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A DISCRETE TIME MODEL FOR PRICING TREASURY BILLS, FORWARD AND FUTURES CONTRACTS

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RESUME

Cet article développe un modèle temporel discret d'évaluation cohérente des bons du Trésor, ainsi que des contrats à terme de gré à gré ou des "futures" souscrits sur ces bons. Le modèle utilise un processus binominal multiplicatif de tracé de l'évolution du taux et le calcul de l'évolution impliquée de la structure de termes, pour générer des expressions analytiques des prix des contrats à terme de gré à gré et des "futures", ainsi que leurs variances conditionnelles et leurs primes de risque. Des formules dérivées du modèle fournissent un riche ensemble d'hypothèses de tests sur des données financières - tests dont nous avons l'intention de publier ulttrieurement les rtsultats. Nous soulignons également l'utilité potentielle des formules dans des applications de gestion.

A DISCRETE TIME MODEL FOR PRICING TREASURY BILLS, FORWARD AND FUTURES CONTRACTS

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ABSTRACT

This paper develops a discrete time model for consistently valuing treasury bii and either forward or futures contracts written against them. Using a multiplicative binomial process of spot rate evolution, and calculating the implied evolution of the term structure, the model generates analytical expressions for forward and futures prices, as well as for their conditional variances and risk premia. Modelderived formulae prwide a rich set of hypotheses for testing against financial data, and we intend subsequently to report the results of tests. We also outline the potential usefulness of the formulae in managerial applications.

1- INTRODUCTION

This paper uses a discrete time multiplicative binomial model of spot interest rate evolution, and its implied evolution of the term structure, to derive consistent pricing formulae for treasury bills, and forward and futures contracts written against them. All results are developed under assumptions of zero arbitrage profits.

Although the model treats initially estimated forward rates parametrically, the parameters appear in the analytic expressions, permitting ready assessment of their influence. Contract delivery dates and the maturity dates of underlying instruments also figure in the formulae. In addition, expressions for the conditional variance and conditional risk premia of bill values, as well as of both forward and futures prices, are developed for each point in time. Finally, we develop functional relations between conditional variances and risk premia, providing new theoretical support for relations used in the empirical literature (Engle [1982] ; Engle, Lilien, and Robins [1987]).

1 - 1 Organization of paper

The paper is organized as follows. The rest of this section reviews relevant literature. The model and its underlying assumptions are described in Section 2, which specifies how both the *spot* rate and the term structure evolve. Section 3 develops formulae for treasury bill prices using a contingent claims approach. It also shows how the conditional probabilities defined by contingent claims are restricted by initial assumptions regarding the term structure. Section 4 develops formulae f a consistently calculating forward prices, their conditional variances, and conditional risk premia. Section 5 does the same f a futures contracts. The formulae lead to the kinds of testable hypotheses and management applications outlined in the conclusion, section 6.

1 - 2 Review of literature

The theory of commodities futures pricing is initially developed in Black [1976], and further examined in, for example, Gay and Manaster [1984]. Important properties of forward and futures contracts and prices are developed in Cox, Ingersoll, and Ross [1981], Jarrow and Oldfield [1981], and Richard and Sundaresan [1981]. The pricing theories developed in the literature just cited, as well as those of the present paper, do not recognize delivery options. Discussions of how delivery options reduce futures prices appear in Kane and Marcus [1986] and in Boyle [1989].

2 - ANALYTICAL MODEL

Our discrete time model is based on an approach to options pricing originally proposed by Sharpe. We follow the development in Cox and Rubinstein [1985].

2 - 1 Useful model properties

Our model uses spot interest rates as a state variable. It allows the term structure to evolve with fewer constraints than do related papers such as Ho and Lee [1986]. This paper also relaxes the Ho and Lee assumption leading to the possibility of negative spot rates.

A model similar to our own but designed to reflect both interest rate and asset price uncertainty in contingent claims has recently been presented by Kishimoto [1989]. However, Kishimoto does not develop explicit formulae, as does this paper. Bliss and Ronn [1989] offer a trinomial version of Ho and Lee applied to interest rate claims, but again do not develop explicit formulae. However, Bliss and Ronn test their model against monthly data, arguing that interest rate data are better fitted by a trinomial as opposed to a binomial model. If they are correct, our model can be used in this way, since it can interpreted as a trinomial model by considering the formulae we develop only for even-numbered time periods.

Thus, our model permits examining the consequences of zero arbitrage profits more explicitly, and under less restrictive assumptions, than do other contemporary studies. Yet the model is not analytically complex, and we do not think it is based on empirically unreasonable assumptions.

2 - 2 Evolution of interest rates

Let R_t be unity plus a *spot* (one - period) interest rate. The term R_0 is taken to be the initial spot rate, and the terms $\{R_0\}$ $t \in (1, 2, \ldots, T-1)$ will be interpreted below as a series of forward rates on bills, as estimated from data at time zero.

The realized spot rate at time t is the period t riskless rate. These spot rates are assumed to evolve stochastically about (R_t) according to a multiplicative factor $u > 1$ in the following way. Suppose *one* plus the time t spot rate is $u^j R_t$, $j \in J_t$, where

$$(2.2.1) \quad J_t = \{t, t-2, t-4, \ldots, -t+2, -t\}; \quad t \in \{1, 2, \ldots, T\}.$$

Then (one plus) the time t + 1 spot rate will be either $u^{j+1} R_{t+1}$ or $u^{j-1} R_{t+1}$; the realizations occur with probability p and $1 - p \equiv q$ respectively. Thus between times t and t + 1 the spot rate moves either up or down. It *cannot* remain at the same level over successive periods, but it can return to the same level over the next two periods.[1]

In order for rates to be capable of either increasing or decreasing from one period to the next, it is necessary that

$$(2.2.2) \quad [R_{t+1} / u] < R_t < [u \cdot R_{t+1}]$$

for all $t \in (1, 2, \ldots, T)$. Without assumption (2.2.2), some rate patterns could stochastically dominate others, contrary to our assumption of zero arbitrage profit opportunities.

The difference equation generating (one plus) the spot rate is :

$$(2.2.3) \quad S_{t+1} = R_{t+1} S_t U / R_t,$$

where S_{t+1} represent one plus the spot rate at time t + 1. By recursive substitution, and recalling that by definition $S_0 = R_0$, it follows immediately from (2.2.3) that

$$(2.2.4) \quad S_{t+1} = R_{t+1} U_{t+1},$$

where U_{t+1} is the random variable[2] generated by T + 1 successive realizations of U.

The mean and variance of U are respectively given by

$$(2.2.5) \quad E(U) = pu + q / u \text{ and}$$

$$(2.2.6) \quad V(U) = pq[(u^2 - 1) / u]^2.$$

---

[1]. *We could generalize the model to allow for a multiplicative increase u and a multiplicative decrease d, not necessarily equal to 1/u. If subsequent empirical work suggests the generalization would be fruitful, the formulae derived in this paper are not difficult to amend accordingly.*

[2]. *For example, U_3 has the outcomes u^3, u, u^{-1}, and u^{-3}, with probabilities p^3, $3p^2q$, $3pq^2$, and p^3 respectively.*

The drift of the process (2.2.4) is determined[3] by E(U). If E (U) = 1, the process has a constant mean except for the influence of any exogenous changes in R_t. The process has a lower bound of zero and, for finite values of u, t and R_t, a finite upper bound of $u^t R_t$.

The succeeding discussion shows that assuming a particular spot rate process and zero arbitrage profits implies the term structure evolves in a particular manner dependent on the realized spot rates. The model makes this dependence explicit on economic data rather than assuming it in a priori as do Ho and Lee [1986].

3 - SECURITIES AND CLAM PRICES

For structuring the analysis and for recognizing data availability, explicit analysis of the interest rate process is truncated at time T. This section relates bill prices, contingent claim prices and interest rates for a truncated process. Given a set of forward rates, these relations define a particular set of claim values and related conditional probabilities[4] The same contingent claims values are used to derive futures and forward prices consistent with the zero arbitrage profits assumption.

3 - 1 Assumed structure for bill prices

Bills are defined to have a value of unity at maturity ; bills maturing at M > T to have time T values

$$1 / R(j, T, M)^{M-T},$$

where $R(j, T, m) \equiv u^j R_{T,m}$ is an average one - period discount factor for the periods

$$T, \ldots, M: \quad j \in J_T.$$

In other words, beyond time T the term structure is assumed to be flat for any given realization of the rate process.[5]

Let $B_t(j, T, M)$ represent the market price at time t, when the spot rate is $u^j R_t$, of a bill with maturity M > T,

$$t \in \{0, 1, 2, \ldots, T-1\}; \; j \in J_t.$$

(The cases M ≤ T are discussed below.) Then

$$(3.1.1) \quad B_T(j, T, M) = 1 / [u^j R_{T,M}]^{M-T}.$$

---

3. *Apart from the influence of the parameters Rt.*

4. *In turn, the conditional probabilities define the underlying equivalent martingale mesure.*

5. *This does not mean that the average return on the bills necessarily equal to the spot rate. Suppose, for instance, that one plus the spot rate at time T is given by $u^j R^*_T$. Then to reflect a term premium on a long maturity bond, the average yield can be defined as $u^j R_T$, where $R_T = R^*_T + \epsilon$, where ϵ reflects the term premium.*

In addition,

(3.1.2) $B_T(j - 2, T, M) / B_T(j, T, M) = u^{2(M - T)}$.

Except when needed f a expositional clarity*the notation will usually be shortened ; for example, we often write $R_{T,M}$ as R_T and $B_t(j, T, M)$ as $B_t(j)$.

3 - 2 Relations between interest rates and claim prices

To proceed, let $c_t(j, +)$ be the time t value of a (one - period) unit claim to be paid at time t if the spot rate rises, and let $c_t(j, -)$ be the time t value of a (one - period) unit claim to be paid if the spot rate falls. Since only the two outcomes are possible, the sum of claim values equals the present value of a risk free payment of unity, and the discount factor is the reciprocal of the one plus the current risk free rate :

(3.2.1) $c_t(j, +) + c_t(j, -) = 1 / u^j R_t$.

The conditional probabilities defining the equivalent martingale measure in a world of zero arbitrage profits are :

(3.2.2) $p_t(j) = c_t(j, +) / c_t(j, +) + c_t(j, -)$ and

(3.2.3) $q_t(j) = c_t(j, -) / c_t(j, +) + c_t(j, -)$;

cf., for example, Huang and Litzenberger [1988, pp. 223-2341. Hence

(3.2.4) $c_t(j, +) = p_t(j) / u^j R_t$, and

(3.2.5) $c_t(j, -) = q_t(j) / u^j R_t$.

Note also that [6]

$c_t(j, +) / c_t(j - 2, +) = 1 / u^2$, and

$c_t(j, -) / c_t(j - 2, -) = 1 / u^2$.

This mean $p_t(j) / p_t(j - 2) = 1$, implying that $p_t(j) = p_t$, and hence

(3.2.6) $c_t(j, +) = p_t / u^j R_t$, and

(3.2.7) $c_t(j, -) = q_t / u^j R_t$,

for all $j \in J_t$.

3 - 3 Bill price formulae

Bill prices at time t are related to bill prices at t + 1 by the possible rate outcomes and the values of the contingent claims associated with them. Hence the bill prices can be written recursively as

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[6]. *Observe that the process starting at $w^{-2} R_t$ and that starting at $w R_t$ are identical except for the multiplicative factor $u^2 R_t$. Since under the marginale the pseudo probabilities $P+(j)$ will be independant of scale changes in the gambles, we have $P+(j) = P+$ for all $j \in J_t$.*

$B_t(j) = c_t(j, +)\cdot B_{t+1}(j+1) + c_t(j, -)\cdot B_{t+1}(j-1)$.

Substituting from (3.2.6) and (3.2.7),

$$(3.3.1)\quad B_t(j) = [p_t\cdot B_{t+1}(j+1) + q_t\cdot B_{t+1}(j-1)] / u^j R_t.$$

Then starting at time T - 1, proceeding by backward induction and using the form of prices B_T (j, T, M) given in (3.1.1), the following expression can be derived:

$$(3.3.2)\quad B_t(j) = \{ [p_t + u^{2(T-t-1)}v^2 q_t] / u^j R_t \}\cdot B_{t+1}(j+1),$$

where $v = u^{M-T}$ and $t \in \{1, 2, \ldots, T-1\}$. The method is not given explicitly here, since it is a special case[7] of valuing the forward contracts discussed in section 4.

Further recursive application of (3.3.2) with $j = t$ gives

$$(3.3.3)\quad B_0(0, T, M)$$

$$= \{ [p_0 + u^{2(M-1)}q_0] / u^0 R_0 \}\cdot\{ [p_1 + u^{2(M-2)}q_1] / u^1 R_1 \}\cdot \ldots\cdot$$

$$\cdot\{ [p_{T-1} + u^{2(M-T)}q_{T-1}] / u^{T-1} R_{T-1} \}\cdot B_T(T, T, M)$$

where $v = u^{M-T}$ has been eliminated to simplify the formula.

In the sequel it will also be convenient to use the following special case of (3.3.4) when $M \le T_S$:

$$(3.3.4)\quad B_0(0, S, S) =$$

$$\{[p_0 + u^{2(S-1)}q_0] / R_0\}$$

$$\cdot \{[p_1 + u^{2(S-2)}q_1] / uR_1\} \cdot \ldots$$

$$\cdot \{[p_{S-1} + q_{S-1}] / u^{S-1}R_{S-1}.$$

3.4 Implied conditional probabilities

The bonds values given by (3.3.4) must also equal their values computed according to the forward rates in the data set, restricting the values of the conditional probabilities[8] p_t. For example. let T = 3, and calculate time zero bill prices using the terms $\{R_0, R_0, R_0\}$. Then

$$(3.4.1)\quad B_0(0, 1, 1) = 1/R_0 = \{[p_0 + q_0] / R_0\};$$

$$(3.4.2)\quad B_0(0, 2, 2) = 1/R_0\cdot R_1$$

$$= \{ [p_0 + u^2 q_0]/R_0 \}\cdot\{ [p_1 + q_1]/uR_1 \};\ \text{and}$$

$$(3.4.3)\quad B_0(0, 3, 3) = 1/R_0\cdot R_1\cdot R_2 =$$

$$\{ [p_0 + u^4 q_0] / R_0 \}\cdot\{ [p_1 + u^2 q_1] / uR_1 \}\cdot\{[p_2 + q_2] / u^2 R_2 \}.$$

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7. The value of the bond is the same as the value of the forward contract if the latter has a delivery (exercise) price of zero.

8. In empirical work, the term structure employed will be derived from the values of treasury bills in the data set; the values of $\{R_t\}$, $\{p_t\}$ and u will be set accordingly.

The values of p_t and q_t satisfying (3.4.1), (3.4.2), and (3.4.3) are

$$p_0 = u / [u + 1], \quad p_1 = u^3 / [u^3 + 1];$$

independent of $\{R_t\}$, $t \in \{0,1,2\}$. In general, the restrictions implied by determining the bill prices consistently with both (3.3.4) and the $\{R_t\}$ imply the conditional probabilities

$$(3.4.4) \quad p_t = u^{1+2t} / [u^{1+2t} + 1],$$

$t \in \{0,1,2,\ldots, T-2\}$, as can be verified using (3.3.4). As the example indicates p_{T-1} and q_{T-1} are unrestricted, apart from the requirement that they *sum* to unity.

The term structure also evolves in a particular manner, since at time t it is defined by [9] the bill price formulae (3.3.3) and (3.3.4), the probabilities (3.4.4), and the originally specified values of the $\{R_t\}$; $t \in \{t, t = 1\ldots, T-1\}$;

$t \in \{1, 2, \ldots, T-1\}$.

3 - 5 Conditional variance of bill prices

Using (3.3.1) and (3.3.2) to express $B_{t+1}(j-1)$ in terms of $B_{t+1}(j+1)$ we obtain

$$(3.5.1) \quad B_{t+1}(j-1) = u^{2(M-t-1)} B_{t+1}(j+1),$$

after eliminating $V = u^{M-T}$. Since $B_{t+1}(j+1)$ occurs with probability p and $B_{t+1}(j-1)$ with probability q, the results of Appendix I imply that the conditional variance of the bill price is :

$$(3.5.2) \quad V_{B,t}(j) = pq(u^{2(M-t-1)} - 1)^2 B_{t+1}(j+1).$$

The conditional variance of the bill return is found by rewriting (3.5.2) to express $B_{t+1}(j+1)$ in terms of $B_t(j)$, and then dividing by $B_t(j)$:

$$(3.5.3) \quad V^*_{B,t}(j) =$$

$$pq(u^{2(M-T-1)} - 1)^2 \cdot (u^j R_t)^2 / (p_t + u^{2(M-t-1)} q_t)^2.$$

---

[9]. *As a practical matter, the path dependent term structure for time t can then be calculated using the forward rates implicit in the ratios of bill prices. It is also possible to* ***compute*** *the implied variance of bond prices for times $t \in \{1, 2, \ldots, T\}$; recall we have already* ***specified*** *circumstances under which the variance of the interest rate process increases.*

In essence we are working, as are Ho-Lee ***[1986] and*** *Bliss-Ronn* ***[1989],*** *with particular forms of perturbation functions. However, our approach begins with underlying* ***economic variables*** *- the spot rate process and bill prices - and provides* ***explicit*** *formulae* ***relating*** *these* ***fundamentals*** *to* ***the financial instruments*** *we study.* ***In contrast*** *to these other studies, we* ***incorporate*** *time-dependence in relative bill prices* ***and*** *a high degree* ***of flexibility*** *in our choices of* ***term*** *structures.*

3 · 6 Conditional risk premia

We define the conditional risk premium as the conditionally expected price at time t + 1 less the bill price at time t times the current spot rate

$$(3.6.1) \quad P_{B,t}(j) = E_t B_{t+1}(J) - B_t(j) u^j R_t,$$

or, more simply as the excess rate of return

$$(3.6.2) \quad P^*_{B,t}(j) = [E_t B_{t+1}(J) - B_t(j) u^j R_t] \;/\; B_t(j),$$

where J is a random variable assuming one of the values

$$\{ j + 1,\; j - 1 \}.$$

By expressing all bill prices in terms of B, (j), and using (3.3.1) and (3.3.2), (3.6.2) simplifies to

$$(3.6.3) \quad P^*_{B,t}(j)$$

$$= \{ [(p + u^{2(M-T-1)} q) / (p_t + u^{2(M-T-1)} q_t)] - 1 \} u^j R_t .$$

3 · 7 Relation between risk premia and conditional variance

The excess rate of return on the bill can be expressed in terms of the conditional standard deviation $[V^* B, (j)]^{1/2}$. Combining (3.5.3) and (3.6.3) gives

$$(3.7.1) \quad P^*_{B,t}(j) / [V^*_{B,t}(j)]^{1/2}, \; = \; K^*_{B,t}$$

$$\text{where } K^*_{B,t} = \{ [(p + u^{2(M-T-1)} q) / (p_t + u^{2(M-T-1)} q_t)] - 1 \} \cdot$$

$$\cdot \{ p_t + u^{2[M-t-1]} q_t \} / \{ pq(u^{2(M-T-1)} - 1) \}.$$

Equation (3.7.1) says that the relationship between the conditional risk premium and the conditional standard deviation is one of strict proportionality. There is also a constant proportionality relationship between conditional risk premium and conditional variance, depending on R_t as well as on the variables determining $K^*_{B,t}$.

Equation (3.7.1) offers theoretical support for relations assumed in the literature. Specifically, the ARCH-M model of Engle, Lilien, and Robins [1987] based on Engle [1982], postulates an ex ante relationship between the conditional mean and conditional variance of returns when there is one risky and one riskless asset. They estimate models in which the conditional mean of bill returns is a linear function of the conditional standard deviation and a linear function of its logarithm Different assumptions about investors' utility functions lead to the different functional forms.

4 · FORWARD PRICES

This section values forward contracts and develops expressions for forward prices, their conditional variances, and risk premiums.

4 - 1 Recursive calculation of forward prices

Let $G_t(j, T, M)$ be the forward price at time t, when the spot rate is $u^j R_t$, on a contract written at time t, with delivery date T, against a bill maturing at time $M \geq T$. On the delivery date, the forward price equals the value of the underlying instrument ; cf. Cox, Ingersoll, Ross [1981]. Therefore,

$$(4.1.1) \quad G_T(j, T, M) = B_T(j, T, M).$$

As before, the arguments T and M will be suppressed whenever no ambiguity results, and the forward price will usually be written

$$G_t(j); \quad t \in \{ 0, 1, 2, \ldots, T \}; \quad j \in J_t.$$

Next, let the value at time t of a forward contract written at time 0, with exercise (delivery) price X_T, and when the spot rate is $u^j R_t$, be defined as $F_{0,t}(j, X_T, T, M)$. As usual, the last two arguments will be suppressed unless needed for clarity, and the value of the forward contract will normally be written [10]

$$F_{0,t}(j, X_T).$$

Consider first the problem of valuing a forward contract with an arbitrary delivery price ; it will then be easy to calculate the forward price for that contract. Proceeding by backward induction, on the delivery date the contract value is the difference between the bond price and X_T, the delivery (exercise) price. Thus, if the interest rate is $u^T R_T$:

$$(4.1.2) \quad F_{0,T}(T, X_T) = B_T(T) - X_T = [\, 1/ [u^T R_T]^{M-T} \,] - X_T.$$

Then it follows immediately that

$$(4.1.3) \quad F_{0,T}(T - 2 \cdot k, X_T) = v^{2k} B_T(T) - X_T.$$

where $v = v(T, M) = u^{M-T}$, and $k \in \{0, 1, 2, \ldots, T\}$.

Next,

$$\begin{aligned}(4.1.4) \quad F_{0,T-1}(T-1, X_T) &= c_{T-1}(T-1, +) \cdot F_{0,T}(T, X_T) \\ &\quad + c_{T-1}(T-1, -) \cdot F_{0,T}(T-2, X_T) \\ &= (p_{T-1} + v^2 q_{T-1}) \cdot F_T(T, X_T) \,/\, u^{T-1} R_{T-1} \\ &= \{ [(p_{T-1} + v^2 q_{T-1}) \cdot B_T(T)] - X_T \} \,/\, u^{T-1} R_{T-1}. \\ &= A_{T-1} - X_{T-1}.\end{aligned}$$

---

10. *We shall show below how the notation can accommodate forward contracts written at arbitrary times t. It is convenient to define the value of the forward contract as well as the forward price so that bond prices, forward prices, and futures prices can all be rehed using the same methodology. The notation for forward and futures prices (G_t (j) and H_t (j) ,respectively) is consistent with Cox Ingersoll Ross [1981], and G_t (j) is the special value of X_t such that the value of the forward contract is zero when it is written ; cf. Jarrow and Oldfield [1981].*

Note that X_{T-1} is the present value of X_T, calculated using (one plus) the appropriate spot rate $u^{T-1} R_{T-1}$.

It is then easy to see that

$$(4.1.5) \quad F_{0,T-1}(T - 1 - 2k,\ X_T) = (u \cdot v)^{2k} A_{T-1} - u^{2k} X_{T-1},$$

$k \in \{0, 1, 2, \ldots, T - 1\}$.

Recursive application of (4.1.4) and (4.1.5) gives

$$\begin{aligned}(4.1.6) \quad F_{0,t-1}(t - 1,\ X_T) &= (p_{t-1} + v^2 q_{t-1}) F_{0,t}(t,\ X_T) \ / \ u^{t-1} R_{t-1} \\ &= [(p_{t-1} + v^2 q_{t-1}) \cdot A_t - X_t] \ / \ u^{t-1} R_{t-1}. \\ &= A_{t-1} - X_{t-1}, \text{ and}\end{aligned}$$

$$(4.1.7) \quad F_{0,t-1}(t - 1 - 2k,\ X_T) = (u \cdot v)^{2k} A_{t-1} - u^{2k} X_{t-1},$$

$k \in \{0, 1, 2, \ldots, t - 1\}$.

It is then possible to write the explicit form

$$\begin{aligned}(4.1.8) \quad F_{0,0}(0,\ X_T) = \ & \{[p_0 + u^{2[M-1]} q_0] \ / \ R_0\} \\ & \cdot \{[p_1 + u^{2[M-2]} q(1)] \ / \ uR_1\} \cdot \ldots \\ & \cdot \{[p_{T-1} + u^{2[M-T]} q_{T-1}] \ / \ u^{T-1} R_{T-1}\} \ B_T(T) \\ & - \{[p_0 + u^{2[T-1]} q_0] \ / \ R_0\} \\ & \cdot \{[p_1 + u^{2[T-2]} q(1)] \ / \ uR_1\} \cdot \ldots \\ & \cdot \{[p_{T-1} + q_{T-1}] \ / \ u^{T-1} R_{T-1}\} \cdot X_T\end{aligned}$$

eliminating $v = u^{M-T}$ as before.

Equation (4.1.8) can be written more simply:

$$(4.1.9) \quad F_{0,0}(0,\ X_T) = B_0(0,\ T,\ M) - B_0(0,\ T,\ T) \cdot X_T.$$

Next, equation (4.1.9) implicitly defines $G_0(0)$ by the condition

$F_{0,0}(0,\ G_0(0)) = 0$, or

$$(4.1.10) \quad G_0(0) = B_0(0,\ T,\ M) \ / \ B_0(0,\ T,\ T);$$

cf. Jarrow and Oldfield [1981, p. 381, eqn.(13)]. A forward contract originating at time t, when one plus the interest rate is $u^j R_t$, has a value at time $s > t$, when one plus the interest rate is $u^k R_s$, given by

$F_{t,s}(k, G_t(j))$. The explicit form of $G_t(j)$ is:

$$(4.1.11) \quad G_t(j) = B_t(j,\ T,\ M) \ / \ B_t(j,\ T,\ T).$$

4 • 2 Conditional variance of forward prices

The forward price on a contract written at time t does not change before the contract delivery date, time T. However, new contracts can be written at times s > t, and the conditional variance of forward prices refers to the possible variations in the prices on these new contracts, which will be written to reflect the newly prevailing time and interest rate environment.

For theoretical purposes, assume a new contract is written at each point in times, and that all contracts have the same delivery date T. Given the forward price $G_t(j)$, the forward price at time $t+1$ is either

$G_{t+1}(j+1)$, with probability p, or

$G_{t+1}(j-1)$ with probability q.

Rewrite the second of these outcomes:

$G_{t+1}(j-1) = u^{2[M-t-1]} \cdot G_{t+1}(j+1)$ with probability q.

Appendix I shows, by taking

$y = u^{2[M-t-1]} \cdot G_{t+1}(j+1)$ and

$x = G_{t+1}(j+1)$

that the conditional variance of this binomial distribution is

$$(4.2.1) \quad V_{G,t}(j) = pq \cdot \{ [u^{2[M-t-1]} - 1] \cdot [G_{t+1}(j+1)] \}^2.$$

Then, using (4.1.11), (4.2.1) can be rewritten as

$$(4.2.2) \quad V_{G,t}(j) = pq \cdot \{ [u^{2[M-t-1]} - 1] \cdot [G_t(j)] \}^2 \cdot \{[p_t + u^{2(T-t-1)} q_t] / [p_t + u^{2(M-t-1)} q_t]\}^2.$$

Denoting the conditional variance of the rate of change of the forward price by $V^*_{G,t}(j) = V_{G,t}(j) / [G_t(j)]^2$, it follows immediately from (4.2.2) that

$$(4.2.4) \quad V^*_{G,t}(j) = pq \cdot [u^{2[M-t-1]} - 1]^2 \cdot \{[p_t + u^{2(T-t-1)} q_t] / [p_t + u^{2(M-t-1)} q_t]\}^2,$$

independent of j.

4 • 3 Conditional risk premia in forward prices

Define the conditional risk premium in a forward price by

$$(4.3.1) \quad P_{G,t}(j) = E_t\{ G_{t+1}(J) \} - G_t(j),$$

where E_t denotes the time t conditional expectation of the time t + 1 forward price, and J is a random variable,

$J \in \{ j + 1, j - 1 \}$.

Condition (4.3.1) can be rewritten

$$(4.3.2)\quad P_{G,t}(j) =$$

$$= \{ [(p + u^{2[M-t-1]}q)\cdot(p_t + u^{2[T-t-1]}q_t) / (p_t + u^{2[M-t-1]}\cdot q_t)] - 1\}\cdot$$

$$\cdot G_t(j)$$

An expression for the risk premium in rate of return form can also be found using

$$(4.3.3)\quad P^*_{G,t}(j) = P_{G,t}(j) / G_t(j).$$

4 - 4 Conditional risk premium and standard deviation

As with bill prices, a proportional relationship between $P^*_{G,t}(j)$ and $[V^*_{G,t}(j)]^{1/2}$ by using (4.2.4) and (4.3.3) to define an appropriate proportionality constant.

$$(4.4.1)\quad P^*_{G,t}(j) / [V^*_{G,t}(j)]^{1/2} = K^*_{G,t}(j), \text{ where}$$

$$(4.4.2)\quad K^*_{G,t}(j) =$$

$$\{ [p + u^{2[M-t-1]}q]\cdot[p_t + u^{2[T-t-1]}q_t] - [p_t + u^{2[M-t-1]}q_t] \} /$$

$$(pq)^{1/2}(u^{2[M-t-1]} - 1).$$

5 - FUTURES PRICES

This section develops expressions far futures prices, theu conditional variances, and risk premiums.

5 - 1 Recursive calculation of futures prices

Let $H_t(j, T, M)$ be the futures price at time t, when the *spot* rate is $u^j R_t$, on a Contract with delivery date T written against a bill maturing at time M. On the delivery date, the futures price equals the value of the underlying instrument; cf. Cox, Ingersoll, Ross [1986]. Therefore,

$$(5.1.1)\quad H_T(j, T, M) = B_T(j, T, M).$$

In periods $t, t \in \{0, 1, 2, \ldots, T - 1\}$, the futures price is defined as $H_t(j, T, M)$. However as with the underlying bills, the arguments T and M will be suppressed whenever no ambiguity results, and the futures price will usually be written $H_t(j)$;

$t \in \{ 0, 1, 2, \ldots, T \}$.

Under the perfect markets, zero arbitrage opportunities assumptions of this paper, futures prices satisfy the condition

$$[c_t(j,+) + c_t(j,-)] \cdot H_t(j)$$

$$= c_t(j,+) \cdot H_{t+1}(j+1) + c_t(j,-) \cdot H_{t+1}(j-1);$$

(5.1.2) $t \in \{1, 2, \ldots, T-2\}$, and

$$[c_{T-1}(j,+) + c_{T-1}(j,-)] \cdot H_{T-1}(j) =$$

$$c_{T-1}(j,+) \cdot B_T(j+1) + c_{T-1}(j,-) \cdot B_T(j-1);$$

cf., e.g., Cox - Ingersoll - Ross - [1981, P. 337, eqn (42)].

Using (3.3.2) and (3.3.3) condition (5.1.2) can be rewritten

(5.1.3) $H_t(j) = p_t H_{t+1}(j+1) + q_t H_{t+1}(j-1);$

$t \in \{1, 2, \ldots, T-1\};$

and $H_T(j) = B_T(j)$, $j \in J_T$.

Moreover $H_T(T) = B_T(T) = 1/[u^T R_T]^{M-T}$.

Then it follows immediately that

(5.1.4) $H_T(T-k) = v^{2k} H_T(T)$,

where[11] $v = v(T, M) = u^{M-T}$, and $k \in \{0, 1, 2, 3, \ldots, T-1\}$.

Taking (5.1.3) with t = T - 1 and using the results of (5.1.4) gives

(5.1.5) $H_{T-1}(j) =$

$$= p_{T-1} H_T(j+1) + q_{T-1} H_T(j-1)$$

$$= p_{T-1} H_T(j+1) + v^2 q_{T-1} H_T(j+1)$$

$$= [p_{T-1} + v^2 q_{T-1}] \cdot H_T(j+1).$$

Similarly,

(5.1.6) $H_{T-2}(j) =$

$$= p_{T-2} H_{T-1}(j+1) + q_{T-2} H_{T-1}(j-1)$$

$$= p_{T-2}[p_{T-1} + v^2 q_{T-1}] \cdot H_T(j+2) + q_{T-2}[p_{T-1} + v^2 q_{T-1}] \cdot H_T(j)$$

$$= [p_{T-2} + v^2 q_{T-2}] \cdot [p_{T-1} + v^2 q_{T-1}] \cdot H_T(j+2).$$

---

11. In 6.2, below it will be convenient to recognize that v (T, M) can be expressed as v(M - T); i.e., as a function of the difference between delivery and maturity dates.

Moreover,

$$(5.1.7)\quad H_{T-2}(T - 2) =$$

$$= [p_{T-2} + v^2 q_{T-2}]\cdot[p_{T-1} + v^2 q_{T-1}]\cdot H_T(T)$$

$$= [p_{T-2} + v^2 q_{T-2}]\cdot H_{T-1}(T - 1).$$

It follows that

$$(5.1.8)\quad H_t(j) = v^2 H_t(j \; t \; 2), \text{ and}$$

$$(5.1.9)\quad H_t(j) = [p_t + v^2 q_t]\cdot H_{t+1}(j + 1),$$

for all $j \in J_t$ and for all t, $t = 0, 1, \ldots, T-1$.

Finally, setting j = t and applying **the condition** (5.1.8) **recusively** gives an explicit formula for the futures price at time zero:

$$(5.1.10)\quad H_0(0) = [p_0 + v^2 q_0]\cdot[p_1 + v^2 q_1]\cdot \ldots$$

$$\cdot[p_{T-1} + v^2 q_{T-1}]\cdot B_T(T).$$

The futures price depends on M, T, $B_M(T)$, and u, but not on $\{R_t\}$, $t \in \{0, 1, 2, \ldots, T - 1\}$. The following properties will also prove useful.

Proposition 5.1.1 The futures **prices** $H_t(T)$ are decreasing in t. Inspection of (5.1.7) and (5.1.9) shows that

$$(5.1.11)\quad H_t(t) > H_{t+1}(t \; t \; 1) > H_{t+2}(t \; t \; 2).$$

To see the effect of time variation on the futures prices when j is held constant, consider

Proposition 5.13. Let the spot rate remain unchanged between periods. **Then the** ratio of **futures** prices decreases with time if and only if $M \geq M^*$, **where**

$$M^* = T \; t \; \{[\ln(p_{t-2}p_{t-1} / q_{t-2}q_{t-1})] / 2\cdot\ln(u)\}.$$

To establish this **proposition,** use (5.1.8) and (5.1.9) to write

$$5\ .1.12)\quad H_t(j) / H_{t-2}(j) = v^2 / [p_{t-2} + v^2 q_{t-2}]\cdot[p_{t-1} + v^2 q_{t-1}]\cdot$$

The behaviour of the ratio on the right hand side of (5.1.12) is revealed by defining

$x = v^2$, $a = p_{t-2}\cdot p_{t-1}$, $b = p_{t-2}\cdot q_{t-1} + p_{t-1}\cdot q_{t-2}$, and $c = q_{t-2}\cdot q_{t-1}$,

and **considering** the equation

$$x = a \; t \; bx \; t \; cx^2.$$

Noting that a + b + c = 1, rewrite the quadratic as

$$(5.1.13)\quad [x - a/c]\cdot[x - 1] = 0.$$

Given the values of $p_{t-1}, p_{t-2}, p, q_{t-1}$, and q_{t-2} as assumed in (3.4.4), it follows that $a > c$. Thus when v^2 lies between unity and a/c, the ratio (5.1.12) is greater than unity, and f a values of $v^2 > a/c$, (5.1.12) is less than unity. Since v^2 is an increasing function of **M,** there is a critical value M* which determines whether (5.1.12) is increasing a decreasing in t. Straightforward calculation shows M* is defined by

$$M^* = T + \{[\ln(p_{t-2}p_{t-1} / q_{t-2}q_{t-1})] / 2\cdot\ln(u)\}.$$

Note *that* M* is not necessarily an integer, as are M and T.

5 - 2 Conditional variance of futures prices

Conditional on a realization H, (j), the futures price at time t + 1 is either

$H_{t+1}(j + 1)$, with probability p, or

$H_{t+1}(j - 1$ with probability q.

Rewrite the second of these outcomes:

$H_{t+1}(j - 1) = v^2\cdot H_{t+1}(j + 1)$ with probability q.

Appendix I shows, by taking

$y = v^2\cdot H_{t+1}(j + 1)$ and

$x = H_{t+1}(j + 1)$

that the conditional variance of this binomial distribution is

$$(5.2.1)\quad V_{H,t}(j) = pq\cdot\{ [v^2 - 1]\cdot[H_{t+1}(j + 1)] \}^2.$$

Then, using (5.1.9), (5.2.1) can be rewritten as

$$(5.2.2)\quad V_{H,t}(j) = pq\cdot\{ [v^2 - 1]\cdot[H_t(j)] \}^2 / [p_t + v^2q_t]^2.$$

To see the effect on $V_{H,t}(j)$ when t increases while interest rates are held constant, recall from (5.1.8) that

$$H_{t+2}(j) = v^2\cdot H_{t+2}(j + 2)$$

Then by (5.1.3) and (5.2.2)

$$(5.2.3)\quad V_{H,t}(j) = pq\cdot\{ (v^2 - 1)\cdot H_{t+1}(j + 1) \}^2$$

$$= pq\cdot\{ (v^2 - 1)[p_t + v^2q_t]H_{t+2}(j + 2) \}^2,$$

$$= pq\cdot\{ (v^2 - 1)[p_t + v^2q_t]H_{t+2}(j) / v^2 \}^2,$$

Then, whether (5.2.3) increases or decreases in t depends on the behavior of both $p_t + v^2 q_t$ and $H_{t+2}(j)$, as well as on their relative sizes. Thus the change in $V_{H,t}(j)$ is in general ambiguous; cf. Proposition 5.1.2.

Defining the conditional variance of the rate of change of the futures price by

$$V^*_{H,t}(j) = V_{H,t}(j) / [H_t(j)]^2,$$

it follows immediately from (5.2.2) that

$$(5.2.4) \quad V^*_{H,t}(j) = pq\cdot\{ (v^2 - 1) / (p_t + v^2 q_t) \}^2,$$

independent of j. Note that $V^*_{H,t}(j)$ increases in t if and only if the denominator $p_t + v^2 q_t$ decreases in t.

5 - 3 Conditional risk premia in futures prices,

Define the conditional risk premium in a futures price by

$$(5.3.1) \quad P_{H,t}(j) = E_t\{ H_{t+1}(J) \} - H_t(j),$$

where E_t denotes the time t conditional expectation of the time t +1 futures price, and J is a random variable,

$$J \in \{ j + 1, j - 1 \}.$$

Condition (5.3.1) can be rewritten

$$(5.3.2) \quad P_{H,t}(j) = \{ [p + v^2 q] - [p_t + v^2 q_t] \}\cdot H_{t+1}(j+1).$$

The risk premium can also be expressed in terms of the rate of change of futures prices,

$$(5.3.3) \quad P^*_{H,t}(j) = P_{H,t}(j) / H_t(j).$$

Then since

$$p_t + v^2 q_t > p_{t+1} + v^2 q_{t+1},$$

the risk premia are positive whenever

$$p + v^2 q \geq p_0 + v^2 q_0.$$

5 - 4 Relations between conditional risk premia and variances

Recalling (5.3.2),

$$P_{H,t}(j) = \{ [p + v^2 q] - [p_t + v^2 q_t] \}\cdot H_{t+1}(j+1),$$

which, using (5.2.1) can be rewritten as

$$(5.4.1) \quad P_{H,t}(j) = \{[p + v^2 q] - [p_t + v^2 q_t]\}\cdot \cdot\{V_{H,t}(j)/pq\}^{1/2}/(v^2 - 1).$$

Similarly, using (5.3.3) and (5.4.1),

$$(5.4.2) \quad P^*_{H,t}(j) = \{[p + v^2 q] - [p_t + v^2 q_t]\}\cdot \cdot\{W_t(j)/pq\}^{1/2} / (v^2 - 1)\cdot H_t(j)$$
$$= \{[p + v^2 q] - [p_t + v^2 q_t]\}\cdot \cdot\{V^*_{H,t}(j)/pq\}^{1/2} / (v^2 - 1).$$

Then it can readily be seen that

$$P^*_{H,t}(j) \;/\; [V^*_{H,t}(j)]^{1/2} = K^*_{H,t}$$

where the constant of proportionality $K^*_{H,t}$ is given by

$$K^*_{H,t} \quad \{ [p + u^{2(M-T)}q] - [p_t + u^{2(M-T)}q_t] \} \;/\; (pq)^{1/2} \cdot (u^{2(M-T)} - 1).$$

Thus $K^*_{H,t}$ depends an : p, u, T, and M, but not on the $\{R_t\}$. A similar proportionality constant can be defined to describe the relation between the conditional risk premium and the conditional variance of the futures price rate of change. Note that $K^*_{H,t}$ will be positive if

$$[p + v^2 q] > [p_t + v^2 q_t].$$

5 - 5 RELATIONS BETWEEN FORWARD AND FUTURES PRICES

The formulae for futures and forward prices permit explicit comparisons. Recall :

(4.1.10) $\quad G_0(0, T, M) = B_0(0, T, M) \;/\; B_0(0, T, T)$ and

(5.1.10) $\quad H_0(0, T, M) = [p_0 + v^2 q_0] \cdot [p_1 + v^2 q_1] \cdot \ldots$

$$\cdot [p_{T-1} + v^2 q_{T-1}] \cdot B_T(T),$$

from which the ratio $G_0(0) / H_0(0)$ can readily be calculated. In addition, by using

(3.3.3) $\quad B_0(0, T, M)$

$$= (PR(t = 0, T-1) \; \{[p_t + u^{2(T-1)} v^2 q_t] \;/\; u^t R_t\} \;) \cdot B_T(T, T, M)$$

forming the ratio $H_0(0, T, M), / B_0(0, T, M)$, where PR ( ) denotes a product recalling that $u > 1$, and that

$$B_0(0, T, T) = 1/R_0 \cdot uR_1 \cdot \ldots \cdot u^{T-1}R_{T-1},$$

it is easy to see that

$$G_0(0, T, M) = B_0(0, T, M) \;/\; B_0(0, T, T) > H_0(0, T, M).$$

The last condition is a special case of Cox - Ingersoll - Ross Proposition 9 [1986, pp 331-3321.

6 - CONCLUSIONS

This section briefly outlines possibilities for both empirical tests and managerial applications of our theory. Both these applications are to be explaed in future wak.

Empirical work necessarily involves tests of joint hypotheses of rational expectations or market efficiency and the underlying model of security pricing.

Our proposed estimations will use methods designed for time series which display persistence in heteroscedasticity. So fat, the main testable implications we have developed involve the relations between conditional means and conditional standard deviations.

Further testable implications will follow from individual pricing formulae and from the relations between price series which they define. Data availability will restrict some tests, and not all logical implications are actually testable.

The paper's formulae can be used to calculate approximate values for instruments issued by financial institutions. In addition, the relations between futures and forward contracts are useful in informing managers about how different instruments related in a practical sense. Finally, the formulae make clear the important variables affecting valuation of a given instrument.

7 - SELECTED REFERENCES

Bollerslev, T., 1986, "Generalized autoregressive conditional heteroscedasticity", Joumal of Econometrics 31, 307-327.

Bliss, Robert R Jr., and Ehud I. Ronn, 1989, "Arbitrage-based estimation of nonstationary shifts in the term structure of interest rates", Joumal of Finance 44, 591 410.

Cox, J.C., J. Ingersoll and S.A. Ross, 1981, "The relation between forward prices and futures prices", Joumal of Financial Economics 9,321-346.

Cox, John C., and Mark Rubinstein, 1985, Options Markets, Englewood Cliffs. N.J. : Prentice - Hall. :

Cox. J.C., S.A. Ross, and M. Rubinstein, 1979, "Option pricing : a simplified approach", Joumal of Financial Economics 7, 229-263.

Duffie, Darrell, 1989, Futures Markets, Englewood Cliffs, N. J. : Prentice -Hall.

Engle, R.F., 1982, "Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation", Econometrica 50, 987-1007.

Engle, R.F., D.M. Lilien and R.P. Robins, 1987, "Estimating time varying risk premia in the term structure", Econometrica 55, 391-407.

Fama, E. F., and R R Bliss, 1985, "The information in long maturity forward rates", University of Chicago Graduate School of Business Working Paper 170.

Ho, T.S.Y. and S. Lee, 1986, 'Term structure movements and pricing interest rate" contingent claims, Journal of Finance 41,1011-1029.

Hull, John, and Alan White, 1989, "Pricing interest-rate sensitive securities", University of Toronto Graduate School of Management Working Paper, presented to Northern Finance Association, Ottawa, 1989.

Kalymon, Basil A., The Future of Canada's Financial System, London : University of Western Ontario, 1988.

Kishimoto, Naoki, 1989. "Pricing amtingent claims under interest rate and asset price risk", Journal of Finance 44, 571-590.

McCurdy, T.H. and I.G. Morgan, 1989, "Evidence of risk premia in foreign currency futures markets", Econometric Society European meetings, August.

Poitras, Geoffrey, 1988, "Hedging Canadian **Treasury Bill** Positions **with** US **Money** Market Futures Contracts", Review of **Futures** Markets **7, 176-191.**

Richard, **S.F.** and M. **Sundaresan, 1981, "A** continuous time equilibrium model of forward prices and futures prices in a **multigood** economy", Journal of Financial **Economics** 9,347-372.

Appendix I. Variance of a binomial distribution

Let Z be a binomial random variable assuming the values x and y with probabilities **p** and q = 1 - p respectively. Suppose also that y **>** x. **Then**

$$Z^* = (Z - x) / (y - x)$$

is a standardized random variable with outcomes **0 and 1** with probabilities **p and** q respectively, **and** hence has variance pq. **Then** since

$V(Z^*) = pq = V(Z) / (y - x)^2$, it follows immediately that

$$V(Z) = pq \cdot (y - x)^2.$$

Appendix II. Conditional variance of interest rate process

The conditional variance of the interest rate **process** is determined as follows. Given the prevailing **spot** rate $u^{j-1} R_{t-1}$, the spot in the next period **increases to** $u^j R_t$ **or decreases to** u^{j-2} & with **probabilities** p **and** q respectively. Thus, using the results of Appendix I,

$$V_t(u^j) = pq \cdot [(u^{j+1} - u^{j-1}) R_{t+1}]^2.$$

For any fixed value of t, V_t is **an increasing** function of j.