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INTEREST RATES RISK IMMUNIZATION BY LINEAR PROGRAMMING

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IMMUNISATION CONTRE LE RISQUE DE TAUX D'INTERET PAR LA PROGRAMMATION LINEAIRE - UNE METHODE SANS RISQUE

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RESUME

Les auteurs proposent une méthode sans risque pour assurer la couverture du flux de trésorerie d'un contrat à terme donné, en utilisant un nombre restreint d'actifs. La méthode est sans risque en ce sens que le taux d'intérêt est supposé suivre un scénario de pire cas. Elle est fondée sur la programmation linéaire et donne une courbe des taux endogène, comme solution du problème inverse. Les applications possibles sont la titrisation et essentiellement les réaménagements d'engagements.

INTEREST RATES RISK IMMUNIZATION BY LINEAR PROGRAMMING*

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ABSTRACT :

We propose a riskfree method to hedge a given future cash stream using a restricted number of assets. The method is riskfree in the sense that the interest rate is assumed to follow a worst case scenario.It is based on linear programming and we find an endegeneous yield curve as a solution of the dual problem.Potential applications are securitization and in substance defeasance.

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1. INTRODUCTION

A fixed income liablity, or asset, can be represented by a sequence of negative, or positive, deterministic cash-flows. The present value of such cash-flows could easily be computed using ad hoc zero-coupon bonds if they existed for every maturity. One would simply duplicate each cash-flow by a zero-coupon bond. The given sequence of cash-flows thereby generates a portfolio of zero-coupons, the market value of which gives the present value of the liability.

Since the zero-coupon portfolio perfectly duplicates the given sequence of cash-flows, its holder will always be able to finance the liability, regardless of the particular path the interest rates follow. In the real world, unfortunately, it will usually be impossible to duplicate the liability with the assets that happen to be available. However, it may still be possible to construct a portfolio that enables its holder to finance the given liability in every conceivable circumstance.

For this purpose, we envision the situation as a game between the market, on the one hand, and the portfolio holder, on the other. The market fixes the path of interest rates, subject to the restriction that they do not rise above some prescribed r_{max} nor fall below some other value r_{min}. The other player chooses his portfolio among available assets. We then define the present value of the liability to be the min-max value of the game, that is, the lowest value of a portfolio which will enable its holder to finance the liability whatever the path of interest rates. As the reader will see, determining this value then boils down to a problem in linear programming.

In other words, we assume that the holder of the portfolio always faces a worst-case scenario, that is, that whenever he has to borrow, he borrows at the highest r_{max}, and whenever he lends, he does so at the lowest rate r_{min}. Therefore, provided the interest rates remain between r_{min} and r_{max} the immunization is absolute. We, in fact, compute an upper bound for the cost of the service of the liability.

As a solution of the dual of the optimization problem we solve, we find an endogeneous shadow yield curve. This yield curve depends on rmax but also on the pool of assets selected to finance the liability.

A test of the robustness of our results is the evaluation of the incidence of changes in r_{min} and r_{max} on the outputs.

Our basic motivation for this work is of course immunization, either to perform an in-substance defeance, or to securitize assets. We computed an actual example and summarized the results in the appendix.

a) Notation

The assets, or liabilities, cash-flows which are assumed to continue for n days, are represented by n-dimensional column vectors, the i-th element being the cash flow at date i.

We let $S = {}^t(s_1,...,s_n)$ represent the liability the holder has to finance using p available market assets with prices $\rho^1,\dots,\rho^i,\dots,\rho^p$ and cash flows e^i_j for the i-th asset at date $j, i=1\dots p, j = 1\dots n$.

Set :

$E^i = {}^t(e^i_1,\dots,e^i_n) \in R^n$ for $i = 1,\dots,p$

$E = (E^1,\dots,E^i,\dots E^p)$ the n x p matrix of cash flows

$p = {}^t(\rho^1,\dots,\rho^i,\dots,\rho^p) \in R^p$

We assume that the holder *can* buy any proportion α^i $(0 \le \alpha^i < 1)$ of the i-th asset E^i *at* price $\alpha^i\rho^i$. The corresponding cash flows are then : $\alpha^i E^i = {}^t(\alpha^i e^i_n,\dots,\alpha^i e^i_n)$. In reality there are N^i indivisible unit assets available in the market with cash flows E^i/N^i and price ρ^i/N^i : the numbers N^i are supposed large enough to consider α^i as a continous variable. On the other hand the amounts E^i are chosen small enough to have a negligible effect on the market prices.

In order to complete the markets, we also assume that the holder of the portfolio is able to lend (or borrow) any amount q_i (or q'_i when borrowing) between the (i-1)th and the i-th day of the period with return $a_i = 1+r_{min}$ (a $a'_i = 1 + r_{max}$ when borrowing) for $i = 1,\dots,n$. That is to say that she always faces the worst situation : the lowest possible interest rate when lendiig and the highest when borrowing.

The corresponding cash cash flows are represented f a the date. $i = 2,\dots,n$ by column vectors of R^n :

$q_i d_i = q_i {}^t(0,\dots,0,-1,a_i,0,\dots,0)$

$q'_i d'_i = q'_i {}^t(0,\dots,0,+1,-a'_i,0,\dots,0)$

Where the -1 (or the +1) is the (i-1)th coordinate.

At date $i = 1$, the cash flows vectors are :

$q_1 d_1 = q_1 {}^t(a_1, 0, \dots, 0)$

$q'_1 d'_1 = q'_1 {}^t(-a'_1, 0, \dots, 0)$

but the holder will have to pay at date 0 the amounts q_1 and $-q'_1$ respectively to include those cash flows in the portfolio. By convention, positive flows correspond to receiving cash and negative flows to paying out.

As a_i is the worst possible return *on* a lending and a'_i the highest possible yield on a borrowing :

$1 < a_i < a'_i$ for $i = 1,\dots,n$.

A *strategy* $(a^1,\dots,\alpha^p, q_1,\dots,q_n, q'_1,\dots,q'_n)$ consist of buying the proportions α^j of assets E^j for $j = 1,\dots$ p and lending (or borrowing) the amounts q_i (or q'_i when borrowing) between dates i-1 and i for $i = 1,\dots,n$.

b) Admissible strategies

A *strategy* $(a^1,\dots, \alpha^p, q_1,\dots, q_n; q'_1,\dots q'_n)$ is *admissible* if :

$0 \leq \alpha^i \leq 1 \quad i = 1,\dots,p$

$0 \leq q_i \quad i = 1,\dots,n$

$0 \leq q'i \quad i = 1,\dots,n$

$$S = \Sigma_{(i=1,\dots,p)}\, \alpha^i E^i + \Sigma_{(i=1,\dots,n)}\, q_i d_i + \Sigma_{(i=1,\dots,n)}\, q'_i d'_i \quad (1)$$

that is the holder adopting an admissible strategy exactly finances the liability at every date 1,...,n when facing the worst case interest rate scenario, and therefore when facing any interest rate senario.

c) A linear programming problem

The total cost at date 0 of an admissible strategy $(a^1,\dots,\alpha^p, q_1,\dots, q_n, q'_1,\dots q'_n)$ is :

$$V = q_1 - q'_1 + \Sigma_{(i=1,\dots,p)}\, \alpha^i p^i \quad (2)$$

the problem is to minimize this cost subject to the constraints of admissibility.

Let us introduce the slack variable β^i such that $0<\beta^i<1$ and $\alpha^i+\beta^i=1$ for $i = 1,\dots,p$. Those variables have no financial meaning but allow us to set our optimization problem in a standard way. Write :

$$c = {}^t(\rho^1,\dots, \rho^p, 0,\dots, 0, +1, 0,\dots, -1,\dots 0) \in R^{2(n+p)}$$

with p,n-1 and finally n-1 zeros.

$$b = {}^t(s_1,\dots, s_n, 1, \dots, 1) \in R^{n+p}$$

$$A = \left(\begin{array}{cccccccccc} & & a_1 & -1 & & 0 & -a'_1 & +1 & & 0 \\ E^1,\dots,E^p & 0 & & a_2 & -1 & & & -a'_2 & +1 & \\ & & & & \ddots & & & & \ddots & \\ & & 0 & & & -1 & 0 & & & +1 \\ & & & & & a_n & & & & a'_n \\ \hline Id^p & Id^p & & 0 & & & & 0 & & \end{array}\right)$$

or using compact notations :

$$A = \begin{pmatrix} E & Op & Z(a) & -Z(a') \\ & & & \\ Id^p & Id^p & O^p & O^p \end{pmatrix}$$

Note that the matrix $A \in L(R^{2(n+p)}, R^{n+p})$ has full rank n+p. We finally write :

$$u = {}^t(\alpha^1,\dots, \alpha^p, \beta^1,\dots, \beta^p, q_1,\dots,q_n, q'_1,\dots,q'_n).$$

Condition (1) of admissibility can be written $Au = b$, and the cost of strategy u given by (2) is $V=(c,u)$ the scalar product of vectors c and u. We should thus solve the linear program (p):

$$\begin{array}{l} \operatorname{Inf}\,(c,u) \\ u \in R^{2(n+p)} \\ Au = b \\ u \geq 0 \end{array}$$

The dual program (P') is :

$$\begin{array}{l} \operatorname{Sup}\,(b,h) \\ h \in R^{n+p} \\ {}^tAh < c \end{array}$$

We shall write $h = {}^t(k,m)$ with $k \in R^n$ and $m \in R^p$.

The dual program is the optimization problem faced by the market. We shall interpret it in section 4.

Recall the basic theorem of linear programming (see [2]):

(i) (P) admits an optimal solution if and only if (P') admits an optimal solution.

(ii) (P) admits an optimal solution if the sets of admissible strategies :

$$K = (u \in R^{2(n+p)} / u \geq 0,\ Au = b)$$
$$K' = (h \in R^{n+p} / {}^tAh \leq c)$$

are both nonempty.

(iii) The optimal strategies (u,h) of the programs (P) and (P') are characterized by :

$u \in K$, $h \in K'$ et $(u, {}^tAh-c) = 0$

3.EXISTENCE AND FIRST PROPERTIES OF A SOLUTION

a) Construction of an admissible strategy for the holder

The strategy $u_0 = {}^t(0,\dots,0,1,\dots,1,q_1,\dots,q_n,0,\dots,0)$ with p zeroes, p ones, n zeroes at the end and :

$${}^t(q_1,\dots,q_n) = Z(a)^{-1}s = \begin{pmatrix} a_1 & -1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ 0 & & & a_n \end{pmatrix}^{-1} \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$$

is admissible

Actually one can show recursively that :

$$Z(a)^{-1} = \begin{pmatrix} a_1 & -1 & & \\ & \ddots & \ddots & 0 \\ & & \ddots & \ddots \\ 0 & & & -1 \\ & & & a_n \end{pmatrix}^{-1} = \begin{pmatrix} 1/a_1 & 1/a_1a_2 & \cdots & 1/a_1\ldots a_n \\ & \ddots & & \\ & & \ddots & \\ 0 & & 1/a_{n-1} & 1/a_{n-1}a_n \\ & & & 1/a_n \end{pmatrix}$$

and this is a nonnegative matrix, therefore : $0 \leq (q_1,\ldots,q_n)$

Moreover the equality $Au_0 = b$ can easily be checked.

The suggested strategy consists in investing at the lowest possible yields the present value of the liability calculted with these lowest possible yields.

b)Efficient strategies

A strategy $u = (a^1,\ldots,\alpha^p,\beta^1,\ldots,\beta^p,q_1,\ldots,q_n,q'_1,\ldots,q'_n)$ is said to be *efficient* if :

$0 \leq u$ and $q_iq'_i = 0$ for all $i = 0,\ldots,n-1$.

Using $0 \leq u$ and letting $q = {}^t(q_1,\ldots,q_n)$ and $q' = {}^t(q'_1,\ldots,q'_n)$ this is equivalent to :

$0 \leq u$ and $(q,q') = 0$.

Following an efficient strategy one cannot borrow and lend money at the same time at the worst possible yields.

c)Existence of an optimal strategy

We have already proved in a) that the set K of admissible solutions of (P) is nonempty.

We shall now prove that the set K' of admissible solutions of the dual problem (P') is also nonempty.

It then follows from property (i) section 2 c) that there exist optimal strategies for both problem (P) and (P').

$h=(k,m)$ is element of K' if and only if :

${}^tAh < c$

that is :
(j) ${}^tE.k+m \leq \rho$
(jj) $m \leq 0$
(jjj) ${}^tZ(a).k \leq {}^t(1,0,\ldots,0) \leq {}^tZ(a').k$

Property (jjj) can be restated :

$1/a'_1 \leq k_1 < 1/a_1$
$1/a'_i \leq k_i/k_{i-1} < 1/a_i$ for $i = 2,\ldots,n$.

Clearly k can be chosen to satisfy (jjj) because $a_i < a'_i$ for every $i = 1,...,n$. One can then obviously choose a negative m in order to statisfy (j)and(jj) : we conclude that the set K' is nonempty and therefore that there are optimal strategies for (P) and (P').

4. THE SHADOW YIELD CURVE

a) the shadow yield curve and the liquidity premia

Let $V(b,c)$ be the optimal value of problem (P) :
$$V(b,c) = \inf_{\substack{Au = b \\ u > 0}} (c,u)$$

If $h \in R^{n+p}$ is a solution of the dual problem (P') :
$$\sup_{\substack{h \in R^{n+p} \\ {}^tAh \leq c}} (b,h)$$

then : $\forall\, y \in R^{n+p}, V(y,c) \geq V(b,c) + (h,y-b)$

i.e. h is a subgradient of V (a convex function of y) at point b (See (1)).

If V admits fust partial derivatives then by the enveloppe theorem :

$\forall\, i \in (1,^{n+p})\ \partial V/\partial b_i = h_i = k_i$

For $i = 1,...,n$ this property has a financial interpretation : in a d e r to pay one additionnal franc at date (a marginal variation of $b_i = s_i$), one needs an additionnal $k_i = h_i$ franc at date 0. Thus $k_i=\partial V/\partial b_i$, the marginal cost at date 0 of the franc payable at date i can be interpreted as a discount factor between 0 and i, or alternatively as the price at date 0 of a zero coupon bond paying 1 franc at date i.

Therefore the vector $k = {}^t(k_1,...,k_n)$ defines an endogeneous zero coupon bond yield curve. Let us underline that this yield curve depends on the upper (and lower) bounds a'_i (and a_i) assumed for the returns on each period but also depends on the liability we have to finance and on the pool of assets chosen to carry on the optimization. Moreover

$\forall\, i \in [1,p]\ \partial V/\partial b_{n+i} = m^i$

m^i is the marginal cost at dale 0 of the constraint $\alpha^i \leq 1$. By (jj), this cost is nonpositive : it represents the gain on the objective function from the availability on the market of an additonnal unit of asset i at the same price.

The vector $m = {}^t(m^1,...,m^p)$ will be referred to hereafter as the liquidity premium vector.

It turns out as we shall see hereafter that one obtains the present value of any liability in a neighbourhood of s, net of liquidity constraints, by discounting the liability cash flows using the discount factors k_i.

b)Interpretation and properties of the solutions

Let (u,(k,m)) be solutions of (p) and (p') respectively. For the sake of simplicity, let us assume that (k,m) is unique.

Property (iii) of section **2.c)** can be written :

(v) $a^i\,(k,E^i)+m^i-\rho^i) = 0$ for $i = 1,\dots,p$

(vv) $\beta^i m^i = 0$ for $i = 1,\dots,p$

(vvv) $q_1(a_1k_1-1) = 0$ $q'_1(a'_1k_1-1) = 0$

$q_i(a_ik_{i-}k_i-1) = 0$ $q'_i(a'_ik_i-k_i-1) = 0$ for $i = 2,\dots,n$

But because of (jjj) section 3.a) :

$1/a'_1 \le k_1 \le 1/a_1$

$1/a'_1 < k_i / k_{i-1} < 1/a_i$ for $i = 2,\dots,n$.

That is : the forward rate between date **i-1** and i on the shadow yield curve lies between the **ex ante** bottom rate a_i and ceiling rate a'_i.

Morever,condition (vvv)implies that, if the optimal strategy u of the holder involves between date **i-1** and i borrowing (or **lending)** at the ceiling **(bottom** when lending) rate :

$q_1 \neq 0$ implies $k_i / k_{i-1} = 1/a_i$

$q'_i \neq 0$ implies $ki / k_{i-1} = 1/a'_i$

As a consequence, it cannot be optimal to have $q_i \neq 0$ and $q'_i \neq 0$ **(because** $ai < a'_i$**)**, that is : **every optimal strategy** *is efficient*.

For an asset i, there are three possible situations :

1) $m^i < 0$

The liquidity premium is strictly negative. **Condition** (vv) **then** shows that asset i is entirely bought by the **holder,** and **condition** (v) can be written :

$(k,E^i) - \rho^i = -m^i > 0$

The present value of the asset calculated by using the shadow discount **factors** is strictly superior to its market price : it appears **then** to be natural that the holder buys asset **i** entirely. Moreover the liquidity premium equals precisely the difference between the market price of the asset and its present value.

2) $m^i = 0$ and $\beta^i > 0$

The **asset** is **partly** included in the optimal portfolio but the availability of additional units of the asset **neither** changes the optimal strategy of **the holder** nor the value of the game. Moreover, condition (v) implies :

$(k,E^i) = \rho^i$

The present value of the asset calculated with the shadow discount factors factors equals its market price.

3) $m^i = 0$ and $\alpha^i = 0$

The asset is not part of the optimal portfolio, its market price is too high :

$(k,E^i) < \rho^i$

c) Extention of the set of assets

The classification presented in section b) can help us to foresee the potentiel effects of adding a new asset E^{p+1} to the set of available assets on the market. There are two cases :

1) $(k,E^{p+1}) < \rho^{p+1}$

The new asset is too expensive. Letting $\alpha^{p+1} = 0, \beta^{p+1} = 1$ and $m^{p+1}=0$, one gets a saddle point for the extended problem, as can be verified by condition (iii) of section 2.c), with the same immunization cost. This new asset is useless for our problem.

2) $(k,E^{p+i}) > \rho^{p+i}$

Letting $m^{p+1} = \rho^{p+1} - (k, E^{p+1})$, $h' = (k,m,m^{p+1})$, one gets an admissible stategy for the extended dual problem Using natural notations one has : $(b',h') = (b,h) + m^{p+1} \leq V'$, where V' is the new immunzation cost.

This inequality gives a restriction on the potential gains on the immunization cost from including the new asset. Not surprisingly, the potential gains are bounded by the difference between the market price of the new asset and its present value calculated on the shadow yield curve that is the liquidity premium on the new asset.

d) interpretation of the dual problem

The dual problems (P) and (P') have the same value :

$V = (b,h) = (c,u)$

i.e. $V = (k,s) + \Sigma_{(i=1,\dots,p)} m^i = q_1 - q'_1 + \Sigma_{(i=1,\dots,p)} \alpha^i \rho^i$

or,using condition (v) :

$V - ;\Sigma_{(i=1,\dots,p)} m^i = (k,s) = \Sigma_{(\text{asset of types 1) and 2)})} \alpha^i (k,E^i) + q^1 - q'^1$

Therefore the liability and the portfolio of assets (including borrowing and lending flows at the date 0) have the same present value when discounted by the shadow yield curve.

When assets are available in any quantity, the liquidity premia are zero and the cost of immunization of a liability is exactly equal to its present value discounting by tfe shadow yield curve.moreover in this case the interpretation of the dual problem :

$$\begin{array}{l} \text{Sup } (b,h) \\ h \in R^{n+p} \\ {}^tAh \leq c \end{array}$$

is easier. The present value of the liability (s,k) is maximized on the set of admissible shadow yield curves. Admissibility is defined by :

(j) ${}^tE.k \leq \rho$

(jjj) $1/a'1 < k1 < 1/a1$

$1/a'i < ki / ki\text{-}1 < 1/ai$ for $i = 2,\dots,n$.

which means that the present value of an asset discounted on the shadow yield curve must not exceed its market price and that the shadow yield curve must lie betwen the imposed lower and upper bounds. If there are liquidity premia the constraint (j) becomes:

(j) ${}^tE.k + m \leq \rho$

but (1,m) is added to the objective function. Therefore the shadow yield curve chosen can be such that the present value of an asset is larger than its market price, the difference however is a cost in the optimal strategy of the market.

It is a classical result of linear programming that when u is a nondegenerate extreme point of K, the solution of the dual problem, that is the shadow yield curve and the liquidity premium vector, is unique and does not change in a neighbourhood of b.

5. ROBUSTNESS OF THE RESULTS

a) The ceiling-bottom rate hypothesis

We now consider the optimum $V(a,a')$ of the function of the bottom and ceiling returns a and a'. Let $u = (\alpha^1,\dots,\alpha^p,\beta^1,\dots,\beta^p,q_1,\dots,q_n,q'_1,\dots,q'_n)$ and $h = (k,m) \in R^{n+p}$ be the solutions of (P) and (P') respectively, for the parameters a and a'. Then one has by an enveloppe theorem whenever V admits first derivatives (See(1)):

$$\forall i \in [1,n],\ \partial V/\partial a_i\,(a,a') = -k_i q_i$$

$$\partial V/\partial a'_i\,(a,a') = k_i q'_i$$

Suppose as an example that $a_i(r) = 1+r$ and $a_i(r') = 1+r'$ for $i=1,\dots,n$, that is the ceilingand bottom returns are constant over the period, then:

$$\partial V/\partial r\,(a(r),a'(r')) = \Sigma_{(i=1,\dots,n)}\,\partial V/\partial a_i\ \partial a_i/\partial r$$

$$= \Sigma_{(i=1,\dots,n)}\ -k_i q_i$$

$$\partial V/\partial r'\,(a(r),a'(r')) = \Sigma_{(i=1,\dots,n)}\,\partial V/\partial a'_i \partial a'_i/\partial r'$$

$$= \Sigma_{(i=1,\dots,n)}\,k_i q'_i$$

Let $D = (r' - r)/2$ and $M = (r' + r)/2$, represent respectively the "volatily" and "mean" of the possible yields. It is then fallows that:

$$\partial V/\partial D\,(a,a') = \Sigma_{(i=1,\dots,n)}\,k_i\,(q_i + q'_i)$$

$$\partial V/\partial M\,(a,a') = \Sigma_{(i=1,\dots,n)}\,k_i\,(q_i + q'_i)$$

6. IMPLEMENTATOIN AND EXAMPLE

We used the simplex algorithm for linear programming on its simplest form, described for example in [3]. The program was written in (*) C language on a workstation (*) SUN 4/110.

a) Data form

It turns out that in this type of situation, the vectors and matrices contain mainly zeroes. In the example we computed, the assets were short term (less than five years) annual coupon bonds. Thus the cash flows vectors have at most five nonzero coordinates. Moreover the borrowing and lending vectors have by definition at most two non zero coordinates. We let **n** represent the number of dates which have a nonzero cash flow. a_i (or a'_i) represents the bottom (or ceiling return between the (i-1)-th and the i-th date. We represent the vectors dynamically.

In the numerical treatment we standardize the units of the numbers involved (using 1 franc as a unit) in order to make the approximation errors easier to evaluate. In particular we replace E^i by E^i/ρ^i.

b) Example

In the actual example, where we performed the calculations, we took r_{min} = 5% and r_{max} = 15% for France in the next 5 years. The example is based on a real liability faced by a French firm in June 1988, and on actual fixed-income securities available at that time on the French market. We used the simplex algorithm with 476 variable and 238 constraints. This is a relatively small linear program given the large number of zero elements. It takes a few minutes to converge on our workstation but more powerful versions of the simplex algorithm are commercialized.

For this computed example we provide in the appendix :

- The set of available market assets plotted in a plane (maturity, yield to maturity).

- The shadow yield curves for two cases : bottom rate = 5% and ceiling rate = 15%, or bottom rate = 5% and ceiling rate = infinity (this is for the particular case where you are not allowed to be short at any date in the future).

7. CONCLUSION

Let us end with some possible limitations and extensions of our method

We have assumed so far that the market assets were available in any amount and divisible. If one wants to include such indivisible assets in the market portfolio, then the function to minimize turns out to be one with integer arguments, and one would have to resort to integer programming, with its well-known complexity.

When there is no uncertainty on the future yields, there still exists a spread between the borrowing and lending rates (ceiling and bottom rates) which corresponds to the margin taken by financial intermediaries. The method still applies in this case.

To conclude, the immunization problem being fundamentally caused by the incompleteness of financial markets, our solution introduces ceiling and bottom rates to complete the markets.

However, there are many other possibilities, for instance :

- Minimizing every nonzero balance at every date, using for instance a quadratic criterion :

$$\operatorname*{Inf}_{0 \leq \alpha^i \leq 1} \quad Q\left(\Sigma_{(i=1,\dots,p)} \alpha^i E^i - S\right) + \Sigma_{(i=1,\dots,p)} \alpha^i \rho^i$$

with Q a positive definite quadratic form.

- The EIPIS methodology (see [4]) which balances cash flows and durations of the assets with that of the liability on many different time periods.

The comparative appeal of our approach is twofold First, it completely eliminates the risk associated with changes in interest rates, provided they stay within the prescribed bottom and ceiling rates. Second, it provides an immunization cost including *all cost,* and not only an immunization portfolio.

8, BIBLIOGRAPHY

[1] La théorie des jeux et ses Applications à l'Economie Mathématique
Ivar EKELAND (Presses Universitaires de France 1974).

[2] Introduction à l'Analyse Numérique Matricielle et à l'Optimisation
P.G.CIARLET (Masson 1982).

[3] Linear Programming
Vasek CHVATAL (W. H Freeman and Company New York).

[4] EIPIS, une méthode d'immunisation
P.Y GEOFFARD, J.M LASRY (Caisse Autonome de Refinancement discussion paper).