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DURATION : ITS ROLE IN IMMUNIZATION

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LA DUREE : SON ROLE DANS L'IMMUNISATION

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RESUME

L'utilisation de la durée pour réaliser l'immunisation présente plusieurs difficultés résultant de la nécessité de faire des hypothèses très restrictives sur la forme et le comportement de la structure des termes des taux d'intérêt.

Le présent article pate sur ce problème et a un double objectif :

Premièrement, on élabore formule de durée qui est compatible avec n'importe quelle hypothèse sur la structure des termes des taux d'intérêt, mais qui implique une parfaite connaissance des mouvements futurs de ces taux d'intérêt. Cet handicap renvoie toutefois au problème réel posé par la réalisation de l'immunisation : la nécessité de prévoir l'évolution future des taux d'intérêt.

Deuxièmement, on prouve que la durée est, en fait, une approximation linéaire d'une autre variable que l'on appelle, plus facile à calculer que la durée, en absence de toute hypothèse sur la structure des termes des taux d'intérêt. La relation entre la durée et peut, en même temps, aider à comprendre le rôle joué par la durée dans les stratégies d'immunisation.

Enfin, on étudie rapidement les relations entre la durée et, en précisant les avantages et les inconvénients relatifs de ces deux variables, dans le contexte d'une stratégie d'immunisation.

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SUMMARY

The use of duration to achieve immunization presents several problems derived from the need of making very retrictive assumptions about the shape and behaviour of the term structure of interest rates.

Closely related with that problem, the aim of this paper is twofold. First, we develope a duration formula which is compatible with any assumption about the term structure of interest rates although it implies a perfect knowledge of future movements of interest rates. This handicap, however, points out the real problem involved in the achievement of immunization : the need of forecasting future movements of interest rates.

Second, we prove that duration is, in fact, a linear approximation to another variable, that we call T, which is much easier to compute than duration under the absence of any assumption about the term structure of interest rates. At the same time, this relationship between duration and T, may help to understand the role played by duration in immunizing strategies.

Finally, a brief study of the relationships between duration and T, is made, pointing out the advantages and desadvantages of each variable with respect to the other in the context of a immunizing strategy.

DURATION : ITS ROLE IN IMMUNIZATION.

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GENERAL DURATION FORMULA[1]

Let's assume an investor who buys at time t = o a fixed-income portfolio that generates the folowing payments stream:

	c_1	c_2		c_p		c_n	AMOUNTS
0	t_1	t_2	. . .	t_p	. . .	t_n	TIME

The current value of that portfolio can be calculated from the term structure of interest rates that is described by the function h (O,t) which takes the following values:

$$h(0,t) = \begin{cases} r_1 & \text{at } t=t_1 \\ r_2 & \text{at } t=t_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \\ r_p & \text{at } t=t_p \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \\ r_n & \text{at } t=t_n \end{cases}$$

where h (O, t) is the current rate of growth of an investment which is mantained for a period of length t, i.e. an investment of one money unit now, will grow to an amount exp [h (O ,t). t] at time t[2]

Under those assumptions, the present discounted value of a portfolio as the one described above, is:

$$V_0 = \sum_{i=1}^{n} c_i \cdot \exp[-r_i \cdot t_i] \qquad [1]$$

where V_0 can be regarded as a real function of n variables - $r_1, r_2, \ldots, r_n$ - which is infinitely differentiable.

1 A discussion about the assumptions and implications of this preliminaries can be found in MENU. V and NAVARRO. E. : ¿ Es posible la inmunizacion finaciera ? Expresion general de la duration inmunizadora. Transactions of the 23rd International Congress af Actuarides. Vol. 5 Helsinki, Finland July 1.988.

2 This magnitude is called 'logarithmic density" for its relation with the normal interest rate is
$h(0,t) = \ln[(1+i_t)^t]/t = \ln(1+i_t)$
where i_t is the mean interest rate for the period [O,t] . In any case we will refer to h(O,t) as the "interest rate."

Additionaly we assume that the investor has a planning period of length t_p and that he wishes to buy a portfolio in order to abtain, at least, the return that the current term structure of interest rates offers for the period $[0, t_p]$ i. e. $h(0, t_p)$.

The portfolio value at time t_p (if interest rates do not change during the period $[0, t_p]$) would be :

$$V_p = V_0 . \exp[r_p . t_p] = \sum_{i=1}^{n} c_i . \exp[-r_i . t_i + r_p . t_p] \qquad [2]$$

As before, V_p can be viewed as a real function of n variables - $r_1, r_2, \ldots, r_n$ - infinitely differentiable.

Now we are going to analyse how a change in interest rates affects the portfolio final value, i. e. the portfolio value at t_p. So that, we assume that after $t = 0$ there is a shift in the term structure of interest rates such that its new values are :

$$h^*(0,t) = \begin{cases} r_1^* & \text{at } t = t_1 \\ r_2^* & \text{at } t = t_2 \\ \cdots & \\ r_p^* & \text{at } t = t_p \\ \cdots & \\ r_n^* & \text{at } t = t_n \end{cases}$$

The new portfolio present value, after that change in interest rates is :

$$V_0^* = \sum_{i=1}^{n} c_i . \exp[-r_i^* . t_i]$$

and if there were no more changes in the term structure during the period $(0, t_p)$ the portfolio value at t_p would be :

$$V_p^* = V_0^* . \exp[r_p^* . t_p] = \sum_{i=1}^{n} c_i . \exp[-r_i^* . t_i + r_p^* . t_p] \qquad [3]$$

In order to analyse the change in V_p due to a given change in interest rates we will expand the function V_p at $(r_1, r_2, \ldots, r_n)$ using Taylor's formula Then we have :

$$V_p^* - V_p = \nabla V_p . (\Delta r_1, \Delta r_2, \ldots, \Delta r_n)' +$$

$$+ 1/2\ (\Delta r_1, \Delta r_2, \ldots, \Delta r_n) . H_p . (\Delta r_1, \Delta r_2, \ldots, \Delta r_n)'$$

where ∇V_p is the gradient vector of V_p at $(r_1, r_2, \ldots, r_n)$ and H_p

is the hessaian matrix of V_p at a point $(\theta_1, \theta_2, \dots, \theta_n)$ in the linear segment $L(r, r^*)$.

It is not difficult to prove that Hp is a positive semidefinite matrix so that

$$V_p^* \geq \nabla V_p . (\Delta r_1, \Delta r_2, \dots, \Delta r_n)' + V_p \qquad [4]$$

An immunizing strategy would consist, thus, in choosing a portfolio such that $\nabla V_p . (\Delta r_1, \Delta r_2, \dots, \Delta r_n)' = 0$. If that condition held we would be able to assure that if any change in interest rates hapens just after t = 0 , the portfolio value at $t = t_p$ would be at least V_p.

If we develope (4), we obtain :

$$\nabla V_p . (\Delta r_1, \Delta r_2, \dots, \Delta r_n)' = \sum_{i=1}^{p-1} -t_i . c_i . \exp[-r_i . t_i + r_p . t_p] . \Delta r_i$$

$$+ t_p \sum_{i=1}^{p-1} c_i . \exp[-r_i . t_i + r_p . t_p] \Delta r_p + t_p \sum_{i=p+1}^{n} c_i . \exp[-r_i . t_i + r_p . t_p] \Delta r_p +$$

$$+ \sum_{i=p+1}^{n} -t_i . c_i . \exp[-r_i . t_i + r_p . t_p] \Delta r_i$$

Rearranging terms and making it equal to zero :

$$\sum_{i=1}^{n} -t_i . c_i . \exp[-r_i . t_i + r_p t_p] \Delta r_i + t_p \Delta r_p \sum_{i=1}^{n} c_i . \exp[-r_i . t_i + r_p . t_p] =$$

and if $\Delta r_p \neq 0$

$$t_p = \frac{\sum_{i=1}^{n} t_i . c_i . \exp[-r_i . t_i] \quad \Delta r_i}{\Delta r_p . \sum_{i=1}^{n} c_i . \exp[-r_i . t_i]} \qquad [5]$$

Condition (5) is then a necessary and suficient condition for immunizing a portfolio that generates a payments stream as initially described.

Then if we called the right hand side of (5) generalized duration - D_g - we have that an investor can be immunized against a given change in interest rates by building a porfolio with a generalized duration equal to its planning period, i. e. $D_g = t_p$.

The great advantage of D_g over other duration formulae is that D_g is compatible with any assumption about the future behaviour of the term strucuture of interest rates ; in fact all other alternative duration formulae can be obtained from D_g if the hypothesis made to obtain all those alternative formulae are applied in D_g[3].

But unfortunately, the applicability of this duration formula depends on the investor ability to forecast, as precisely as possible, future movements of interest rates, for we need to know the vazlues of r_1 (far all i) in order to compute the value of D_g

THE NATURE OF DURATION.

Now we are going to prove that duration is in fact an approximation to another variable. that we call s, wich can be regarded as an alternative to duration in immunizing strategies. Intuitively, the idea behind this new variable is the following.

Let's suppose that just after t = O interest rates change from h (O, t) to h*(O, t). Now, let V_t be the portofolio value at time t calculated at the initial term structure of interest rates, i. e. h (O, t), and V_t^* be the portofolio value at time t calculated at the new interest rates, i. e. h* (O, t) (we also assume that there are no more future interest rates changes after t = O). If that change consists of a general increase of the term structure of interest rates, then the portfolio present value will decrrease from V_0 to V_0^* (see figure I), but from t = O onwards portfolio value will grow faster for the coupon and redemption payments will be reinvested at higher rates. So that, there will be a future instant t = τ at which portfolio value V_t^* will be equal t V_t i.e. the value that the portofolio would have had if interest rates had not change (see figure I).

Alternatively, if the interest rates change consists of a general decrease of the term structure of interest rates, the portfolio present value will increase from V_0 to V_0^* (see figure II), but its value will grow at a lower rate. So there will be an instant t = τ at which the portfolio will have the same value that it would have had if that change in interest rates had not taken place (see figure II).

FIGURE. I: Value of τ[a]
General increase of interest rates

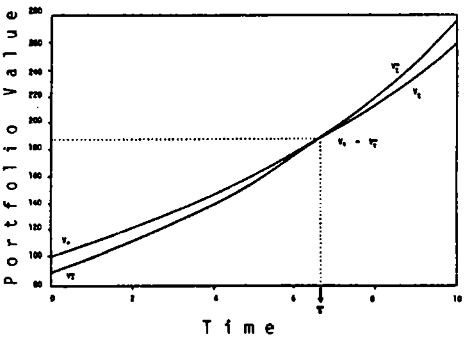
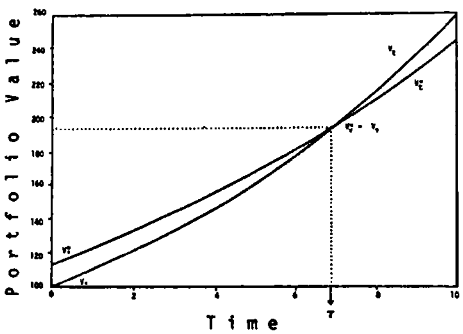


FIGURE II: Value of τ
General decrease of interest rates



[a] For example, Macaulay's duration can be obtained introducing the assumption $r_1 = r_2 = \ldots = r_n =$ r and $r_1 = r_2 = \ldots = r_n = r$. In that case,

$$D'_g = \frac{\sum_{i=1}^{n} t_i . c_i . \exp[-r.t_i]. \, Ar}{\sum_{i=1}^{n} c_i . \exp[-r.t_i]. \, Ar} = \frac{\sum_{i=1}^{n} t_i . c_i . \exp[-r.t_i]}{\sum_{i=1}^{n} t_i . \exp[-r.t_i]}$$

It is that instant - after a change in interest rates - at which portfolio values, V_t and V_t^*, are equal what we call T.

Mare formally we define the τ, value (s) of portfolio corresponding to a given change of the term structure of interest rates as[4] :

$$\tau = \frac{1}{\Delta r_\tau} \ln \frac{V_0}{V_0^*} \qquad [6]$$

where V_0 is the present portfolio value at the current term structure of interest rates,

V_0^* is the present portofolio value after the interest rates change and

$\Delta r_t = h^*(0,\tau) - h(0,t)$, i. e. the change of interest rate corresponding to period $[0,\tau]$

Now we are going to prove that Dg is, as we have claimed, a linear approximation to T.

Let $F : R^n \longrightarrow R$ be a real function of n variables defined as

$$F(r_1,\ldots,r_n) = \ln(V_0) = \ln\left(\sum_{i=1}^{n} c_i . \exp[-r_i . t_i]\right)$$

where the independent variables are $r_1, r_2, \ldots, r_n$. $F(r_1, \ldots, r_n)$ is then a infinitely differentiable function.

We can expand F in a neighbourhood of $(r_1, r_2, \ldots, r_n)$ using again Taylor's formula. Thus :

$$F(r_1^*, \ldots, r_n^*) = F(r_1, \ldots, r_n) + \nabla F(r_1,\ldots,r_n).(\Delta r_1,\ldots,\Delta r_n)' +$$

$$+ 1/2\,(\Delta r_1,\ldots,\Delta r_n).H_F.(\Delta r_1,\ldots,\Delta r_n)' \qquad [7]$$

where $\nabla F(r_1, \ldots, r_n)$ is the gradient vector of F at $(r_1, \ldots, r_2)$ and H_f its hessain matrix at point $(\theta_1, \ldots, \theta_n)$ in the linear segment $L(r,r^*)$.

Substituying and dropping the last term of the right hand side of [7] can rewrite it as :

$$\ln(V_0^*) - \ln(V_0) \approx \nabla F(r_1,\ldots,r_n).(\Delta r_1,\ldots,\Delta r_n)'$$

and appliying the chain rule,

$$\ln\left(\frac{V_0^*}{V_0}\right) \approx \frac{1}{V_0} \nabla V_0(r_1,\ldots,r_n).(\Delta r_1,\ldots\Delta r_n)' =$$

$$= -\frac{\sum_{i=1}^{n} t_i . c_i . \exp[-r_i . t_i] \;.\; r_i}{V_0} = -D_g \; \Delta r_\tau$$

We obtain (6) from the fact that t i τ, as we have said, the instant(s) at which $V_\tau = V_\tau^*$, i.e. $V_\tau = V_0 . \exp[r_\tau . \tau] = V_0^* . \exp[r^*_\tau . \tau] = V_0^*$

and finally we get:

$$D_g = \frac{1}{\Delta r_0} \ln [V_0/V_0^*] \qquad [8]$$

This feature of duration, summarised by [8], may help to understand its role in immunization strategies; if τ is the period of time at the end of which a portfolio value remains unchanged after a interest rates shift then, duration, as an approximation to it, will inherit this characteristic. So if $D_g = t_p$ the portfolio value at t_p will remain approximately unchanged despite the interest rates change, i. e. the portfolio is immunized against that change.

Let's analyse now some other characteristique of τ and its relationsips with D_g.

First, we would like to point oout the fact that the value of τ is much easier to compute than D_g, involving only the portfolio present values before and after the interest rates change and the increment in the interest rate corresponding to a period of length τ. On the contrary, the value of D_g, as we have seen before, depends on all interest rate changes[5].

However, under the assumption of parallel shifts of the term structure of interest rates the values of τ depend on the sign and the size of that change whereas duration[6] does not depend on them (see table I)

Another interesting point arising from [6] is that it is an equation that has not necessarily one single solution, although there are some important cases where the value of τ is unique: when all interest rates move simultaneously upwards or downwards. Moreover we can see that if the payments streamgenerated by the portfolio consists of an unique payment then $D_g = \tau = t_n$. In any case, when the shift in interest rates consist of a "twist" of the term structure of interest rates with some interest rates moving upwards and some others moving downwards. the value of generalized duration is not unique either.

With respect to the goodness of D_g as an approximation of τ, it must be pointed out that the difference between τ and D_g can be significant if the extent of interest rate changes or the length of the planning period are big. We can see in Table I the values of D_g an τ corresponding to a portfolio consisting of bond with half-yearly coupon payments under the assumption of a flat term structure of interest rates. As we said before the values of τ depend on the size and sign of the interes rate change whereas the value of Dg is unique.

From table I we can also see that differences between duration and τ increase as time to maturity and the interes rate changes (in absolute value) do, although it is not greater than 0.7 years in any of the considered case (see also figures III, IV and V).

Another point that can be obtained From [7] is that portofolio value at $t = D_g$ is greater if interest rates change than if they don't, i. e.

$$V_0^g = V_0 \cdot \exp[D_g \cdot r_0^g] \leq V_0^* \cdot \exp[D_g \cdot r_0^{*g}] = V_0^{*g} \qquad [9]$$

This can be easily proved. Rearranging terms we can see that [9] is equivalent to:

$$\ln(V_0) - \ln(V_0^*) \leq D_g \, \Delta r_0$$

and this is true for H_f is a positive semidefinite matrix.

fact, portfolio present values can be obtained from each portfolio asset price before and after interest rate change, an information which is very easy to obtain; on the contrary, to obtain an accurate estimation of the term structure of interest rates may be a very hard task.

W e can use Macaulay's formula if we assume that the term structure of interest rates is flat or Fisher-Weil's formula otherwise.

TABLA I

DURATION AND VALUES OF τ FOR DIFFERENT INTEREST RATE CHANGES FOR A BOND WITH HALF-YEARLY COUPON PAYMENTS

ANNUAL INTEREST RATE = 5 %; ANNUAL COUPON = 8 %

TERM TO	τ VALUES				τ VALUES		
	INTEREST RATE CHANGES				INTEREST RATE CHANGES		
MATURITY	(-0,01)	(-0,005)	(-0,001)	DURATION	(0,001)	(0,005)	(0,01)
0,5	0,500000	0,500000	0,500000	0,500000	0,500000	0,500000	0,500000
1,0	0,981085	0,981063	0,981045	0,981042	0,981037	0,981019	0,980998
3,0	2,740513	2,739523	2,738731	2,738534	2,738335	2,737543	2,736552
5,0	4,281941	4,277633	4,274179	4,273315	4,272450	4,268987	4,264648
10,0	7,450354	7,424830	7,404335	7,399202	7,394064	7,373473	7,347649
20,0	11,96794	11,85184	11,75907	11,73591	11,71274	11,62023	11,50491
30,0	15,06972	14,82411	14,62981	14,58158	14,53346	14,34245	14,10704
50,0	18,91061	18,37745	17,96620	17,86554	17,76573	17,37516	16,90636

ANNUAL INTEREST RATE = 8 %; ANNUAL COUPON = 8 %

TERM TO	τ VALUES				τ VALUES		
	INTEREST RATE CHANGES				INTEREST RATE CHANGES		
MATURITY	(-0,01)	(-0,005)	(-0,001)	DURATION	(0,001)	(0,005)	(0,01)
0,5	0,500000	0,500000	0,500000	0,500000	0,500000	0,500000	0,500000
1,0	0,980825	0,980804	0,980787	0,980783	0,980778	0,980761	0,980740
3,0	2,728567	2,727561	2,726756	2,726555	2,726354	2,725549	2,724542
5,0	4,229457	4,224998	4,221425	4,220531	4,219636	4,216053	4,211567
10,0	7,137544	7,110865	7,089475	7,084122	7,078765	7,057316	7,030456
20,0	10,58913	10,47597	10,38606	10,36369	10,3413)	10,25235	10,14205
30,0	12,31314	12,10123	11,93528	11,89430	11,85352	11,69244	11,49511
50,0	13,63271	13,28090	13,01281	12,94759	12,88306	12,63186	12,33282

ANNUAL INTEREST RATE = 12 %; ANNUAL COUPON = 8 %

TERM TO	τ VALUES				τ VALUES		
	INTEREST RATE CHANGES				INTEREST RATE CHANGES		
MATURITY	(-0,01)	(-0,005)	(-0,001)	DURATION	(0,001)	(0,005)	(0,01)
0,5	0,500000	0,500000	0,500000	0,500000	0,500000	0,500000	0,5
1,0	0,980486	0,980465	0,980448	0,980444	0,980439	0,980423	0,980402
3,0	2,712348	2,711322	2,710501	2,710297	2,710091	2,709270	2,708244
5,0	4,156764	4,152118	4,148396	4,147465	4,146533	4,142804	4,138135
10,0	6,703048	6,675404	6,653282	6,647751	6,642220	6,620095	6,592442
20,0	8,872355	8,771727	8,692298	8,672593	8,652948	8,574982	8,47892
30,0	9,394758	9,241209	9,121917	9,092580	9,063434	8,948760	8,809634
50,0	9,473219	9,286585	9,144243	9,109579	9,075274	8,941534	8,781827

This point has some interesting implications : if it happens a general increase of the term strucuture of interest rates then Dg ≥ τ and, inversely, if it happens a general decrease of interest rates then Dg ≤ τ . this point can be seen at tables I and figures III, IV and V.

FIGURE III
DURATION; τ AND TIME TO MATURITY
Interest Rate = 5%; Coupon Rate = 8 %

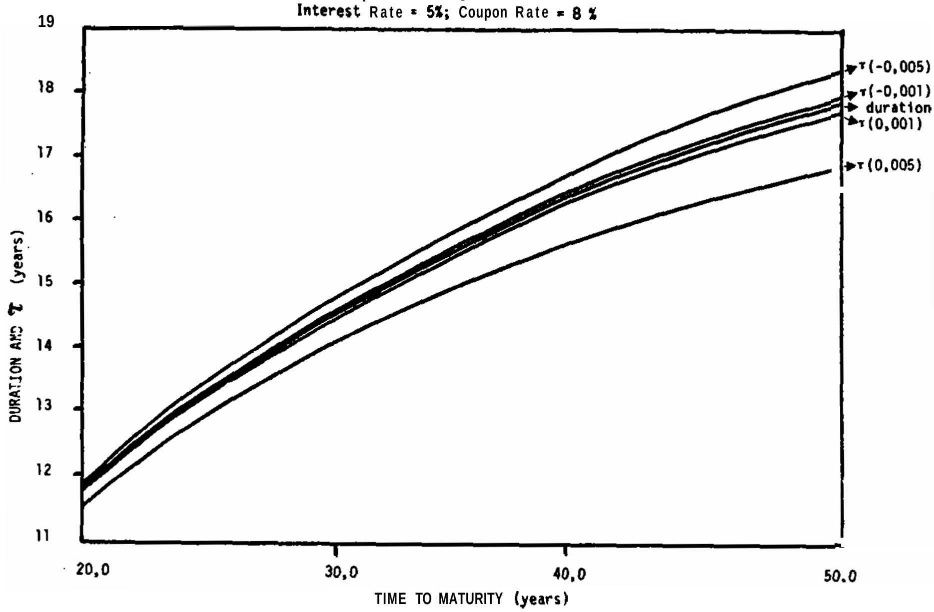


FIGURE IV
DURATION; τ AND TIME To MATURITY
Interest Rate = 8 8; Coupon Rate = 8 8

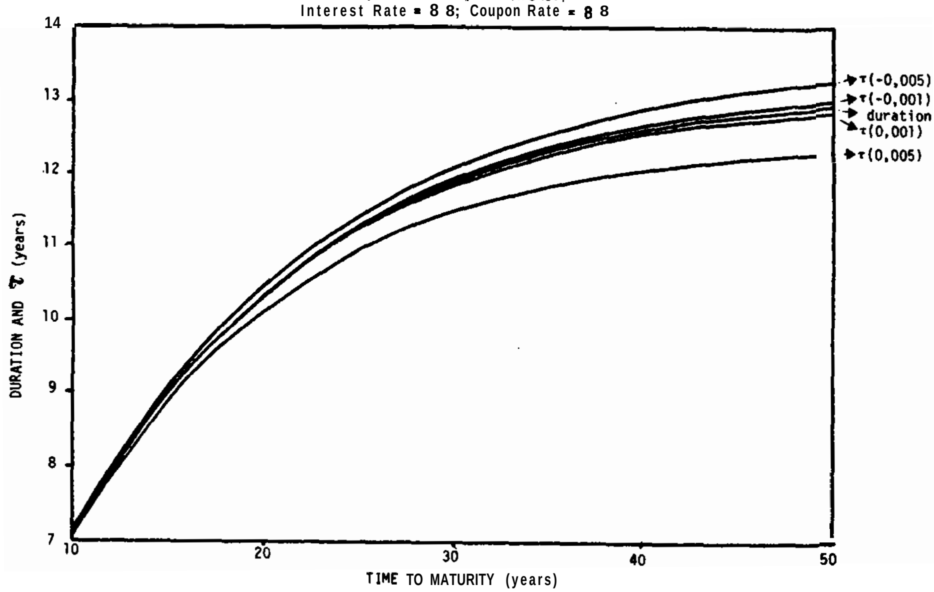
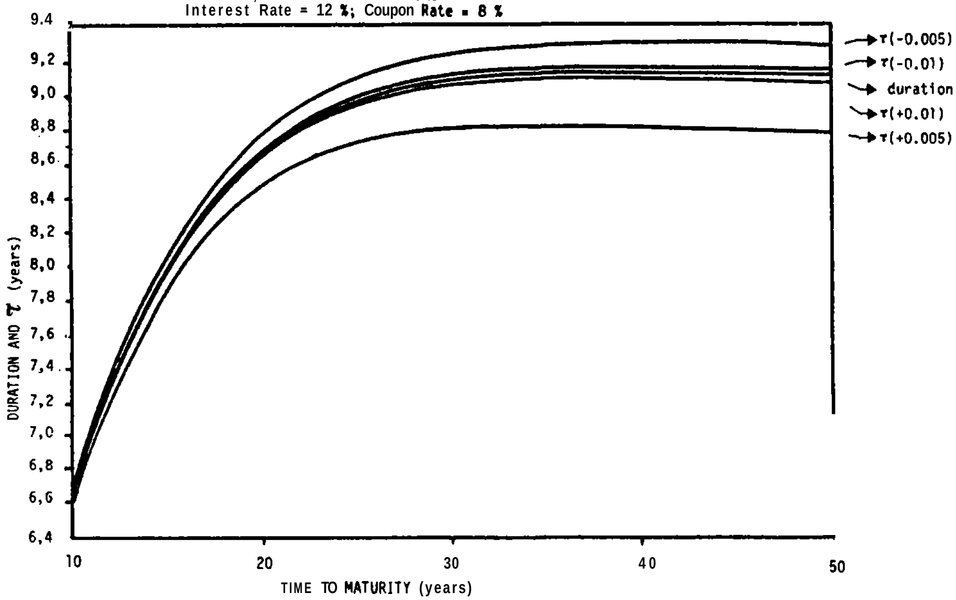


FIGURE V
DURATION; τ AND TIME TO MATURITY
Interest Rate = 12 %; Coupon Rate = 8 %



CONCLUSIONS

There are many results arising from this paper that we would like to point out. First, the existing relationship between duration and τ which, in our opinion throws light on the real role played by duration in immunization techniques. It has been shown that duration is in, fact, an approximation to the period of time that - after a change in interest rates - a portfolio takes to have the same value that the portfolio would have had if that change in interest rates would not have taken place. We have also seen that τ values and duration are very close if the length of the payments stream generated by the portfolio is not too big (less than ten years) or if the change in interest rates is less than seventy basic points.

Second, we have, on one hand, that τ is easier to compute than duration for to calculate τ only involves the portfolio present values before and after the interest rate change, and the change in the interest rate corresponding to a period of length τ, i.e. the planning period if we follow a immunizing strategy. On the other hand, computing the "correct" duration formula implies to know the level of interest rates for all periods and all its corresponding variations, a task really cumbersome.

These two points make τ an interesting alternative to duration in immunizing strategies. However, under the assumption of parallel shifts in the term structure of interest rates, duration values do not depend on the sign and size of interest rates changes but, as we could see, it may differ significantly from the "correct" immunizing value which is given by the general duration formula and so making any immunizing strategy based on that duration formulae wanting.

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