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A DISCRETE TIME MODEL FOR PRICING TREASURY BILLS, FORWARD AND FUTURES CONTRACTS

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UN MODELE TEMPOREL DISCRET D'EVALUATION DES PRIX DES BONS DU TRESOR, DES CONTRATS A TERME DE GRE A GRE DES FUTURES
Cet article développe un modèle temporel discret d'évaluation cohérente des bons du Trésor, ainsi que des contrats à terme de gré à gré ou des "futures" souscrits sur ces bons. Le modèle utilise un processus binomial multiplicatif de tracé de l'évolution du taux et le calcul de l'évolution impliquée de la structure de termes, pour générer des expressions analytiques des prix des contrats à terme de gré à gré et des "futures", ainsi que leurs variances conditionnelles et leurs primes de risque. Des formules dérivées du modèle fournissent un riche ensemble d'hypothèses de tests sur des données financières - tests dont nous avons l'intention de publier ultérieurement les résultats. Nous soulignons également l'utilité potentielle des formules dans des applications de gestion.
A DISCRETE TIME MODEL FOR PRICING TREASURY BILLS, FORWARD AND FUTURES CONTRACTS

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ABSTRACT

This paper develops a discrete time model for consistently valuing treasury bills and either forward or futures contracts written against them. Using a multiplicative binomial process of spot rate evolution, and calculating the implied evolution of the term structure, the model generates analytical expressions for forward and futures prices, as well as for their conditional variances and risk premia. Model-derived formulae provide a rich set of hypotheses for testing against financial data, and we intend subsequently to report the results of tests. We also outline the potential usefulness of the formulae in managerial applications.

1. INTRODUCTION

This paper uses a discrete time multiplicative binomial model of spot interest rate evolution, and its implied evolution of the term structure, to derive consistent pricing formulae for treasury bills, and forward and futures contracts written against them. All results are developed under assumptions of zero arbitrage profits.

Although the model treats initially estimated forward rates parametrically, the parameters appear in the analytic expressions, permitting ready assessment of their influence. Contract delivery dates and the maturity dates of underlying instruments also figure in the formulae. In addition, expressions for the conditional variance and conditional risk premia of bill values, as well as of both forward and futures prices, are developed for each point in time. Finally, we develop functional relations between conditional variances and risk premia, providing new theoretical support for relations used in the empirical literature (Engle [1982]; Engle, Lilien, and Robins [1987]).
1.1 Organization of paper

The paper is organized as follows. The rest of this section reviews relevant literature. The model and its underlying assumptions are described in Section 2, which specifies how both the spot rate and the term structure evolve. Section 3 develops formulae for treasury bill prices using a contingent claims approach. It also shows how the conditional probabilities defined by contingent claims are restricted by initial assumptions regarding the term structure. Section 4 develops formulae for consistently calculating forward prices, their conditional variances, and conditional risk premia. Section 5 does the same for futures contracts. The formulae lead to the kinds of testable hypotheses and management applications outlined in the conclusion, section 6.

1.2 Review of literature

The theory of commodities futures pricing is initially developed in Black [1976], and further examined in, for example, Gay and Manaster [1984]. Important properties of forward and futures contracts and prices are developed in Cox, Ingersoll, and Ross [1981], Jarrow and Oldfield [1981], and Richard and Sundaresan [1981]. The pricing theories developed in the literature just cited, as well as those of the present paper, do not recognize delivery options. Discussions of how delivery options reduce futures prices appear in Kane and Marcus [1986] and in Boyle [1989].

2. ANALYTICAL MODEL

Our discrete time model is based on an approach to options pricing originally proposed by Sharpe. We follow the development in Cox and Rubinstein [1985].

2.1 Useful model properties

Our model uses spot interest rates as a state variable. It allows the term structure to evolve with fewer constraints than do related papers such as Ho and Lee [1986]. This paper also relaxes the Ho and Lee assumption leading to the possibility of negative spot rates.

A model similar to our own but designed to reflect both interest rate and asset price uncertainty in contingent claims has recently been presented by Kishimoto [1989]. However, Kishimoto does not develop explicit formulae, as does this paper. Bliss and Ronn [1989] offer a trinomial version of Ho and Lee applied to interest rate claims, but again do not develop explicit formulae. However, Bliss and Ronn test their model against monthly data, arguing that interest rate data are better fitted by a trinomial as opposed to a binomial model. If they are correct, our model can be used in this way, since it can be interpreted as a trinomial model by considering the formulae we develop only for even-numbered time periods.

Thus, our model permits examining the consequences of zero arbitrage profits more explicitly, and under less restrictive assumptions, than do other contemporary studies. Yet the model is not analytically complex, and we do not think it is based on empirically unreasonable assumptions.
2 Evolution of interest rates

Let \( R_t \) be unity plus a spot (one period) interest rate. The term \( R_0 \) is taken to be the initial spot rate, and the terms \( \{ R_t \} \) \( t \in \{ 1, 2, \ldots, T - 1 \} \) will be interpreted below as a series of forward rates on bills, as estimated from data at time zero.

The realized spot rate at time \( t \) is the period \( t \) riskless rate. These spot rates are assumed to evolve stochastically about \( (R_t) \) according to a multiplicative factor \( u > 1 \) in the following way. Suppose one plus the time \( t \) spot rate is \( u R_t \), \( j \in J_t \), where

\[
J_t = \{ t, t - 2, t - 4, \ldots, -t + 2, -t \};
\]

(2.2.1)

\( t \in \{ 1, 2, \ldots, T \} \).

Then (one plus) the time \( t + 1 \) spot rate will be either \( u^i + 1 R_{t+1} \) or \( u^{-i} R_{t+1} \); the realizations occur with probability \( p \) and \( 1 - p = q \) respectively. Thus between times \( t \) and \( t + 1 \) the spot rate moves either up or down. It cannot remain at the same level over successive periods, but it can return to the same level over the next two periods.

In order for rates to be capable of either increasing or decreasing from one period to the next, it is necessary that

\[
(2.2.2) \quad [R_{t+1} / u] < R_t < [u R_{t+1}]
\]

for all \( t \in \{ 1, 2, \ldots, T \} \). Without assumption (2.2.2), some rate patterns could stochastically dominate others, contrary to our assumption of zero arbitrage profit opportunities.

The difference equation generating (one plus) the spot rate is:

\[
(2.2.3) \quad S_{t+1} = R_{t+1} S_t U / R_t,
\]

where \( S_{t+1} \) represent one plus the spot rate at time \( t + 1 \). By recursive substitution, and recalling that by definition \( S_0 = R_0 \), it follows immediately from (2.2.3) that

\[
(2.2.4) \quad S_{t+1} = R_{t+1} U_{t+1},
\]

where \( U_{t+1} \) is the random variable generated by \( T + 1 \) successive realizations of \( U \).

The mean and variance of \( U \) are respectively given by

\[
(2.2.5) \quad \mathbb{E}(U) = pu + q / u \quad \text{and} \quad \mathbb{V}(U) = pq \left[ \frac{u^2 - 1}{u^2} \right].
\]

1. We could generalize the model to allow for a multiplicative increase \( u \) and a multiplicative decrease \( d \), not necessarily equal to \( 1/u \). If subsequent empirical work suggests the generalization would be fruitful, the formulae derived in this paper are not difficult to amend accordingly.

2. For example, \( U_3 \) has the outcomes \( u_3, u, u^{-1}, \) and \( u^{-3} \), with probabilities \( p^3, 3p^2q, 3pq^2, \) and \( p^3 \) respectively.
The drift of the process (2.2.4) is determined by $E(U)$. If $E(U) = 1$, the process has a constant mean except for the influence of any exogenous changes in $R_t$. The process has a lower bound of zero and, for finite values of $u$, $v$, and $R_t$, a finite upper bound of $u^1 R_t$. The succeeding discussion shows that assuming a particular spot rate process and zero arbitrage profits implies the term structure evolves in a particular manner dependent on the realized spot rates. The model makes this dependence explicit on economic data rather than assuming it in a priori as do Ho and Lee [1986].

3. SECURITIES AND CLAIM PRICES

For structuring the analysis and for recognizing data availability, explicit analysis of the interest rate process is truncated at time $T$. This section relates bill prices, contingent claim prices, and interest rates for a truncated process. Given a set of forward rates, these relations define a particular set of claim values and related conditional probabilities. The same contingent claims values are used to derive futures and forward prices consistent with the zero arbitrage profits assumption.

3.1 Assumed structure for bill prices

Bills are defined to have a value of unity at maturity; bills maturing at $M > T$ to have time $T$ values

$$1 / R(j, T, M)^{M - T},$$

where $R(j, T, m)$ is an average one-period discount factor for the periods $T, ..., M$: $j \in J_T$.

In other words, beyond time $T$ the term structure is assumed to be flat for any given realization of the rate process. Let $B_t(j, T, M)$ represent the market price at time $t$, when the spot rate is $u^1 R_t$, of a bill with maturity $M > T$,

$$t \in \{0, 1, 2, ..., T - 1\}; \; j \in J_t.$$

(The cases $M \leq T$ are discussed below.) Then

$$(3.1.1) \quad B_T(j, T, M) = 1 / [u^1 R_{T, M}]^{M - T}. $$

3. Apart from the influence of the parameters $R_t$.

4. In turn, the conditional probabilities define the underlying equivalent martingale measure.

5. This does not mean that the average return on the bills necessarily equal to the spot rate. Suppose, for instance, that one plus the spot rate at time $T$ is given by $u^1 R^* T$. Then to reflect a term premium on a long maturity bond, the average yield can be defined as $u^1 R_T$, where $R_T = R^* T + \varepsilon$, where $\varepsilon$ reflects the term premium.
In addition,

\[ B_T(j - 2, T, H) / B_T(j, T, M) = u^2(M - T). \]

Except when needed for expositional clarity, the notation will usually be shortened; for example, we often write \( R_{T,M} \) as \( R_T \) and \( B_t(j, T, M) \) as \( B_t(j) \).

### 3.2 Relations between interest rates and claim prices

To proceed, let \( c_t(j, +) \) be the time \( t \) value of a (one-period) unit claim to be paid at time \( t \) if the spot rate rises, and let \( c_t(j, -) \) be the time \( t \) value of a (one-period) unit claim to be paid if the spot rate falls. Since only the two outcomes are possible, the sum of claim values equals the present value of a risk-free payment of unity, and the discount factor is the reciprocal of the one plus the current risk-free rate:

\[ c_t(j, +) + c_t(j, -) = 1 / u^2 R_t. \]

The conditional probabilities defining the equivalent martingale measure in a world of zero arbitrage profits are:

\[ p_t(j) = c_t(j, +) / c_t(j, +) + c_t(j, -) \]

\[ q_t(j) = c_t(j, -) / c_t(j, +) + c_t(j, -); \]

cf., for example, Huang and Litzenberger [1988, pp. 223-234]. Hence

\[ c_t(j, +) = p_t(j) / u^2 R_t, \]

\[ c_t(j, -) = q_t(j) / u^2 R_t. \]

Note also that:

\[ c_t(j, +) / c_t(j - 2, +) = 1 / u^2, \]

\[ c_t(j, -) / c_t(j - 2, -) = 1 / u^2. \]

This means \( p_t(j) / p_t(j - 2) = 1 \), implying that \( p_t(j) = p_t \), and hence

\[ c_t(j, +) = p_t / u^2 R_t, \]

\[ c_t(j, -) = q_t / u^2 R_t, \]

for all \( j \in J_t \).

### 3.3 Bill price formulae

Bill prices at time \( t \) are related to bill prices at \( t + 1 \) by the possible rate outcomes and the values of the contingent claims associated with them. Hence the bill prices can be written recursively as

6. **Observe that the process starting at \( wR_t \) and that starting at \( wR_t \) are identical except for the multiplicative factor \( u^2 R_t \). Since under the marginal the pseudo probabilities \( P^+ \) (j) will be independent of scale changes in the gambles, we have \( P^+ (j) = P^+ (j) = P^+ \) for all \( j \in J_t \).**
Then starting at time $T - 1$, proceeding by backward induction and using the form of prices $B_T(j, T, M)$ given in (3.1.1), the following expression can be derived:

$$B_t(j) = \frac{[p_t \cdot B_{t+1}(j + 1) + q_t \cdot B_{t+1}(j - 1)]}{u^T r_t},$$

where $v = u^{M - T}$ and $t \in \{1, 2, \ldots, T - 1\}$. The method is not given explicitly here, since it is a special case of valuing the forward contracts discussed in section 4.

Further recursive application of (3.3.2) with $j = t$ gives

$$(3.3.3) \quad B_0(0, T, M) = \{ [p_0 + u^{2(M-1)}q_0] / u^{0R_0} \} \cdot \{ [p_1 + u^{2(M-2)}q_1] / u^{1R_1} \} \cdot \ldots \cdot \{ [p_{T-1} + u^{2(M-T)}q_{T-1}] / u^{T-1R_{T-1}} \} \cdot B_T(T, T, M)$$

where $v = u^{M - T}$ has been eliminated to simplify the formula.

In the sequel it will also be convenient to use the following special case of (3.3.4) when $M \leq T$:

$$B_0(0, s, s) = \left\{ \frac{[p_0 + u^{2(s-1)}q_0]}{u^{0R_0}} \right\} \cdot \left\{ \frac{[p_1 + u^{2(s-2)}q_1]}{u^{1R_1}} \right\} \cdot \ldots \cdot \left\{ \frac{[p_{s-1} + u^{2s-1}q_{s-1}]}{u^{(s-1)R_{s-1}}} \right\}.$$
The values of $p_t$ and $q_t$ satisfying (3.4.1), (3.4.2), and (3.4.3) are

$$p_t = u / (u + 1), \quad p_1 = u^2 / (u^3 + 1);$$

independent of $(R_t)$, $t \in \{0,1,2\}$. In general, the restrictions implied by determining the bill prices consistently with both (3.3.4) and the $(R_t)$ imply the conditional probabilities

*(3.4.4)* \[ p_t = u^{1+2t} / (u^{1+2t} + 1), \]

$t \in \{0,1,2, \ldots, T - 2\}$, as can be verified using (3.3.4). As the example indicates $p_{T-1}$ and $q_{T-1}$ are unrestricted, apart from the requirement that they sum to unity.

The term structure also evolves in a particular manner, since at time $t$ it is defined by $9$ the bill price formulae (3.3.3) and (3.3.4), the probabilities (3.4.4), and the originally specified values of the $(R_t)$; $t \in \{t, t = 1 \ldots, T - 1\};$

$t \in \{1, 2, \ldots, T - 1\}.$

### 3.5 Conditional variance of bill prices

Using (3.3.1) and (3.3.2) to express $B_{t+1}(j+1)$ in terms of $B_{t+1}(j+1)$ we obtain

*(3.5.1)* \[ B_{t+1}(j) = u^{2(M-t-1)}b_{t+1}(j+1), \]

after eliminating $V = u^M T$. Since $B_{t+1}(j+1)$ occurs with probability $p$ and $B_{t+1}(j+1)$ with probability $q$, the results of Appendix I imply that the conditional variance of the bill price is

*(3.5.2)* \[ \nu_{B_t}(j) = pq(u^{2(M-t-1)} - 1)^2b_{t+1}(j+1). \]

The conditional variance of the bill return is found by rewriting (3.5.2) to express $B_{t+1}(j+1)$ in terms of $B_t(j)$, and then dividing by $B_t(j)$:

*(3.5.3)* \[ \nu_{B_t}(j) = \frac{pq(u^{2(M-t-1)} - 1)^2(u^{1}b_t)^2}{(p_t + u^{2(M-t-1)}q_t)^2}. \]

---

9. As a practical matter, the path dependent term structure for time $t$ can then be calculated using the forward rates implicit in the ratios of bill prices. It is also possible to compute the implied variance of bond prices for times $t \in \{1, 2, \ldots, T\}$; recall we have already specified circumstances under which the variance of the interest rate process increases.

In essence we are working, as are Ho-Lee [1986] and Bliss-Ronn [1989], with particular forms of perturbation functions. However, our approach begins with underlying economic variables - the spot rate process and bill prices - and provides explicit formulae relating these fundamentals to the financial instruments we study. In contrast to these other studies, we incorporate time-dependence in relative bill prices and a high degree of flexibility in our choices of term structures.
3.6 Conditional risk premia

We define the conditional risk premium as the conditionally expected price at time $t+1$ less the current spot rate times the bill price at time $t$.

$$P_{B,t}(j) = E_t B_{t+1}(j) - B_t(j) u_t R_t,$$

or, more simply as the excess rate of return

$$P^*_{B,t}(j) = \{ E_t B_{t+1}(j) - B_t(j) u_t^2 R_t \} / B_t(j),$$

where $J$ is a random variable assuming one of the values {$j+1, j-1$}.

By expressing all bill prices in terms of $B_t(j)$, and using (3.3.1) and (3.3.2), (3.6.2) simplifies to

$$P^*_{B,t}(j) = \{ [ (p + u^2(M-T-1)q) / (p_t + u^2(M-T-1)q_t) ] - 1 \} u^t R_t.$$

3.7 Relation between risk premia and conditional variance

The excess rate of return on the bill can be expressed in terms of the conditional standard deviation $\{V^* B_t(j)\}^{1/2}$. Combining (3.5.3) and (3.6.3) gives

$$K^*_{B,t} = \{ (p + u^2(M-T-1)q) / (p_t + u^2(M-T-1)q_t) \} - 1 \} u_t R_t.$$

Equation (3.7.1) says that the relationship between the conditional risk premium and the conditional standard deviation is one of strict proportionality. There is also a constant proportionality relationship between conditional risk premium and conditional variance, depending on $R_t$ as well as on the variables determining $K^* B_t$.

Equation (3.7.1) offers theoretical support for relations assumed in the literature. Specifically, the ARCH-M model of Engle, Lilien, and Robins [1987] based on Engle [1982], postulates an expected relationship between the conditional mean and conditional variance of returns when there is one risky and one riskless asset. They estimate models in which the conditional mean of bill returns is a linear function of the conditional standard deviation and a linear function of its logarithm. Different assumptions about investors' utility functions lead to the different functional forms.

4 - FORWARD PRICES

This section values forward contracts and develops expressions for forward prices, their conditional variances, and risk premiums.
4 - 1 Recursive calculation of forward prices

Let $G_t(j, T, M)$ be the forward price at time $t$, when the spot rate is $uR_t$, on a contract written at time $t$, with delivery date $T$, against a bill maturing at time $M \geq T$. On the delivery date, the forward price equals the value of the underlying instrument; cf. Cox, Ingersoll, Ross [1981]. Therefore,

$G_t(j, T, M) = B_t(j, T, M).$

As before, the arguments $T$ and $M$ will be suppressed whenever no ambiguity results, and the forward price will usually be written

$G_t(j); \ t \in \{0, 1, 2, \ldots, T\}; \ j \in J_t.$

Next, let the value at time $t$ of a forward contract written at time $0$, with exercise (delivery) price $X_T$, and when the spot rate is $uR_t$, be defined as $F_0, T(j, X_T, T, M)$. As usual, the last two arguments will be suppressed unless needed for clarity, and the value of the forward contract will normally be written 10

$F_0, T(j, X_T).$

Consider first the problem of valuing a forward contract with an arbitrary delivery price; it will then be easy to calculate the forward price for that contract. Proceeding by backward induction, on the delivery date the contract value is the difference between the bond price and $X_T$, the delivery (exercise) price. Thus, if the interest rate is $uT R_T$:

$F_0, T(T, T) = B_T(T) - X_T = \left[ \frac{1}{uT R_T} [u^T R_T]^M T - 1 \right] - X_T.$

Then it follows immediately that

$F_0, T(T - 2k, X_T) = v^{2k} B_T(T) - X_T.$

where $v = v(T, M) = u^{T - M}$, and $k \in \{0, 1, 2, \ldots, T\}$. Next,

$F_0, T-1(T - 1, X_T) = c_{T-1}(T-1, +) F_0, T(T, X_T) + c_{T-1}(T-1, -) F_0, T(T - 2, X_T)$

$= (p_{T-1} + v^2 q_{T-1}) F_0, T(T, X_T) / u^{T-1} R_{T-1}$

$= \{(p_{T-1} + v^2 q_{T-1}) B_T(T)\} / u^{T-1} R_{T-1}$

$= A_{T-1} - X_T.$

10. We shall show below how the notation can accommodate forward contracts written at arbitrary times. It is convenient to define the value of the forward contract as well as the forward price so that bond prices, forward prices, and futures prices can all be rederived using the same methodology. The notation for forward and futures prices ($G_t(j)$ and $H_t(j)$, respectively) is consistent with Cox, Ingersoll, Ross [1981], and $G_t(j)$ is the special value of $X_t$ such that the value of the forward contract is zero when it is written; cf. Jarrow and Oldfield [1981].
Note that $X_{T+1}$ is the present value of $X_T$, calculated using (one plus) the appropriate spot rate $u^{T-1}R_{T-1}$.

It is then easy to see that

$$(4.1.5) \quad F_{0,k-1}(t-1 - 2k, X_t) = (u^v)A_{t-1} - u^{2k}X_{t-1},$$

$k \in \{0, 1, 2, \ldots, T-1\}$.

Recursive application of (4.1.4) and (4.1.5) gives

$$(4.1.6) \quad F_{0,t-1}(t - 1, X_t) = (p_{t-1} + v^2q_{t-1})F_{0,t}(t, X_t) / u^{t-1}R_{t-1}$$

$$= [(p_{t-1} + v^2q_{t-1})A_{t-1} - X_{t-1}] / u^{t-1}R_{t-1},$$

$$= A_{t-1} - X_{t-1}, \text{ and}$$

$$(4.1.7) \quad F_{0,t-1}(t - 1 - 2k, X_t) = (u^v)A_{t-1} - u^{2k}X_{t-1},$$

$k \in \{0, 1, 2, \ldots, t-1\}$.

It is then possible to write the explicit form

$$(4.1.8) \quad F_{0,0}(0, X_t) = \{[p_0 + u^{2^{[M-2]}q_0}] / R_0\}$$

$$\cdot \{[p_1 + u^{2^{[M-2]}q_1}] / uR_1\} \cdot \ldots$$

$$\cdot \{[p_{T-1} + u^{2^{[M-2]}q_{T-1}}] / u^{T-1}R_{T-1}\} / u^{T-1}R_{T-1}.$$ 

Eliminating $v = u^{M-T}$ as before.

Equation (4.1.8) can be written more simply:

$$(4.1.9) \quad F_{0,0}(0, X_t) = B_0(0, T, M) / B_0(0, T, T) \cdot X_t.$$ 

Next, equation (4.1.9) implicitly defines $G_0(0)$ by the condition

$$F_{0,0}(0, G_0(0)) = 0, \text{ or}$$

$$(4.1.10) \quad G_0(0) = B_0(0, T, M) / B_0(0, T, T);$$

cf. Jarrow and Oldfield [1981, p. 381, eqn. (1.3)]. A forward contract originating at time $t$, when one plus the interest rate is $u^{R_t}$, has a value at time $s > t$, when one plus the interest rate is $u^{R_s}$, given by

$$F(t, G_t(j)) = B_t(j, T, M) / B_t(j, T, T).$$
4.2 Conditional variance of forward prices

The forward price on a contract written at time \( t \) does not change before the contract delivery date, time \( T \). However, new contracts can be written at times \( S > t \), and the conditional variance of forward prices refers to the possible variations in the prices on these new contracts, which will be written to reflect the newly prevailing time and interest rate environment.

For theoretical purposes, assume a new contract is written at each point in times, and that all contracts have the same delivery date \( T \). Given the forward price \( G_t(j) \), the forward price at time \( t+1 \) is either

\[ G_{t+1}(j+1) \] with probability \( p \), or

\[ G_{t+1}(j-1) \] with probability \( q \).

Rewrite the second of these outcomes:

\[ G_{t+1}(j-1) = u^{2(T-t-1)} G_{t+1}(j+1) \] with probability \( q \).

Appendix I shows, by taking

\[ y = u^{2(T-t-1)} G_{t+1}(j+1) \] and

\[ x = G_{t+1}(j+1) \]

that the conditional variance of this binomial distribution is

\[
V_{G_{t+1}}(j) = pq \cdot \left\{ \left[ u^{2(T-t-1)} - 1 \right] \cdot [G_{t+1}(j+1)]^2 \right\}.
\]

Then, using (4.1.11), (4.2.1) can be rewritten as

\[
V_{G_{t+1}}(j) = pq \cdot \left\{ \left[ u^{2(T-t-1)} - 1 \right] \cdot [G_t(j)]^2 \right\}.
\]

Denoting the conditional variance of the rate of change of the forward price by \( V^*_{G_{t+1}}(j) = V_{G_{t+1}}(j) / [G_t(j)]^2 \), it follows immediately from (4.2.2) that

\[
V^*_{G_{t+1}}(j) = pq \cdot [u^{2(T-t-1)} - 1]^2 \cdot \left( \frac{P_t + u^{2(T-t-1)} q_t}{P_t + u^{2(T-t-1)} q_t} \right)^2.
\]

It is independent of \( j \).

4.3 Conditional risk premia in forward prices

Define the conditional risk premium in a forward price by

\[
F_{G_{t+1}}(j) = \mathbb{E}_t \{ G_{t+1}(j) \} - G_t(j),
\]
where $E_t$ denotes the time $t$ conditional expectation of the time $t + 1$ forward price, and $J$ is a random variable,

$J \in \{ j + 1, j - 1 \}$.

Condition (4.3.1) can be rewritten

$$P_{G,t}(j) = (1 - \mu_r) \mu_{r+1} (p_t + u^2(M-t-1)q_t) / (p_t + u^2(M-t-1)q_t) - 1 \cdot G_t(j).$$

An expression for the risk premium in rate of return form can also be found using

$$P_{G,t}(j) = P_{G,t}(j) / G_t(j).$$

4.4 Conditional risk premium and standard deviation

As with bill prices, a proportional relationship between $P^* G_t(j)$ and $[V^* G_t(j)]^{1/2}$ by using (4.2.4) and (4.3.3) to define an appropriate proportionality constant.

$$(4.4.1) \quad P_{G,t}(j) / [V_{G,t}(j)]^{1/2} = K_{G,t}(j),$$

where

$$(4.4.2) \quad K_{G,t}(j) = \{ [p + u^2(M-t-1)q_t] \cdot [p_t + u^2(T-t-1)q_t] - [p_t + u^2(M-t-1)q_t] \} / (pq)^{1/2}(u^2(M-t-1) - 1).$$

5. FUTURES PRICES

This section develops expressions for futures prices, their conditional variances, and risk premiums.

5.1 Recursive calculation of futures prices

Let $H(i,t,M)$ be the futures price at time $t$, when the spot rate is $u \Gamma_t$, on a Contract with delivery date $T$ written against a bill maturing at time $M$. On the delivery date, the futures price equals the value of the underlying instrument; cf. Cox, Ingersoll, Ross [1986]. Therefore,

$$H(i,t,M) = B_t(j, T, M).$$

In periods $i, t \in \{ 0, 1, 2, \ldots, T - 1 \}$, the futures price is defined as $H(i, t, M)$. However, as with the underlying bills, the arguments $T$ and $M$ will be suppressed whenever no ambiguity results, and the futures price will usually be written $H(i, t)$,

$$t \in \{ 0, 1, 2, \ldots, T \}.$$
\[ (5.1.2) \quad t \in \{ 1, 2, \ldots, T-2 \}, \text{ and} \]
\[
(c_{T-1}(j, +) + c_{T-2}(j, -)) \cdot H_{T-1}(j) = c_{T-1}(j, +) \cdot B_T(j+1) + c_{T-1}(j, -) \cdot B_T(j - 1);
\]
cf., e.g., Cox - Ingersoll - Ross - [1981, P. 337, eqn (42)].

Using (3.3.2) and (3.3.3) condition (5.1.2) can be rewritten
\[
(5.1.3) \quad H_{t}(j) = p_{t} H_{t+1}(j + 1) + q_{t} H_{t+1}(j - 1);
\]
\[ t \in \{ 1, 2, \ldots, T-1 \}; \]
and \( H_T(j) = B_T(j), \ j \in J_T. \)

Moreover \( H_T(T) = B_T(T) = 1/ [u^T R_T]^M - T. \)

Then it follows immediately that

\[
(5.1.4) \quad H_T(T - k) = v^{2k} H_T(T),
\]
where \( v = v(T, M) = u^M - T, \) and \( k \in \{ 0, 1, 2, 3, \ldots, T - 1 \}. \)

Taking (5.1.3) with \( t = T - 1 \) and using the results of (5.1.4) gives

\[
(5.1.5) \quad H_{T-1}(j) = \]
\[
= p_{T-1} H_T(j + 1) + q_{T-1} H_T(j - 1)
\]
\[
= p_{T-1} H_T(j + 1) + v^2 q_{T-1} H_T(j + 1)
\]
\[
= [p_{T-1} + v^2 q_{T-1}] \cdot H_T(j + 1).
\]

Similarly,

\[
(5.1.6) \quad H_{T-2}(j) = \]
\[
= p_{T-2} H_T(j + 1) + q_{T-2} H_T(j - 1)
\]
\[
= p_{T-2} [p_{T-1} + v^2 q_{T-1}] \cdot H_T(j + 2) + q_{T-2} [p_{T-1} + v^2 q_{T-1}] \cdot H_T(j)
\]
\[
= [p_{T-2} + v^2 q_{T-2}] \cdot [p_{T-1} + v^2 q_{T-1}] \cdot H_T(j + 2).
\]

II. In 6.2 below it will be convenient to recognize that \( y(T, M) \) can be expressed as \( y(M - T) \) or \( y(M - T) \), i.e., as a function of the difference between delivery and maturity dates.
Moreover,

\begin{align*}
(5.1.7) \quad & H_{t-2}(T - 2) = \\
& = [P_{t-2} + v^2 q_{t-2}] \cdot [P_{t-1} + v^2 q_{t-1}] \cdot H_T(T) \\
& = [P_{t-2} + v^2 q_{t-2}] \cdot H_{t-1}(T - 1).
\end{align*}

It follows that

\begin{align*}
(5.1.8) \quad & H_t(j) = v^2 H_{t}(j + 2), \text{ and} \\
(5.1.9) \quad & H_t(j) = [P_t + v^2 q_t] \cdot H_{t+1}(j + 1),
\end{align*}

for all \( j \in J_t \) and for all \( t, t = 0, 1, \ldots, T - 1 \).

Finally, setting \( j = t \) and applying the condition \(5.1.8\) recursively gives an explicit formula for the futures price at time zero:

\begin{align*}
(5.1.10) \quad & H_0(0) = [P_0 + v^2 q_0] \cdot [P_1 + v^2 q_1] \cdot \\
& \quad \ldots \cdot [P_{T-1} + v^2 q_{T-1}] \cdot B_(T).
\end{align*}

The futures price depends on \( M, T, B_M(T) \), and \( u \), but not on \( \{R_t\}, \), \( t \in \{0, 1, 2, \ldots, T - 1\} \). The following properties will also prove useful.

**Proposition 5.1.1** The futures prices \( H_t(T) \) are decreasing in \( t \). Inspection of \( (5.1.7) \) and \( (5.1.9) \) shows that

\begin{align*}
(5.1.11) \quad & H_t(t) > H_{t+1}(t + 1) > H_{t+2}(t + 2).
\end{align*}

To see the effect of time variation on the futures prices when \( j \) is held constant, consider

**Proposition 5.1.3.** Let the spot rate remain unchanged between periods. Then the ratio of futures prices decreases with time if and only if \( M \geq M^* \), where

\[
M^* = T \cdot \frac{\ln(p_{t-2}p_{t-1}/q_{t-2}q_{t-1})}{2 \cdot \ln(u)}.
\]

To establish this proposition, use \( (5.1.8) \) and \( (5.1.9) \) to write

\[
5.1.12 \quad H_t(j) / H_{t-2}(j) = v^2 / [P_{t-2} + v^2 q_{t-2}] \cdot [P_{t-1} + v^2 q_{t-1}].
\]

The behaviour of the ratio on the right hand side of \( (5.1.12) \) is revealed by defining

\[
x = v^2, \quad a = p_{t-2}p_{t-1}, \quad b = p_{t-1}q_{t-2} + p_{t-2}q_{t-1}, \quad c = q_{t-2}q_{t-1},
\]

and considering the equation

\[
x = a + b + c = 1, \quad \text{rewrite the quadratic as}
\]

\[
(5.1.13) \quad [x - a/c] \cdot [x - 1] = 0.
\]
Given the values of $p_t$, $q_t$, $p_{t+1}$, $q_{t+1}$, and $q_{t-2}$ as assumed in (3.4.4), it follows that $a > c$. Thus when $v^2$ lies between unity and $a/c$, the ratio (5.1,12) is greater than unity, and for values of $v^2 > a/c$ (5.1.12) is less than unity. Since $v^2$ is an increasing function of $M_t$, there is a critical value $M^*$ which determines whether (5.1.12) is increasing or decreasing in $t$. Straightforward calculation shows $M^*$ is defined by

$$ M^* = T + \{ \ln(p_t - 2p_{t-1} \div q_{t-2}q_{t-1}) \} / 2 \cdot \ln(u) \}.$$

Note that $M^*$ is not necessarily an integer, as are $M$ and $T$.

5.2 Conditional variance of futures prices

Conditional on a realization $H$, the futures price at time $t+1$ is either

- $H_{t+1} (j + 1)$, with probability $p$, or
- $H_{t+1} (j + 1)$, with probability $q$.

Rewrite the second of these outcomes:

$$ H_{t+1} (j + 1) = v^2 \cdot H_{t+1} (j + 1) \text{ with probability } q. $$

Appendix I shows, by taking

$$ y = v^2 \cdot H_{t+1} (j + 1) \text{ and }$$

$$ x = H_{t+1} (j + 1) $$

that the conditional variance of this binomial distribution is

$$ (5.2.1) \quad V_{H,t}(j) = pq \cdot \{ [v^2 - 1] \cdot [H_{t+1} (j + 1) \}^2. $$

Then, using (5.1.9), (5.2.1) can be rewritten as

$$ (5.2.2) \quad V_{H,t}(j) \cdot pq \cdot \{ [v^2 - 1] \cdot [H_{t+1} (j + 1) \}^2 / [ p_t + v^2 q_t ]^2. $$

To see the effect on $V_{H,t}(j)$ when $t$ increases while interest rates are held constant, recall from (5.1.8) that

$$ H_{t+2} (j) = v^2 \cdot H_{t+2} (j + 2) $$

Then by (5.1.3) and (5.2.2)

$$ (5.2.3) \quad V_{H,t}(j) = pq \cdot \{ [v^2 - 1] \cdot [p_t + v^2 q_t] \cdot H_{t+2} (j + 2) \}^2, $$

$$ = pq \cdot \{ [v^2 - 1] \cdot [p_t + v^2 q_t] \cdot H_{t+2} (j) \} / v^2 \cdot \{ [v^2 - 1] \cdot [p_t + v^2 q_t] \}^2, $$

Then, whether (5.2.3) increases or decreases in $t$ depends on the behavior of both $p_t + v^2 q_t$ and $H_{t+2} (j)$, as well as on their relative sizes. Thus the change in $V_{H,t}(j)$ is in general ambiguous; cf. Proposition 5.1.2.

Defining the conditional variance of the rate of change of the futures price by

$$ V_{H,t}(j) \cdot = \frac{V_{H,t}(j)}{[H_t (j) \}^2, $$

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it follows immediately from (5.2.2) that

\[(5.2.4) \quad V_{B,t}(j) = pq \cdot \left( \frac{v^2}{p_t + v^2q_t} \right)^2,\]

independent of \(j\). Note that \(V_{B,t}(j)\) increases in \(t\) if and only if the denominator \(p_t + v^2q_t\) decreases in \(t\).

5.3 Conditional risk premia in futures prices,

Define the conditional risk premium in a futures price by

\[(5.3.1) \quad \rho_{B,t}(j) = \mathbb{E}_t (H_{t+1}(j)) - H_t(j),\]

where \(\mathbb{E}_t\) denotes the time \(t\) conditional expectation of the time \(t+1\) futures price, and \(J\) is a random variable,

\(J \in \{j+1, j-1\}\).

Condition (5.3.1) can be rewritten

\[(5.3.2) \quad \rho_{B,t}(j) = \{ [p + v^2q] - [p_t + v^2q_t] \} \cdot H_{t+1}(j+1),\]

The risk premium can also be expressed in terms of the rate of change of futures prices,

\[(5.3.3) \quad \rho^*_{B,t}(j) = \frac{\rho_{B,t}(j)}{H_t(j)}\]

Then since

\[p_t + v^2q_t > p_{t+1} + v^2q_{t+1},\]

the risk premia are positive whenever

\[p + v^2q > p_0 + v^2q_0.\]

5.4 Relations between conditional risk premia and variances

Recalling (5.3.2),

\[(5.4.1) \quad \rho_{B,t}(j) = \{ [p + v^2q] - [p_t + v^2q_t] \} \cdot \frac{V_{B,t}(j)/pq}{(v^2 - 1)}.\]

Similarly, using (5.3.3) and (5.4.1),

\[(5.4.2) \quad \rho^*_{B,t}(j) = \{ [p + v^2q] - [p_t + v^2q_t] \} \cdot \frac{V_{B,t}(j)/pq}{(v^2 - 1)}.\]
Then it can readily be seen that
\[ \frac{P^*_{E,1}(j)}{(v^*_{E,1}(j))} \frac{1}{2} = K^*_{E,1} \]
where the constant of proportionality \( K^*_{H,1} \) is given by
\[ K^*_{H,1} = \left\{ p + u^{2[H-P]}q - [p_t + u^{2[H-P]}q_t] \right\} / (pq)^{1/2} \cdot (u^{2[H-P]} - 1) \cdot 1. \]
Thus \( K^*_{H,1} \) depends on \( p, u, T, \) and \( M_t \) but not on the \( \{R_t\} \). A similar proportionality constant can be defined to describe the relation between the conditional risk premium and the conditional variance of the futures price rate of change. Note that \( K^*_{H,1} \) will be positive if
\[ [p + v^2q] > [p_t + v^2q_t]. \]

5.5 RELATIONS BETWEEN FORWARD AND FUTURES PRICES

The formulae for futures and forward prices permit explicit comparisons. Recall:
\[ (4.1.10) \quad G_0(0, T, M) = \frac{B_0(0, T, M)}{B_0(0, T, T)} \] and
\[ (5.1.10) \quad H_0(0, T, M) = \frac{[p_t + v^2q_t] \cdot [p_t + v^2q_t] \cdots}{[p_{t-1} + v^2q_{t-1}] \cdot B_T(T, T)}. \]

From which the ratio \( G_0(0) / H_0(0) \) can readily be calculated. In addition, by using
\[ (3.3.3) \quad B_0(0, T, T) = \left( \text{PR}(t = 0, T-1) \left\{ [p_t + u^{2[H-P]}q_t] / u^{2[H-P]} \right\} \right) \cdot B_T(T, T) \]
forming the ratio \( H_0(0, T, M) / B_0(0, T, M) \), where \( \text{PR}() \) denotes a product recalling that \( u > 1 \), and that
\[ B_0(0, T, T) = 1/B_0(0, T, T) \cdot \cdots \cdot u^{T-1} \cdot B_T(T, T). \]

The last condition is a special case of Cox - Ingersoll - Ross Proposition 9 [1986, pp 331-3321.

6. CONCLUSIONS

This section briefly outlines possibilities for both empirical tests and managerial applications of our theory. Both these applications are to be explained in future work.

Empirical work necessarily involves tests of joint hypotheses of rational expectations or market efficiency and the underlying model of security pricing.

Our proposed estimations will use methods designed for time series which display persistence in heteroscedasticity. So far, the main testable implications we have developed involve the relations between conditional means and conditional standard deviations.
Further testable implications will follow from individual pricing formulae and from the relations between price series which they define. Data availability will restrict some tests, and not all logical implications are actually testable.

The paper’s formulae can be used to calculate approximate values for instruments issued by financial institutions. In addition, the relations between futures and forward contracts are useful in informing managers about how different instruments related in a practical sense. Finally, the formulae make clear the important variables affecting valuation of a given instrument.

7 - SELECTED REFERENCES


Appendix I. Variance of a binomial distribution

Let $Z$ be a binomial random variable assuming the values $x$ and $y$ with probabilities $p$ and $q = 1 - p$ respectively. Suppose also that $y > x$. Then

$$z^* = \frac{(y - x)}{(y - x)}$$

is a standardized random variable with outcomes 0 and 1 with probabilities $p$ and $q$ respectively, and hence has variance $pq$. Then since

$$V(Z^*) = pq = V(Z) / (y - x)^2,$$

it follows immediately that

$$V(Z) = pq \cdot (y - x)^2.$$  

Appendix II. Conditional variance of interest rate process

The conditional variance of the interest rate process is determined as follows. Given the prevailing spot rate $u^{t-1} R_{t-1}$, the spot in the next period increases to $u^t R_t$ or decreases to $u^{t-2} & with probabilities $p$ and $q$ respectively. Thus, using the results of Appendix I,

$$V_t(u^t) = pq \cdot [(u^t R_t - u^{t-1} R_{t+1})^2].$$

For any fixed value of $t$, $V_t$ is an increasing function of $j$. 