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ON A RISK PROCESS WITH AUTOREGRESSIVE INTEREST RETURN

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RESUME

Le calcul de la probabilité de ruine d'une entreprise à risque est un thème très ancien de la théorie du risque mathématique. Des formules de cette probabilité ont été dérivées d'un grand nombre de modèles fondés sur des hypothèses différentes. En affinant les hypothèses de base, on obtient des modèles plus réalistes, et donc des probabilités de ruine également plus réalistes. On inclut dans un nouveau type de modèle les taux d'intérêt importants. De premiers résultats ont d'abord été obtenus pour des taux d'intérêt déterministes, puis pour des taux de rentabilité financière pouvant être stochastiques. Dans cette contribution, on présente un modèle dans lequel les taux d'intérêt stochastiques suivent un modèle de série chronologique du premier ordre. Des résultats concernant certaines probabilités de ruine sont présentés et discutés.

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ABSTRACT:

A risk process with stochastic interest return is defined and discussed. The interest rates are assumed to follow an autoregressive time series model of order *one*. Certain results on ruin probabilities are given in the proposed model.

1. INTRODUCTION.

A classical part of risk theory is the socalled ruin theory. The scandinavian mathematician H. Cramér developed from 1930 up to 1955 a sound mathematical theory for the risk business of an insurance company, nowadays called ruin theory. In the past twenty years those classical results were generalized into many directions. For example the assumption of a Poisson process for the claims number was generalized to more general stochastic processes (see e.g. Anderson (1957)) and different model types were developed (see e.g. Gerber (1984), Janssen (1981)). New components were included into the models, for example inflationary conditions (see e.g. Taylor (1979)), development effects and interest incomes (see e.g. Harrison (1977)).

With these new components the models and consequently also the resulting ruin probabilities become more realistic. In accordance with classical insurance mathematics the inclusion of interest incomes is most important. In former models the interest return rates were taken to be deterministic, whereas in newer more modem models they are allowed to be of a stochastic nature. Concerning models with deterministic interest return rates the reader is referred to Gerber (1971), Emanuel et. al. (1975) and Harrison (1977). On ruin probabilities in models with stochastic discounting one has already two advanced publications, i.e., Schnieper (1983) and Braun (1986). Schnieper uses certain Markov - processes for modelling the interest return rates, Braun (a former student of me) takes in a fairly general setting methods of the theory of weak convergence of stochastic processes. Only Schnieper mentioned the possibility of taking special cases of the well - known ARMA - or ARIMA - models (see Box & Jenkins (1970)) for modelling the stochastic evolution of the interest return rates.

AFIR - people use those time series models since more than ten years for modelling the stochastic law of the return rates (see e.g. the survey article of Devolder (1987)). In the following the author shows that one can easily derive results on the ruin probability in case of risk processes with interest retum rates following an autoregressive time series model. The author restricts to the case of a model of order one. The possibility of generalization is mentioned.

2. THE RISK PROCESS MODEL.

Suppose in the sequel that everything is based on a probability space (Ω, A, P). Let us look at a risk business, described by the following three characteristics:

(1.) the deterministic premium income c per period.

(2.) the stochastic (total) claims amounts

X_i, i = 1, 2, 3,... in periods i = 1, 2, 3, ...

(3.) the stochastic interest rates D_i, i = 1, 2, 3,....in periods i = 1, 2, 3,...

In the present paper assume that:

(A.1) the random variables Xi, i = 1, 2, 3,... are independent and identically distributed with distribution function F on the positive reals,

(A.2) the stochastic processes (Xi, i = 1, 2, 3,...) and (D_i, i = 1, 2, 3,...) are independent,

(A.3) one has a stochastic process (e_i, i = 1, 2, 3,...) of independent, identically distributed random variables, each with (integrable) distribution function G, having mean zero, and two parameters a, b with $|a|<1$, b positive, such that

$$D_i = a \cdot D_{i-1} + (1-a) \cdot b + e_i$$

for all i = 1, 2, 3,...

The third condition means nothing else but that the stochastic process (D_i, i = 1, 2, 3,...) follows an <u>autoregressive time series model of order one</u>, like defined e.g. in the book of Box & Jenkins (1970). Such recursive models are used in many fields of stochastic applications since centuries, e.g. also in the theory of finance (see e.g. Devolder (1987)). With the definition:

$$C_i = (1+D_1)^{-1} \cdot (1+D_2)^{-1} \cdot \ldots \cdot (1+D_i)^{-1}$$

for i = 1, 2, 3, ..., one gets the coefficients for deflating the money units of year no. i (from the end of the year) down to the starting point i = 0 of the process. In practice one clearly needs that D_i is nonnegative with probability one. For having a realistic model one should in addition assume in (A.3) that the probability $P(\exists\, i : D_i < 0)$ is quite small.

With this notation one gets the <u>deflated loss of period no. n</u>:

$$V_n = \sum_{i=1}^{n} C_i \cdot (X_i - c)$$

and the <u>value of the reserve</u> at the time point n:

$$R_n = (u - V_n) \cdot (1+D_1) \cdot (1+D_2) \cdot \ldots \cdot (1+D_n) \quad ,$$

where u is the socalled <u>initial reserve</u> of the risk business. The stochastic process (R_n, n = 1, 2, 3,...) can be called <u>risk (reserve) process with stochastic interest return.</u> In the special situation that $D_1 = 1$, a = 1, Var (e_i) = 0 one gets as special case the risk (reserve) process without interest return.

3. RUIN PROBABILITIES.

For the defined risk (reserve) process one clearly can calculate certain quantities, e.g.

$$\psi_{m|d}(u) = P\left(\min_{p<k\leq p+m} (R_k) < 0 \;\middle|\; D_{p+1} = d,\ R_p = u \right)$$

the probability of a ruin in a period of length m after continuing the process in period $(p+1)$ with the initial value $R_p = u$ and initial interest rate $D_{p+1} = d$ That this probability is independent of the starting period no. p follows from the fact that the processes $(X_i, i = 1, 2, 3, \ldots)$ and $(D_i, i = 1, 2, 3, \ldots)$ are strongly stationary. With the probability $\Psi_{m|d}(u)$ one gets for the probability of a ruin in the first m periods.

$$\psi_m(u) = \int \psi_{m|t}(u)\, G(dt) \quad . \tag{3.1}$$

Finally the limits

$$\psi_{|d}(u) = \lim_{m\to\infty} \psi_{m|d}(u) \quad ,$$

$$\psi(u) = \lim_{m\to\infty} \psi_m(u)$$

are the probabilities of ultimate ruin in case of a given first interest rate $D_1 = d$ and in case of no given first interest rate. Since f a larger m the $\Psi_{m|d}(u)$, $\Psi_m(u)$ are approximations to $\Psi_d(u)$, $\Psi(u)$ and $\Psi_m(u)$ can be computed from $\Psi_{m|d}(u)$ according to (3.1), one basically likes to have formulas or procedures for computing the ruin probability $\Psi_{m|d}(u)$.

The desired result is given in the :

Theorem 1.

In addition to the above given conditions assume that the distribution function G has a density g with respect to another (simpler) distribution function H.
Then one has the recursion :

$$\psi_{m+1|d} = (1 - F((1+d)\cdot u+c)) + \\ + \int g(t-(d\cdot a+(1-a)\cdot b))\cdot \\ \int_0^{(1+d)\cdot u+c} \psi_{m|t}((1+d)\cdot u+c-y)\; F(dy)\; H(dt)$$

starting with $\Psi_{0|t}(u) = 0$ for all t and u.

"Proof"

One has *that* :

$$\psi_{m+1|d}(u) = P(X_{p+1} \geq (1+d)\cdot u+c) + \\ P\Big(\min_{p+1<k\leq p+m+1} (R_k)<0 \;\Big|\; D_{p+1}=d, R_p=u, X_{p+1}\leq (1+d)\cdot u+c\Big).$$

Obviously

$$P(X_{p+1} > (1+d)\cdot u+c) = 1 - F((1+d)\cdot u+c)$$

and

$$P(\min_{p+1<k\leq p+m+1}(R_k)<0 \mid D_{p+1}=d, R_p=u, X_{p+1}\leq(1+d)\cdot u+c) =$$

$$\int P(\min_{p+1<k\leq p+m+1}(R_k)<0 \mid D_{p+1}=d, D_{p+2}=t, R_u=u, X_{p+1}\leq(1+d)\cdot u+c)$$
$$P^{D_{p+2}|D_{p+1}=d}(dt)$$

hold true. The conditional distribution of D_{p+2}, given $D_{p+1}=d$ has the distribution function G_p defined according:

$$G_d(t) = G(t-(d\cdot a+b\cdot(1-a))$$

and consequently the density g_d with respect to H, defined by:

$$g_d(t) = g(t-(d\cdot a+b\cdot(1-a)))\,.$$

Finally one knows that:

$$P(\min_{p+1<k\leq p+m+1}(R_k)<0 \mid D_{p+1}=d, D_{p+2}=t, R_p=u, X_{p+1}\leq(1+d)\cdot u+c) =$$

$$\int_0^{(1+d)\cdot u+c} P(\min_{p+1<k\leq p+m+1}(R_k)<0 \mid D_{p+1}=d, D_{p+2}=t, R_p=u, X_{p+1}=y)\; F(dy)$$

Since:

$$\psi_{m|t}((1+d)\cdot u+c-y) =$$

$$= P(\min_{p+1<k\leq p+m+1}(R_k)<0 \mid D_{p+1}=d, D_{p+2}=t, R_p=u, X_{p+1}=y)$$

the statement of the theorem is proved. A

Remark.

In the applications the distribution function H defines the Lebesque measure or the counting measure on a countable subset of the real numbers. A

In addition to this basic result one can derive elegant bounds for the ruin probabilities $\Psi_m(u)$ and $\Psi(u)$. Concerning the first probability one gets the

Theorem 2.

Under the conditions of section 2 assume in addition that

$$E(X_i) \geq c\ ,\quad E(C_i) \geq 0\ .$$

For given m suppose that

$$E(V_m) < u\ ,\quad P(V_m > u) > 0\ .$$

and that for all positive r one has:

$$E(\exp(r \cdot \sum_{i=1}^{m} |C_i| \cdot |X_i - c|)) < \infty \qquad .$$

Let R_m be the unique positive solution of:

$$E(\exp(r.V_m))/dr = u \cdot E(\exp(r \cdot V_m))\ .$$

Then one has the inequality:

$$\psi_m(u) \leq \exp(-R_m \cdot u) \cdot E(\exp(R_m \cdot V_m))\ .$$

Proof.

Under the given conditions the stochastic process $(V_n,\ n = 1, 2, 3, \ldots)$ is a **submartingale.** Consequently for each nonnegative, **nondecreasing,** convex function f also **the sequence** $(f\ (V_n),\ = 1, 2, 3, \ldots)$ is a (positive) **submartingale.** One can apply the inequality of **Kolmogoroff** for positive **submartingales** meaning that:

$$P(\max_{i \leq m} f(V_i) \geq f(u)\ ) \leq \frac{E(f(V_m))}{f(u)}$$

Since f is **nondecreasing,** this implies:

$$P(\max_{i \geq m} V_i \geq u\ ) \leq \frac{E(f(V_m))}{f(u)}$$

The left hand side is just Ψ_m (u). Putting:

$$f(v) = \exp(r \cdot v)$$

one arrives at:

$$\psi_m(u) \leq E(\exp(r.V_m)) \cdot \exp(-r \cdot u)\ .$$

The minimum of the upper bound is taken on at $r = R_m$. A

One can prove with the same method of proof a similar **result** for **Ψ(u).** Concerning this *one* can refer to **Papatriadafylon** et. al. **(1984),** p. **220** - 222, since the setup there is **similar** to the present one.

Alternatively one can apply the socalled Martingale method (see Gerber (1980)), yielding another elegant bound for the ultimate ruin probability $\Psi(u)$. The derivation is identical with the procedure given in Gerber (1980), p. 132 - 133.

4. FINAL REMARKS.

One can generalize the autoregressive model of order one (see assumption (A.3)) to a model of higher order and try to proof for that situation a result similar to theorem 1. Concerning models of higher order the author refers to the book of Box & Jenkins (1970). The above defined risk (reserve) model with autoregressive interest return rate should not be mixed with that given in Gerber (1981). The author likes to remember that he used time series models already in credibility rating (see Kremer (1988)) and loss reserving (see Kremer (1984)).

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