,N)|_{t=0} + \beta(t), \tag{B1.16}$$

for some function $\beta(t)$ with $\beta(0) = \beta'(0) = 0$ and $\beta(t)/t$ increasing for $0 \le t \le N$.

For brevity, write $\alpha(t) = \sigma^2(t,N)$, so that the required condition (B1.16) becomes:

$$\beta(t) = \alpha(t) - \alpha(0) - t\alpha'(0)$$

Then

$$\beta'(t) = \alpha'(t) - \alpha'(0),$$

and

$$t\beta'(t) - \beta(t) = \alpha(0) - \alpha(t) + t\alpha'(t)$$

$$= - t\alpha'(\theta t) + t\alpha'(t),\ 0 < \theta < 1. \tag{B1.17}$$

But $\alpha'(t) = (d/dt)\, \sigma^2(t,N)$ which increases with t because of the condition on (d^2/dt^2) $\sigma^4(t,N)$. Hence (B1.17) yields

$$t\beta'(t) - \beta(t) > 0.$$

But this is the condition for $\beta(t)/t$ to increase with t, as required.

Now according to (A.15),

$$\sigma^2(0,t) + 2D(t,N) = \sigma^2(0,N) - \sigma^2(t,N) = t[\alpha(0) + \beta(t)/t], \tag{B1.18}$$

by (B1.16), with $\beta(t)/t$ increasing.

Insertion of (B1.18) in (B1.11) indicates that the effect of the $\sigma^2(0,t)$ and $D(t,N)$ terms in the latter formula is to raise redemption yields at longer terms to maturity relative to those at shorter i.e. to introduce a positive component to the slope of the yield curve.

B2. Algebraic evaluation of discount factor z_t

The fundamental expression for the discount factor z_t is (B1.4). Consider first the expectation $E[y(w)|\{r_s, t \le s \le N\}]$ in that expression.

By (3.2.6) and (3.2.7), $I(t,N)$ summarizes the information in $\{r_s, t \le s \le N\}$, and so

$$E[y(w)|\{r_s, t \le s \le N\}] = E[y(w)|I(t,N)] = c^{-1} E[w^{-a}|I(t,N)], \tag{B2.1}$$

by (3.2.17) and definition of $y(w)$, with

$$c = E[Ew^{-a}|I(t,N)]]. \tag{B2.2}$$

Then

$E[y(w)|\{r_s, t \leq s \leq N\}] = c^{-1} E[\exp - a \log w | \log I(t,N)]$, (B2.3)

where log w and log I(t,N) are jointly normally distributed with parameters as set out in (B1.5) to (B1.9).

For brevity the arguments (t,N) will be temporarily suppressed. Now note that, for given log I,

$\log w \sim N[\nu + \rho\tau/\sigma (\log (I - \mu), \tau(1 - \rho^2)^{1/2}]$. (B2.4)

To simplify notation, write $x = \log w$ and write (A.4) as

$x \sim N(\alpha, \theta^2)$. (B2.5)

Then (B2.3) becomes, via (B2.4) and (B2.5),

$E[y(w)|\{r_s, t \leq s \leq N\}] = c^{-1} E[Ee^{-ax}|\log I]$

$= c^{-1} \exp[- a\alpha + 1/2\, a2\, \theta^2]$ (B2.6)

$= c^{-1} \beta \exp[- \gamma \log I]$, (B2.7)

where, recalling the definition of α, one obtains

$\beta = \exp[- a(\nu - \mu\rho\tau/\sigma) + 1/2 a^2 \theta^2]$, (B2.8)

$= a\rho\tau/\sigma$. (B2.9)

Now c was defined by (B2.2), which reduces (B2.7) to:

$E[y(w)|\{r_s, t \leq s \leq N\}] = \exp[-\gamma \log I]/E \exp[-\gamma \log I]$. (B2.10)

Then, with J = log I, (A.lO) yields:

$E[I\, E[y(w)|\{r_s, t \leq s \leq N\}]] = E[e^{-(\gamma-1)J}]/E[e^{-\gamma J}]$, (B2.11)

with $J \sim N(\mu, \sigma^2)$.

This may be evaluated in the same way as the expression which led to (B2.6), giving:

$E[I\, E[y(w)|\{r_s, t \leq s \leq N\}]]$

$= \exp[-(\gamma-1)\mu + 1/2(\gamma-1)^2 \sigma^2]/\exp[-\gamma\mu + 1/2\gamma^2 \sigma^2]$

$= \exp[\mu + 1/2\sigma^2(1 - 2\gamma)]$

$= \exp[\mu + 1/2\sigma^2 - a\rho\tau\sigma]$, (B2.12)

Note that, by definitions (B1.6) to (B1.9),

$\rho\tau\sigma = \mathrm{Cov}[\log I(t,N), \log w]$. (B2.13)

Denoting this covariance by C(t,N), substituting it in (B2.12), reinstating the arguments of μ and σ, and substituting the result in (B1.4) gives:

$$z_t = \exp[\mu(t,N) + \tfrac{1}{2}\sigma^2(t,N) - a\,C(t,N)$$

$$- \mu(0,N) - \tfrac{1}{2}\sigma^2(0,N) + a\,C(0,N)]$$

$$= \exp[-\mu(0,t) - \tfrac{1}{2}[\sigma^2(0,N) - \sigma^2(t,N)] + a\,C(0,t)], \quad \text{(B2.14)}$$

where definition (B1.5) has been used.

Now, by definition (B1.6),

$$\sigma^2(0,N) = V[\log I(0,t) + \log I(t,N)]$$

$$= \sigma^2(0,t) + \sigma^2(t,N) + 2\,\mathrm{Cov}[\log I(0,t), \log I(t,N)]. \quad \text{(B2.15)}$$

Denoting

$$D(t,N) = \mathrm{Cov}[\log I(0,t), \log I(t,N)] \quad \text{(B2.16)}$$

in (B2.15), and then applying (B2.15) to the inner square bracket of (B2.14), one obtains;

$$z_t = \exp[-\mu(0,t) - \tfrac{1}{2}\sigma^2(0,t)]\exp[a\,C(0,t) - D(t,N)]. \quad \text{(B2.17)}$$

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