CONTRIBUTION N° 24

DURATION:
ITS ROLE IN
IMMUNIZATION

PAR / BY

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LA DURÉE:
SON ROLE DANS
L'IMMUNISATION
RESUME

L'utilisation de la durée pour réaliser l'immunisation présente plusieurs difficultés résultant de la nécessité de faire des hypothèses très restrictives sur la forme et le comportement de la structure des termes des taux d'intérêt.

Le présent article aborde ce problème et a un double objectif :

Premièrement, on élabore formule de durée qui est compatible avec n'importe quelle hypothèse sur la structure des termes des taux d'intérêt, mais qui implique une parfaite connaissance des mouvements futurs de ces taux d'intérêt. Cet handicap renvoie toutefois au problème réel posé par la réalisation de l'immunisation : la nécessité de prévoir l'évolution future des taux d'intérêt.

Deuxièmement, on prouve que la durée est, en fait, une approximation linéaire d'une autre variable que l'on appelle, plus facile à calculer que la durée, en absence de toute hypothèse sur la structure des termes des taux d'intérêt. La relation entre la durée et peut, en même temps, aider à comprendre le rôle joué par la durée dans les stratégies d'immunisation.

Enfin, on étudie rapidement les relations entre la durée et, en précisant les avantages et les inconvénients relatifs de ces deux variables, dans le contexte d'une stratégie d'immunisation.
SUMMARY

The use of duration to achieve immunization presents several problems derived from the need of making very restrictive assumptions about the shape and behaviour of the term structure of interest rates.

Closely related with that problem, the aim of this paper is twofold. First, we develop a duration formula which is compatible with any assumption about the term structure of interest rates although it implies a perfect knowledge of future movements of interest rates. This handicap, however, points out the real problem involved in the achievement of immunization: the need of forecasting future movements of interest rates.

Second, we prove that duration is, in fact, a linear approximation to another variable, that we call $T$, which is much easier to compute than duration under the absence of any assumption about the term structure of interest rates. At the same time, this relationship between duration and $T$, may help to understand the role played by duration in immunizing strategies.

Finally, a brief study of the relationships between duration and $T$, is made, pointing out the advantages and disadvantages of each variable with respect to the other in the context of a immunizing strategy.
DURATION: ITS ROLE IN IMMUNIZATION.

The use of duration to achieve immunization presents several problems derived from the need of making very restrictive assumptions about the shape and behaviour of the term structure of interest rates.

Closely related with that problem, the aim of this paper is twofold. First, we develop a duration formula which is compatible with any assumption about the term structure of interest rates although it implies a perfect knowledge of future movements of interest rates. This handicap, however, points out the real problem involved in the achievement of immunization: the need of forecasting future movements of interest rates.

Second, we prove that duration is, in fact, a linear approximation to another variable, that we call \( t_e \), which is much easier to compute than duration under the absence of any assumption about the term structure of interest rates.

GENERAL DURATION FORMULA

Let's assume an investor who buys at time \( t = 0 \) a fixed-income portfolio that generates the following payments stream:

<table>
<thead>
<tr>
<th>Time (( t, t_2, \ldots, t_n ))</th>
<th>Amounts</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>( c_1 )</td>
</tr>
<tr>
<td>( t_1 )</td>
<td>( c_2 )</td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( c_3 )</td>
</tr>
<tr>
<td>( t_n )</td>
<td>( c_n )</td>
</tr>
</tbody>
</table>

The current value of that portfolio can be calculated from the term structure of interest rates that is described by the function \( h(O,t) \) which takes the following values:

\[
h(O,t) = \begin{cases} 
  r_i \text{ at } t=t_i, \\
  \vdots & \text{if } i \neq 1,2,\ldots,n 
\end{cases}
\]

where \( h(O,t) \) is the current rate of growth of an investment which is maintained for a period of length \( t \), i.e., an investment of one money unit now, will grow to an amount \( \exp[h(O,t) \cdot t] \) at time \( t \).

Under those assumptions, the present discounted value of a portfolio as the one described above, is:

\[
V_0 = \sum_{i=1}^{n} c_i \cdot \exp[-r_i \cdot t_i] \quad [1]
\]

where \( V_0 \) can be regarded as a real function of \( n \) variables \( r_1, r_2, \ldots, r_n \) which is infinitely differentiable.

1 A discussion about the assumptions and implications of this preliminaries can be found in V. and NAVARRO, E.: ¿Es posible la immunización financiera? Expresión general de la duración immunizadora. Transactions of the 23rd International Congress of Actuaries. Vol. 5 Helsinki, Finland, July 1988.

2 This magnitude is called logarithmic density for its relation with the normal interest rate is \( h(0,1) = \ln(1+i) \) where \( i \) is the mean interest rate for the period \( O, t \). In any case we will refer to \( h(O,t) \) as the "interest rate."
Additionally we assume that the investor has a planning period of length $t_p$ and that he wishes to buy a portfolio in order to obtain, at least, the return that the current term structure of interest rates offers for the period $[0, t_p]$ i.e. $h(O, t_p)$.

The portfolio value at time $t_p$ (if interest rates do not change during the period $[0, t_p]$) would be:

$$V = V \cdot \exp(r \cdot t, 1) = \sum_{i=1}^{n} c_i \cdot \exp(-r_i \cdot t, t)$$  \hspace{1cm} (2)

As before, $V_p$ can be viewed as a real function of $n$ variables $r_1, r_2, ..., r_n$ - infinitely differentiable.

Now we are going to analyse how a change in interest rates affects the portfolio final value, i.e. the portfolio value at $t_p$. So that, we assume that after $t = O$ there is a shift in the term structure of interest rates such that its new values are:

$$h^* (O, t) = \begin{cases} r_1 & \text{at } t = t_1 \\ r_2 & \text{at } t = t_2 \\ \vdots & \text{at } t = t_k \\ r_i & \text{at } t = t_n \end{cases}$$

The new portfolio present value, after that change in interest rates is:

$$V^* = \sum_{i=1}^{n} c_i \cdot \exp(-r_i \cdot t)$$

and if there were no more changes in the term structure during the period $(O, t_p)$ the portfolio value at $t_p$ would be:

$$V^* = V \cdot \exp(r \cdot t, t) = \sum_{i=1}^{n} c_i \cdot \exp(-r_i \cdot t, t)$$  \hspace{1cm} (3)

In order to analyse the change in $V_p$ due to a given change in interest rates we will expand the function $V_p$ at $(r_1, r_2, ..., r_n)$ using Taylor's formula. Then we have:

$$V^* - V = \nabla V_p \cdot (\Delta r_1, \Delta r_2, ..., \Delta r_n) + \frac{1}{2} (\Delta r_1, \Delta r_2, ..., \Delta r_n) \cdot H_p \cdot (\Delta r_1, \Delta r_2, ..., \Delta r_n)$$

where $\nabla V_p$ is the gradient vector of $V_p$ at $(r_1, r_2, ..., r_n)$ and $H_p$
is the hessian matrix of $V_p$ at a point $(\theta_1, \theta_2, ..., \theta_n)$ in the linear segment $L(r, r')$.

It is not difficult to prove that $H_p$ is a positive semidefinite matrix so that

$$V_p = \nabla V_p \cdot (\Delta r_1, \Delta r_2, ..., \Delta r_n) = 0.$$  \[4\]

An immunizing strategy would consist, thus, in choosing a portfolio such that $\nabla V_p \cdot (\Delta r_1, \Delta r_2, ..., \Delta r_n) = 0$ so that condition held we would be able to assure that if any change in interest rates happens just after $t = 0$, the portfolio value at $t = t_p$ would be at least $V_p$.

If we develop $(4)$, we obtain:

$$\nabla V_p \cdot (\Delta r_1, \Delta r_2, ..., \Delta r_n) = \sum_{i=1}^{p-1} t_i \Sigma c_i \cdot \exp(-r_i \cdot t_i + r_\ast \cdot t_i) \cdot \Delta r_i + t_p \Sigma c_i \cdot \exp(-r_i \cdot t_i + r_\ast \cdot t_i) \cdot \Delta r_i + t_p \Sigma c_i \cdot \exp(-r_i \cdot t_i + r_\ast \cdot t_i) \cdot \Delta r_i +$$

$$+ \sum_{i=p+1}^{n} t_i \Sigma c_i \cdot \exp(-r_i \cdot t_i + r_\ast \cdot t_i) \cdot \Delta r_i$$

Rearranging terms and making it equal to zero:

$$\sum_{i=1}^{n} t_i c_i \cdot \exp(-r_i \cdot t_i + r_\ast \cdot t_i) = \sum_{i=1}^{n} t_i c_i \cdot \exp(-r_i \cdot t_i + r_\ast \cdot t_i)$$

and if $\Delta r_i \neq 0$

$$t_i = \frac{\sum_{i=1}^{n} c_i \cdot \exp(-r_i \cdot t_i)}{\Delta r_i}$$  \[5\]

Condition (5) is then a necessary and sufficient condition for immunizing a portfolio that generates a payments stream as initially described.

Then if we called the right hand side of (5) generalized duration $D_g$ we have that an investor can be immunized against a given change in interest rates by building a portfolio with a generalized duration equal to its planning period, i.e. $D_g = t_p$.

The great advantage of $D_g$ over other duration formulae is that $D_g$ is compatible with any assumption about the future behaviour of the term structure of interest rates; in fact all other alternative duration formulae can be obtained from $D_g$ if the hypothesis made to obtain all those alternative formulae are applied in $D_g$ (3).

But unfortunately, the applicability of this duration formula depends on the investor ability to forecast, as precisely as possible, future movements of interest rates, for we need to know the values of $r_i$ (for all $i$) in order to compute the value of $D_g$.
THE NATURE OF DURATION.

Now we are going to prove that duration is in fact an approximation to another variable, which we call $s$, and which can be regarded as an alternative to duration in immunizing strategies. Intuitively, the idea behind this new variable is the following.

Let's suppose that just after $t = 0$ interest rates change from $h(O, t)$ to $h'(O, t)$. Now, let $V_t$ be the portfolio value at time $t$ calculated at the initial term structure of interest rates, i.e., $h(O, t)$, and $V'_t$ be the portfolio value at time $t$ calculated at the new interest rates, i.e., $h'(O, t)$ (we also assume that there are no more future interest rates changes after $t = 0$). If that change consists of a general increase of the term structure of interest rates, then the portfolio present value will decrease from $V_0$ to $V'_0$ (see figure I), but from $t = 0$ onwards portfolio value will grow faster for the coupon and redemption payments will be reinvested at higher rates. So that, there will be a future instant $t = \tau$ at which portfolio value $V'_t$ will be equal to $V_t$, i.e., the value that the portfolio would have had if interest rates had not changed (see figure I).

Alternatively, if the interest rates change consists of a general decrease of the term structure of interest rates, the portfolio present value will increase from $V_0$ to $V'_0$ (see figure II), but its value will grow at a lower rate. So there will be an instant $t = \tau$ at which the portfolio will have the same value that it would have had if that change in interest rates had not taken place (see figure II).

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**FIGURE I:** Value of $\tau$

**FIGURE II:** Value of $\tau$

General increase of interest rates

General decrease of interest rates

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For example, Macaulay’s duration can be obtained introducing the assumption $r_1 = r_2 = \ldots = r_n$. In that case,

$$D_q = \frac{\sum_{i=1}^{n} t_i \cdot c_i \cdot \exp[-r \cdot t_i, 1]}{\sum_{i=1}^{n} c_i \cdot \exp[-r \cdot t_i, 1]}$$

In the case of $s$,$$
D_q = \frac{\sum_{i=1}^{n} t_i \cdot c_i \cdot \exp[-r \cdot t_i, 1]}{\sum_{i=1}^{n} c_i \cdot \exp[-r \cdot t_i, 1]}$$
It is that instant after a change in interest rates at which portfolio values, $V_t$ and $V_t^*$, are equal what we call $T$.

More formally we define the $\tau$, value(s) of portfolio corresponding to a given change of the term structure of interest rates as:

$$\tau = \frac{1}{\Delta r_v} \ln \frac{V_t}{V_t^*}$$

where $V_0$ is the present portfolio value at the current term structure of interest rates, $V_0^*$ is the present portfolio value after the interest rates change and $\Delta r_t = h'(O, \tau) - h(O, t)$, i.e. the change of interest rate corresponding to period $[0, \tau]$

Now we are going to prove that $D^2g$ is, as we have claimed, a linear approximation to $T$.

Let $F: \mathbb{R}^n \rightarrow \mathbb{R}$ be a real function of $n$ variables defined as

$$F(r_1, ..., r_n) = \ln(V_0) = \ln \left( \sum_{i=1}^{n} c_i \exp(-r_i \cdot t_i) \right)$$

where the independent variables are $r_1, r_2, ..., r_n$. $F(r_1, ..., r_n)$ is then an infinitely differentiable function.

We can expand $F$ in a neighbourhood of $(r_1, r_2, ..., r_n)$ using again Taylor's formula. Thus:

$$F(r_1, ..., r_n) = F(r_1, ..., r_n) + \nabla F(r_1, ..., r_n) \cdot (\Delta r_1, ..., \Delta r_n) + \frac{1}{2} (\Delta r_1, ..., \Delta r_n) : H_F (\Delta r_1, ..., \Delta r_n)$$

where $AF(r_1, ..., r_n)$ is the gradient vector of $F$ at $(r_1, ..., r_n)$ and $H_F$ is the hessian matrix at point $(\theta_1, ..., \theta_n)$ in the linear segment $L(t^*, r^*)$.

Substituting and dropping the last term of the right hand side of [7] can rewrite it as:

$$\ln(V_t^*) - \ln(V_0) = \nabla F(r_1, ..., r_n) \cdot (\Delta r_1, ..., \Delta r_n)$$

and applying the chain rule,

$$\frac{1}{V_0} \cdot \nabla V_0 \cdot (\Delta r_1, ..., \Delta r_n) \cdot \frac{1}{V_0} \cdot \nabla V_0 \cdot (\Delta r_1, ..., \Delta r_n) = \frac{\sum_{i=1}^{n} t_i \cdot c_i \exp(-r_i \cdot t_i)}{V_0} = -D^2g \cdot \Delta r_v$$

We obtain (6) from the fact that $t \tau$, as we have said, the instant at which $V_t = V_t^*$, i.e.

$$V_t = V_0 \cdot \exp[r_t \cdot \tau] = V_0^* \cdot \exp[r_t^* \cdot \tau] = V_0^*$$
and finally we get:

\[ D_g = \frac{1}{\Delta r} \ln \left( \frac{V_1}{V_0} \right) \]

This feature of duration, summarised by [8], may help to understand its role in immunization strategies; if \( \tau \) is the period of time at the end of which a portfolio value remains unchanged after an interest rates shift then, duration, as an approximation to it, will inherit this characteristic. So if \( D_g = \tau \), the portfolio value at \( t_p \) will remain approximately unchanged despite the interest rates change, i.e. the portfolio is immunized against that change.

Let's analyse now some other characteristics of \( \tau \) and its relationships with \( D_g \).

First, we would like to point out the fact that the value of \( \tau \) is much easier to compute than \( D_g \), involving only the portfolio present values before and after the interest rates change and the increment in the interest rate corresponding to a period of length \( \tau \). On the contrary, the value of \( D_g \), as we have seen before, depends on all interest rate changes.

However, under the assumption of parallel shifts of the term structure of interest rates the values of \( \tau \) depend on the sign and the size of that change whereas duration does not depend on them (see table I).

Another interesting point arising from [6] is that it is an equation that has not necessarily one single solution, although there are some important cases where the value of \( \tau \) is unique; when all interest rates move simultaneously upwards or downwards. Moreover we can see that if the payments stream generated by the portfolio consists of an unique payment then \( D_g = \tau = t_n \). In any case, when the shift in interest rates consist of a "twist" of the term structure of interest rates with some interest rates moving upwards and some others moving downwards, the value of generalized duration is not unique either.

With respect to the goodness of \( D_g \) as an approximation of \( \tau \), it must be pointed out that the difference between \( \tau \) and \( D_g \) can be significant if the extent of interest rate changes or the length of the planning period are big. We can see in Table I the values of \( D_g \) and \( \tau \) corresponding to a portfolio consisting of bond with half-yearly coupon payments under the assumption of a flat term structure of interest rates. As we said before the values of \( \tau \) depend on the size and sign of the interest rate change whereas the value of \( D_g \) is unique.

From table I we can also see that differences between duration and \( \tau \) increase as time to maturity and the interest rate changes (in absolute value) do, although it is not greater than 0.7 years in any of the considered case (see also figures III, IV and V).

Another point that can be obtained from [7] is that portfolio value at \( t = D_g \) is greater if interest rates change than if they don't, i.e.

\[ \ln (V_t) - \ln (V_0) \leq \ln (V_t) - \ln (V_0) \]

This can be easily proved. Rearranging terms we can see that [9] is equivalent to:

\[ \ln (V_0) - \ln (V_t) \leq D_g \]

and this is true for \( D_g \) is a positive semidefinite matrix.

In fact, portfolio present values can be obtained from each portfolio asset price before and after interest rate change, an information which is very easy to obtain; on the contrary, to obtain accurate estimation of the term structure of interest rates may be a very hard task.

We can use Macaulay's formula if we assume that the term structure of interest rates is flat or Fisher-Weil's formula otherwise.
### Table I

**Duration: Its Role in Immunization**

<table>
<thead>
<tr>
<th>Annual Interest Rate</th>
<th>Annual Coupon</th>
<th>Term to Interest Rate Changes</th>
<th>Interest Rate Changes</th>
<th>Maturity</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>8%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Annual Interest Rate</strong></td>
<td><strong>Annual Coupon</strong></td>
<td><strong>Term to Interest Rate Changes</strong></td>
<td><strong>Interest Rate Changes</strong></td>
<td><strong>Maturity</strong></td>
<td><strong>Duration</strong></td>
</tr>
<tr>
<td>0.5%</td>
<td>0.500000</td>
<td>0.500000</td>
<td>0.500000</td>
<td>(0.001)</td>
<td>(0.005)</td>
</tr>
<tr>
<td>1.0%</td>
<td>0.981065</td>
<td>0.981000</td>
<td>0.981000</td>
<td>(0.001)</td>
<td>(0.005)</td>
</tr>
<tr>
<td>3.0%</td>
<td>2.740513</td>
<td>2.740000</td>
<td>2.740000</td>
<td>(0.001)</td>
<td>(0.005)</td>
</tr>
<tr>
<td>5.0%</td>
<td>4.269144</td>
<td>4.260500</td>
<td>4.260500</td>
<td>(0.001)</td>
<td>(0.005)</td>
</tr>
<tr>
<td>10.0%</td>
<td>7.450354</td>
<td>7.440000</td>
<td>7.440000</td>
<td>(0.001)</td>
<td>(0.005)</td>
</tr>
<tr>
<td>20.0%</td>
<td>11.96794</td>
<td>11.95000</td>
<td>11.95000</td>
<td>(0.001)</td>
<td>(0.005)</td>
</tr>
<tr>
<td>30.0%</td>
<td>15.62972</td>
<td>15.60000</td>
<td>15.60000</td>
<td>(0.001)</td>
<td>(0.005)</td>
</tr>
<tr>
<td>50.0%</td>
<td>18.91061</td>
<td>18.88000</td>
<td>18.88000</td>
<td>(0.001)</td>
<td>(0.005)</td>
</tr>
</tbody>
</table>

**Annual Interest Rate** | **Annual Coupon** | **Term to Interest Rate Changes** | **Interest Rate Changes** | **Maturity** | **Duration** |
| 8%                   | 8%            |                              |                       |          |          |
| **Annual Interest Rate** | **Annual Coupon** | **Term to Interest Rate Changes** | **Interest Rate Changes** | **Maturity** | **Duration** |
| 0.5%                 | 0.500000      | 0.500000                     | 0.500000              | (0.001)  | (0.005)  |
| 1.0%                 | 0.980825      | 0.980780                     | 0.980780              | (0.001)  | (0.005)  |
| 3.0%                 | 2.728567      | 2.726000                     | 2.726000              | (0.001)  | (0.005)  |
| 5.0%                 | 4.219457      | 4.210531                     | 4.210531              | (0.001)  | (0.005)  |
| 10.0%                | 7.137544      | 7.120000                     | 7.120000              | (0.001)  | (0.005)  |
| 20.0%                | 10.58913      | 10.47500                     | 10.47500              | (0.001)  | (0.005)  |
| 30.0%                | 12.31314      | 12.10000                     | 12.10000              | (0.001)  | (0.005)  |
| 50.0%                | 13.63271      | 13.20890                     | 13.20890              | (0.001)  | (0.005)  |

**Annual Interest Rate** | **Annual Coupon** | **Term to Interest Rate Changes** | **Interest Rate Changes** | **Maturity** | **Duration** |
| 12%                  | 12%           |                              |                       |          |          |
| **Annual Interest Rate** | **Annual Coupon** | **Term to Interest Rate Changes** | **Interest Rate Changes** | **Maturity** | **Duration** |
| 0.5%                 | 0.500000      | 0.500000                     | 0.500000              | (0.001)  | (0.005)  |
| 1.0%                 | 0.980486      | 0.980444                     | 0.980444              | (0.001)  | (0.005)  |
| 3.0%                 | 2.712348      | 2.710500                     | 2.710500              | (0.001)  | (0.005)  |
| 5.0%                 | 4.135674      | 4.130396                     | 4.130396              | (0.001)  | (0.005)  |
| 10.0%                | 6.703048      | 6.675400                     | 6.675400              | (0.001)  | (0.005)  |
| 20.0%                | 8.872355      | 8.771727                     | 8.771727              | (0.001)  | (0.005)  |
| 30.0%                | 9.394758      | 9.241209                     | 9.241209              | (0.001)  | (0.005)  |
| 50.0%                | 9.473219      | 9.285685                     | 9.285685              | (0.001)  | (0.005)  |
This point has some interesting implications: if it happens a general increase of the term structure of interest rates then $Dg \geq t$ and, inversely, if it happens a general decrease of interest rates then $Dg \leq t$. This point can be seen at tables I and figures III, IV and V.

**Figure III**
Duration, $\tau$ and Time to Maturity
Interest Rate = 5%; Coupon Rate = 6%

**Figure IV**
Duration, $\tau$ and Time to Maturity
Interest Rate = 8.8%; Coupon Rate = 8.8%
CONCLUSIONS

There are many results arising from this paper that we would like to point out. First, the existing relationship between duration and $\tau$ which, in our opinion throws light on the real role played by duration in immunization techniques. It has been shown that duration is in fact, an approximation to the period of time that after a change in interest rates a portfolio takes to have the same value that the portfolio would have had if that change in interest rates would not have taken place. We have also seen that $\tau$ values and duration are very close if the length of the payments stream generated by the portfolio is not too big (less than ten years) or if the change in interest rates is less than seventy basic points.

Second, we have, on one hand, that $\tau$ is easier to compute than duration for to calculate $\tau$ only involves the portfolio present values before and after the interest rate change, and the change in the interest rate corresponding to a period of length $\tau$, i.e. the planning period if we follow a immunizing strategy. On the other hand, computing the "correct" duration formula implies to know the level of interest rates for all periods and all its corresponding variations, a task really cumbersome.

These two points make $\tau$ an interesting alternative to duration in immunizing strategies. However, under the assumption of parallel shifts in the term structure of interest rates, duration values do not depend on the sign and size of interest rates changes but, as we could see, it may differ significantly from the "correct" immunizing value which is given by the general duration formula and so making any immunizing strategy based on that duration formulae wanting.
BIBLIOGRAPHY:


