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STOCHASTIC APPROACH TO PENSION FUNDING METHODS

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APPROCHE STOCHASTIQUE DES METHODES DE CONSTITUTION DE CAISSE DE RETRAITE

APPROCHE STOCHASTIQUE DES MÉTHODES DE CONSTITUTION DE CAISSE DE RETRAITE

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RESUME

On décrit un modèle mathématique qui facilite la comparaison des différentes méthodes de constitution de caisse de retraite. On suppose d'abord que les taux de rendement sont aléatoires, puis qu'ils sont représentés par un modèle autorégressif de l'amplitude correspondante de l'intérêt. Des expressions de la variabilité des contributions et du niveau du fonds peuvent en être dérivées. Ceci conduit à une discussion de la méthode "optimale" de constitution du fonds. Une description plus détaillée de la méthode est donnée dans la thèse de doctorat de Dufresne (1986) et dans les récents articles d'Haberman et Dufresne (1987) et de Dufresne (1988, 1989). Les résultats mathématiques de la troisième partie sont présentés et discutés plus longuement dans un récent document de travail (Haberman, 1989).

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ABSTRACT

A mathematical model is described which facilitates the comparison of different pension funding methods. Rates of return are assumed firstly to be random and then to be represented by an autoregressive model for the corresponding force of interest. Expressions for the variability of contributions and fund levels can be derived This leads to a discussion of the "optimal" method of funding. A fuller description of the approach is given in Dufresne's doctoral thesis (1986) and in recent papers by Haberman and Dufresne (1987) and Dufresne (1988, 1989). The mathematical results of section 3 are presented and discussed at greater length in a recent working paper (Haberman (1989)).

1. TYPES OF FUNDING METHOD

Broadly, there are two types of funding methods.

With individual funding methods (e.g. Projected Unit Credit and Entry Age Normal), the normal cost (NC) and the actuarial liability (AL) are calculated separately for each member and then summed to give the totals for the population under consideration.

With aggregate funding methods (e.g. Aggregate and Attained Age Normal), there is no hypothecation of normal cost or actuarial liability to individuals ; instead the group is considered as an entity, ab initio.

Let C (t) and F (t) be the overall contribution and Fund level at time t f a a particular pension scheme.

For an individual funding method,

$$C(t) = \sum_{x} NC(x,t) + ADJ(t) \qquad (1)$$

where NC (x,t) is the normal cost for a member aged x at time t, Σ denotes summation over the membership subdivided by attained age and ADJ (t) is an adjustment to the contribution rate at time t, representing the liquidation of the unfunded liability at time t, UL (t). UL (t) is defined by

$$UL(t) = \sum_{x} (AL(x,t)) - F(t)$$

where AL (x,t) is the actuarial liability for a member aged x at time t.

For an aggregate method, the overall contribution is directly related to the difference between the present value of future benefits and the fund. Specifically,

$$C(t) = \left[\frac{PVB(t) - F(t)}{PVS(t)} \right] S(t) \qquad (2)$$

where S (t) is the payroll at time t, PVB (t) is the present value of future benefits (of all members including pensioners) at time t and PVS (t) is the present value of future salaries (of active members) at time t.

This paper considers the behaviour of C (t) and F (t) in the presence of random investment returns.

At any time t, a valuation is carried out to estimate C (t) and F (t) based only on the scheme membership at time t. However, as t changes, we do allow for new entrants to the membership so that the population remains stationary - see assumptions below.

In the subsequent mathematical discussion, we make the following assumptions.

1. All actuarial assumptions are consistently borne out by experience, except for investment returns.

2. The population is stationary from the start.

3. There is no inflation on salaries, and no promotional salary scale. For simplicity, each active member's annual salary is set at 1 unit.

4. The interest rate assumption for valuation purposes is fixed.

5. The real interest rate earned during the period, (t, t+l) is i (t+l). The corresponding force of interest is assumed here to be constant over the interval (t, t+l) and is written as δ (t+1). Thus $1 + i(t+1) = \exp(\delta(t+1))$.

6. $E[1 + i(t)] = E[e^{\delta(t)}] = 1 + i$, where i is the valuation rate of interest. This means that the valuation rate is correct "on average". This assumption is not essential mathematically but it is in agreement with classical ideas on pension fund valuation. Further, we define $\sigma^2 = \mathrm{Var}\, i(t)$.

Assumptions 1., 2., 3., and 4. imply that the following parameters are constant with respect to time t :

NC the total normal contribution
AL the total actuarial liability
B the overall benefit outgo (per unit of time).

Further, assumptions 1., 2., and 6. imply that

$$'AL = (1 + i)(AL + NC - B). \qquad (3)$$

The paper adopts a discrete time approach It is possible to approach this problem using a continuous time formulation ; however, the mathematics requires familiarity with stochastic differential equations and the derails have been omitted here - the interested reader is referred to Dufresne (1986).

Several results are quoted here without proof. Full proofs are available in the references by Dufresne and Haberman given at the end of this paper.

2. RANDOM INDEPENDENT, IDENTICALLY DISTRIBUTED RATES OF RETURN : BASIC RESULTS

It is assumed in this section that the earned rates of return $i(t)$ for $t > 1$ are independent, identically distributed random variables with $i(t) > -1$ with probability 1.

2.1. Moments of C(t) and F(t) : individual funding methods

There are two general ways in which the ADJ (t) term may be computed.

Under the "amortization of losses" method, we consider the loss in each year between valuation dates, for example (t-1,t). ADJ (t) is then the total of the intervaluation losses arising during the last M years (i.e. between t-M and t) divided by the present value of an annuity for a term of M years (i.e. spread over an M year period).

The properties of this method are not pursued here and the interested reader is referred to Dufresne (1986,1989) for a detailed discussion.

Under the "spread" method, we define ADJ(t) = i.e. the adjustment to the normal cost is equal to the overall unfunded liability divided by the present value of an annuity for a term of M years. Then.

$$C(t) = NC + \frac{(AL - F(t))}{a_{\overline{M|}}}. \qquad (4)$$

The paper concentrates on the "spread" method.

Then

$$\begin{aligned} F(t+1) &= (1 + i(t+1))(F(t) + C(t) - B) \\ &= (1 + i(t+1))[F(t) + NC + (AL - F(t))/\ddot{a}_{\overline{M|}} - B] \\ &= [(1 + i(t+1))/(1 + i)](qF(t) + r) \end{aligned} \qquad (5)$$

where $q = (1 + i)(1 - 1/\ddot{a}_{\overline{M|}})$ and $r = (1 + i)(NC - B + AL/\ddot{a}_{\overline{M|}})$.

Then, it can be proved that

$$E\,F(t+1) = q\,E\,F(t) + r\,. \qquad (6)$$

This is a recurrence relation which can be solved to give

$$E\,F(t) = q^t F(0) + r\,(1 - q^t)/(1 - q) \text{ for } t \geq 0.$$

If M > 1 then it can be shown that $0 < q < 1$ and so

$$\lim_{t} E\,F(t) = r/(1-q).$$

Using $AL = (1+i)(AL + NC - B)$, it can be shown that

$$r/(1-q) = AL. \qquad (7)$$

Equation (4) implies that $E\,C(t) = NC + (AL - EF(t))/$ and so

$$\lim_{t} E\,C(t) = NC. \qquad (8)$$

A consequence of equations (6) - (8) is that, if F (0) = AL, then

$$E\ F(t) = AL \text{ and } E\ C(t) = NC \quad \text{for } t > 0.$$

Concerning second moments, it can be proved that

$$\text{Var } F(t+1) = a \text{ Var } F(t) + b\ (E\ F\ (t+1))^2 \tag{9}$$

where $a = q^2 (1 + \sigma^2 (1+i)^{-2})$ and $b = \sigma^2 (1+i)^{-2}$.

Equation (9) is also a recurrence relation which may be solved in successive steps to give

$$\begin{aligned} \text{Var } F(0) &= 0 \\ \text{Var } F(1) &= b\ (E\ F(1))^2 \\ \text{Var } F(2) &= a\ b(E\ F\ (1))^2 + b(E\ F(2))^2 \quad \text{and so on.} \end{aligned}$$

Generally, $\text{Var } f(t) = b \sum_{k=1}^{t} a^{t-k}\ (E\ F(k))^2$ for $t > 1$. (10)

It can then be show that

$$\lim_{t} \text{Var } F(t) \approx \begin{cases} bAL^2/(1-a) & \text{if } a > 1 \\ \infty & \text{if } a \geq 1 \end{cases} \tag{11}$$

Also, $\text{Var}C(t) = \text{Var } F(t)/(\ddot{a}_{\overline{M|}})^2$.

It is also possible to work out covariances. Thus, it can be proved that

$$\text{Cov}(F(t+u),\ F(t)) = q^u \text{Var } F(t), \qquad u > 0.$$

Similarly, $\text{Cov}(C(t+u),\ C(t)) = q^u \text{Var } C(t)$

and

$$\text{Cov}(C(t+u),\ F(t)) = -q^u \text{Var}(F(t)/\ddot{a}_{\overline{M|}}.$$

Thus, if $a < 1$, the correlationcoefficients satisfy

$$\begin{aligned} \lim \text{Cor}(F(t+u),\ F(t)) &= \lim_{t} \text{Cor}(C(t+u),\ C(t)) \\ &= -\lim_{t} \text{Cor}(F(t+u),\ C(t)) \\ &= q^{|u|}. \end{aligned}$$

2.2. Moments of C (t) and F (t) : the aggregate funding method

As noted in equation (2), the Aggregate Funding Method is such that

$$C(t) = (PVB - F(t)) . S / PVS$$

with
S = Pensionable earnings ;
PVB = Present value of future benefits (including pensioners);
PVS = Present value of future earnings.

S, PVB and PVS are aggregate values. relating to the whole population of current members, and here are constants (from assumptions 1., 2., 3. and 4.).

Here

$$F(t+1) = (1 + i(t+1))(F(t) + C(t) - B)$$

$$= (1 + i(t+1))(F(t)(1 - S/PVS) + S . PVB/PVS - B)$$

$$= \{(1 + i(t+1))/(1+i)\}(q'F(t) + r')$$

where $q' = (1+i)(1 - S/PVS)$ and $r' = (1+i)(S . PVB/PVS - B)$.

As before $E F(t+1) = q' E F(t) + r'$.

It can also be shown that $0 < q' < 1$.

Therefore,

$$\lim_{t} E F(t) = r' / (1-q').$$

Clearly, $E C(t) = (PVB - E F(t)) . S / PVS$.

Again

$$\text{Var } F(t+1) = a' \text{ Var } F(t) + b[E F(t+1)]^2$$

with $a' = (q')^2 (1 + \sigma^2(1+i)^{-2})$

Eq (10) still holds, and the earlier result becomes

$$\lim_{t} \text{Var } F(t) = \begin{cases} b[\lim_{t} E F(t)]^2/(1 - a') & \text{if } a' < 1 \\ \infty & \text{if } a' \geq 1. \end{cases} \qquad (12)$$

Clearly, Var C(t) = Var F(t)) . S^2 / PVS ^{2}. Covariances and correlation coefficients are derived in the same fashion, substituting q' for q.

Remarks

(a) Trowbridge (1952) has shown that in some cases the Aggregate and Entry Age Normal methods are asymptotically equivalent. The conditions he supposed are assumptions 1. - 6. inclusive plus

7. There is only one entry age into the scheme ; and

8. σ^2 = Var i(t) = 0.

Clearly, if assumption 7. is maintained but assumption 8. is dropped (i.e. $\sigma^2 > 0$) then Trowbridge's proof still applies, but now to E F(t) and E C (t), yielding

$$\lim_t E\,{}^{AGG}F(t) = \lim_t E\,{}^{EAN}F(t) = {}^{EAN}AL; \qquad (13)$$

$$\lim_t E\,{}^{AGG}C(t) = \lim_t E\,{}^{EAN}C(t) = {}^{EAN}NC.$$

(b) It should be noted that, in this simple framework, the Aggregate method is really a particular case of the Entry Age Normal method (assuming assumption 7. is still in force) ; equation (13) implies

$$\begin{aligned} {}^{AGG}C(t) &= (PVB - {}^{AGG}F(t)).S/PVS \\ &= (PVB - {}^{EAN}AL).S/PVS + ({}^{EAN}AL - {}^{AGG}F(t))S/PVS \\ &= {}^{EAN}NC + ({}^{EAN}AL - {}^{AGG}F(t))/\ddot{a}_{\overline{N|}} \qquad (14) \end{aligned}$$

where N is defined such that $\ddot{a}_{\overline{N|}} = PVS/S$. Equation (14) says that the Aggregate and Entry Age Normal methods are identical, if the latter is applied together with an N-year spread of (AL - F (t)). This fact was previously noted by C.J. Nesbitt in his contribution to the discussion of Trowbridge (1963).

(c) If M = 1, then equation (15) does not apply ; instead

$$\begin{aligned} F(t+1) &= (1 + i(t+1))[F(t) + NC + (AL - F(t)) - B] \\ &= [(1 + i(t+1))/(1 + i)]\,(1 + i)(AL + NC - B) \\ &= [(1 + i(t+1))/(1 + i)]\,AL. \end{aligned}$$

Thus, for each t > 1,

EF (t) = AL
EC (t) = NC

and

$$\text{Var } C(t) = \text{Var } F(t) = \sigma^2(1+i)^{-2} AL^2.$$

2.3. Numerical example

A numerical example is now introduced in order to illustrate how C (t) and F (t) vary about their mean values.

The assumptions are :

Population : English Life Table n° 13 (Males) stationary
Entry age : 30 (only)
Retirement age : 65
No salary scale, or inflation on salaries
Benefits : Level life annuity (2/3 of salary)
Funding methods : 1. Aggregate
2. Entry Age Normal, spreading AL - F (t) over M years.
Valuation interest rate : .01

Given these assumptions, it can be shown that the actuarial liability and normal cost have the following numerical values :

EAN_{AL} = 451 % of payroll

EAN_{NC} = 14.5 % of payroll.

Actual rates of return on assets : $(i(t))_{t \geq 1}$, identically distributed random variables, with $Ei(t) = .01$ and $(\text{Var } i(t))^{1/2} = .05$ (=a).

Table 1 contains the limiting "relative standard deviations"

$$(\text{Var } F(t))^{\frac{1}{2}}/EF(t) \text{ and } (\text{Var } C(t))^{\frac{1}{2}}/EC(t)$$

as $t \to \infty$. In every case

$$\lim_t E\,F(t) = EAN_{AL} \text{ and } \lim_t E\,C(t) = EAN_{NC}$$

(including the Aggregate method - see Remark (b) of Section 2.2. For this particular population and interest rate, the value of N satisfying $\ddot{a}_{\overline{N|}} = PVS/S$ is about 17).

Remarks

(a) Of course, Var F (t) and Var C (t) can be computed for $t < \infty$, using the formulae of Section 2.1.

(b) Leaving aside the case M = 80 (of little practical importance), there appears to be a trade-off between Var F and Var C, e.g. increasing M reduces Var C, but increases Var F. This phenomenon is studied in greater detail in the next section

(c) Both st. dev. F (∞) and st. dev. C (∞) are nearly linear in σ when a is "small" (see equation 9). F a instance, if σ = .10, then for M = 20 we obtain

$$(\text{Var } F(\infty))^{\frac{1}{2}}/AL = 35.0\%$$

and

$$(\text{Var } C(\infty))^{\frac{1}{2}}/NC = 59.8\%,$$

or roughly double the corresponding figures in Table 1. If σ = .025, then for M = 20 the ratios are 8.3 % and 14.2 % a roughly half the corresponding figures in Table 1.

2.4. The trade-off between Var F and Var C

As $\sigma^2 \to 0$ and $M \to \infty$, it can be shown that the following results hold :

(a) if $i \geq 0$,

$$\text{Var } F(\infty) \sim \frac{\sigma^2}{(1+i)^2} \cdot \frac{s_{\overline{M|}}}{2} \cdot AL^2 ,$$

$$\text{Var } C(\infty) \sim \frac{\sigma^2}{(1+i)^2} \cdot \frac{(1+i)^{M-1}}{2\ddot{a}_{\overline{M|}}} \cdot AL^2 ; \qquad (15)$$

(b) if $i < 0$,

$$\text{Var } F(\infty) \sim \frac{\sigma^2}{(1+i)^2} \cdot \frac{s_{\overline{M|}}}{2+1} \cdot AL^2$$

$$\text{Var } C(\infty) \sim \frac{\sigma^2}{(1+i)^2} \cdot \frac{(1+i)^{M-1}}{(2+i)\ddot{a}_{\overline{M|}}} \; AL^2 . \qquad (16)$$

As approximations, these results are quite good ; for example, if a = .05, i = .01 and M = 20, equation (15) yields

$$(\text{Var } F(\infty))^{\frac{1}{2}}/AL = .05(1.01)^{-1}(s_{\overline{20|}}(.01)/2)^{\frac{1}{2}} = 16.4\%$$

while the exact value (from Table 1) is 16.8 %.

When i = 0, the result becomes

$$\text{Var } F(\infty) \sim \sigma^2 \frac{M}{2} . AL^2$$

$$\text{Var } C(\infty) \sim \sigma^2 \frac{1}{2M} . AL^2 \qquad (17)$$

which can be interpreted as follows : st. dev. F (resp. st. dev. C) is approximately

proportional (resp. inversely proportional) to $M^{1/2}$, at least when i is close to **0.** Hence, in **Table** 1, moving from M = 5 to M = 20 approximately doubles st. **dev.** F (∞) and halves st. dev. C (∞).

Figure 1 is a plot of st. dev. F (∞) against st. dev. C (∞), **corresponding** to **Table** 1. It shows that the **trade-off** alluded to above does take place but only up to M* = 60 ; however, the situation is altogether different for larger **M's** : if we intend to minimize variances, then any M **>** M* = **60** is to be rejected for clearly some other M ≤ M* = 60 reduces both Var F and **Var** C. For this reason, **the** range 1 I M ≤ M* may be described as **an** "optimal region".

The particular M* = 60 in Figure 1 has little practical **significance** because **deficiencies** or surpluses are not in practice spread over periods of 50 years or more. However, comparing Figures 1 and 2 shows how sensitive st. dev. F and st. dev. C can be to varying the parameter **i** = Ei (t). If a is still .05 but **i** is now .**10**, only values of M smaller **than** 8 would be considered. But this example is again **artificial**, for it **amounts** to assuming red rates of return to be 10 % on average.

The table below gives numerical values of M* (rounded), as a function of **i** and a It should be bome in mind that **i** is an average real rate of return, when interpreting these figures. These results may have some practical importance if M* turns out to be small - for example, if **i** = 0.03 and a = 0.20, the optimal region is 1 I M I 13. At the time of writing, a valuation (real) rate of interest of 3 % would be common in the United **Kingdon,** as would values of M of between 10 and 20 years : both parameter values **are** consistent with the "optimal" values of M* shown below.

i	-.01	0	.01	.03	.05
.05	-	401	60	23	14
.10	-	101	42	20	13
.15	158	45	28	16	11
.20	41	26	19	13	10
.25	22	17	14	10	8

Explicit formulae for M* can be obtained : the details are not presented here but the interested reader is referred to **Dufresne** (1986) or **Haberman** and Dufresne (1987).

2.5. Extensions

It is possible to investigate the properties of the first two moments of F (t) and C (t) when some of the strict assumptions laid down in Section **1** are relaxed. Thus, **Dufresne** (1986) and Habennan and Dufresne (1987) **consider** weakening these **assumptions** so that :

E **i** (t) = i ≠ i' the valuation interest rate ;
the population is only **asymptotically** stationary ; **and**
salaries grow with inflation (constant or not, but ***not* random).**

In section 3, we move away from the assumption that earned rates of interest are independent and identically distributed random variables.

3. AUTOREGRESSIVE RATES OF RETURN : BASIC RESULTS

It is apparent from the discussion in Section 2 that the equations for the moments of F (t) and C (t) are of the same type for individual and aggregate funding methods. This section, therefore, considers only one type, viz individual funding methods.

In order to investigate the effects of autoregressive models for the earned real rate of return, the paper follows the suggestion of Panjer and Bellhouse (1980) and considers the corresponding force of interest and assume that it is constant over the interval of time (t, t+l) : the notation used will be (t+1).

Now, it is assumed that the (earned real) force of interest is then given by the following autoregressive process in discrete time of order 1 (AR (1)) :

$$\delta(t) = \Theta + \varphi\,[\,\delta(t-1) - \Theta\,] + e(t) \qquad (18)$$

where e (t) for t = 1,2 ... are independent and identically distributed normal random variables each with mean 0 and variance γ^2. Equation (18) replaces assumption 5. introduced earlier. This model suggests that interest rate earned in any year depend upon interest rates earned in the previous year and some constant level. Box and Jenkins (1976) have shown that, under the model represented by equation (18),

$$E[\,\delta(t)\,] = \Theta$$

$$\mathrm{Var}\,[\,\delta(t)\,] = \frac{\gamma^2}{1-\varphi^2} = v^2, \text{ say}$$

$$\mathrm{Cov}\,[\,\delta(t),\ \delta(s)\,] = \left(\frac{\gamma^2}{1-\varphi^2}\right).\varphi^{|t-s|} = \gamma(t,s), \text{ say.}$$

The condition for this process to be stationary is that $|\varphi| < 1$

Boyle (1976) investigated the simpler model :

$$\delta(t) = \Theta + e(t) \qquad (19)$$

where $\varphi = 0$. Clearly this model bears a close resemblance to that considered in Section 2. It can be shown that equation (9) leads to similar results to those presented in section 21. for individual funding methods.

In order to apply the autoregressive model (18) to determine moments of F (t) and C (t), it is necessary to abandon the approach of section 2 whereby recurrence relations between, for example, E F (t+1) and E F (t) were sought. The presence of a dependence on the past in the autoregressive model would make such an approach problematic.

The approach followed here begins with considering the series generated by the recurrence relation (5), which for convenience is rewritten here as

$$F(t+1) = (1+i(t+1))\,(Q\,F(t) + R) \qquad (20)$$

where $Q = 1 - \frac{1}{\ddot{a}_{\overline{M|}}} = vq$, $R = (NC - B + \frac{AL}{\ddot{a}_{\overline{M|}}}) = vr$ and $v = (1+i)^{-1}$.

Then $F(t) = F(0).\ Q^t e^{\Delta(t)} + Q^{t-1}Re^{\Delta(t)} + Q^{t-2}R\, e^{\Delta(t)-\Delta(1)}$

$$+ \ldots\ldots\ldots + R\, e^{\Delta(t)-\Delta(t-1)} \qquad (21)$$

where $\Delta(t) = \sum_{u=1}^{t} \delta(u)$.

In order to obtain an expression for E F (t) it is necessary to consider terms of the form

$$E(e^{\Delta(t)-\Delta(s)}) \text{ for } s=0,1,\ \ldots,\ t-1.$$

Given the distributional assumption for e (t), and that

$$E(\Delta(t)) = E(\sum_{u=1}^{t}\delta(u)) = t\Theta$$

$$Var(\Delta(t)) = Var(\sum_{u=1}^{t}\delta(u)) = \sum_{u=1}^{t}\sum_{w=1}^{t} \gamma(u,w)$$

$$= \nu^2\sum_{u=1}^{t}\sum_{w=1}^{t} \varphi^{|u-w|} \qquad (22)$$

then $E(e^{\Delta(t)-\Delta(s)}) = \exp[(t-s)\Theta + \frac{1}{2}\sum_{u=s+1}^{t}\sum_{w=s+1}^{t} \gamma(u,w)]$

$$= \exp[(t-s)\Theta + \nu^2 G(t,s)] \text{ ,say.}$$

An expression for G (t, s) can be obtained by standard techniques (Haberman (1989)).

Thus $E[e^{\Delta(t)-\Delta(s)}] = \exp[(t-s)(\Theta + \frac{1}{2}\frac{1+\varphi}{1-\varphi}\nu^2) - \nu^2\varphi\frac{1-\varphi^{t-s}}{(1-\varphi)^2}]. \qquad (23)$

If the subsidiary parameters $c = \exp\left(\Theta + \frac{1}{2}\frac{1+\varphi}{1-\varphi}\nu^2\right)$ and $\bar{d} = \nu^2\varphi(1-\varphi)^{-2}$, are introduced,

then $$E[e^{\Delta(t) - \Delta(s)}] = c^{t-s} e^{-\bar{d}(1-\varphi^{t-s})}. \qquad (24)$$

Equation (21), implies that

$$E\,F(t) = F(0)\,Q^t\,E\,e^{\Delta(t)} + R\sum_{s=0}^{t-1} Q^{t-s-1}\,E(e^{\Delta(t)-\Delta(s)})$$

$$= (F(0)Q^t c^t e^{\bar{d}\varphi^t} + \frac{R}{Q}\sum_{s=0}^{t-1}(Q^{t-s}\,c^{t-s}\,e^{d\varphi^{t-s}})e^{-\bar{d}}. \qquad (25)$$

The second term is of the form of the present value in conventional life contingencies of a temporary annuity based on Gompertz's a Makeham's law of mortality.

In section 2.1., it was noted that $0 < q < 1$.

So $$cQ = \exp(-\Theta - \tfrac{1}{2}\nu^2)\exp\left(\Theta + \frac{1+\varphi}{1-\varphi}\nu^2\right)$$

$$= q\exp\left(\frac{\varphi\nu^2}{1-\varphi}\right) = q \text{ if } \varphi = 0.$$

For convergence as $t \to \infty$, we require $cQ < 1$. And we note that $\varphi < 1$.

It can then be shown that

$$\lim_{t\to\infty} E\,F(t)) = \frac{R}{Q}\,\frac{Qc}{1-Qc}\,e^{-\bar{d}}$$

$$= \frac{vrc}{1-vqc}\,e^{-\bar{d}} \qquad (26)$$

If $\varphi = 0$ then $c = \exp(\varphi + \frac{1}{2}\nu^2) = 1+i$ and $\bar{d} = 0$, and

hence $\lim_{t\to\infty} E\,F(t) = \frac{r}{1-q}$ as in equation (7).

Then $E\,C(t) = NC + \frac{AL - E\,(F(t))}{\ddot{a}_{\overline{M|}}}$ from equation (4).

To obtain an explicit expression for Var (F (t)), it will be necessary to consider E (F (t) $)^2$, which itself will depend on terms of the form

$$E(e^{\Delta(t) - \Delta(s) + \Delta(t) - \Delta(r)})$$

for r, s = 0, 1, ..., t-1.

Without loss of generality, we consider r > s.

Given the distributional assumptions for e (t), Haberman (1989) has shown that, for r >s,

$$E\left(e^{\Delta(t)-\Delta(s)+\Delta(t)-\Delta(r)}\right) = \exp\left((t-s)\varphi+(t-r)\varphi+\tfrac{1}{2}\sum_{s+1}^{r}\sum_{s+1}^{r}\gamma(u,w) + 2\sum_{s+1}^{t}\sum_{r+1}^{t}\gamma(u,w)\right)$$

where, in this case of an AR (1) model,

$$\gamma(u,w) = \nu^{2}\ \varphi^{|u-w|}.$$

For convenience, we can write

$$E\left(e^{\Delta(t)-\Delta(s)+\Delta(t)-\Delta(r)}\right) = \exp\left[(t-s)\,\Theta+(t-r)\,\Theta+\nu^{2}H(t,r,s)\right] \quad (27)$$

and it has been shown by Haberman (1989) that H (t, r, s) simplifies to :

$$H(t,r,s) = \tfrac{1}{2}(t-s)\frac{(1+\varphi)}{(1-\varphi)} - \frac{\varphi^{r-s+1}}{(1-\varphi)^{2}} + \frac{2(r-s)\varphi^{r-s}}{(1-\varphi)} + \tfrac{3}{2}(t-r)\frac{(1+\varphi)}{(1-\varphi)} + \frac{2(\varphi^{t-r+1}+\varphi^{t-s+1})}{(1-\varphi)^{2}} - \frac{3\varphi}{(1-\varphi)^{2}}$$

For convenience, we will take F (0) = 0.

$$\text{Then, } E(F(t))^{2} = E\left[\sum_{s=0}^{t-1}\sum_{r=0}^{t-1} e^{\Delta(t)-\Delta(s)}\, e^{\Delta(t)-\Delta(r)}\, Q^{t-1-s}\, Q^{t-1-r}\, R^{2}\right]$$

$$= \frac{2R^{2}}{Q^{2}}\sum_{r=1}^{t-1}\sum_{s=0}^{t-1} Q^{t-s}\, Q^{t-r}\ E\left(e^{\Delta(t)-\Delta(s)+\Delta(t)-\Delta(r)}\right) + \frac{R^{2}}{Q^{2}}\sum_{s=0}^{t-1} Q^{2(t-s)}\ E\left(e^{2(\Delta(t)-\Delta(s))}\right) \quad (28)$$

Then, it has been shown by Haberman (1989) that

$$\lim_{t\to\infty} E(F(t)^{2}) = e^{-3\bar{d}}\,\frac{2R^{2}Qc^{2}w}{(1-Qc)(1-Q^{2}cw)} + e^{-4\bar{d}}\,\frac{R^{2}cw}{(1-Q^{2}cw)}$$

where $d = \frac{\nu^{2}\varphi}{(1-\varphi)^{2}}$, $c = \exp\left(\Theta+\tfrac{1}{2}\frac{(1+\varphi)\nu^{2}}{(1-\varphi)}\right)$, $Q=vq$, $R=vr$ as before and

$$w = \exp\left(\Theta+\tfrac{3}{2}\frac{(1+\varphi)}{1-\varphi}\nu^{2}\right).$$

We note that $\varphi < 1$, by assumption,

and that $Qc = Q \exp\left[\theta + \tfrac{1}{2}\frac{(1+\varphi)}{1-\varphi}\nu^2\right] = q \exp \frac{(\varphi\nu^2)}{1-\varphi}$

and $Q^2c\,w = q^2v^2\,cw = q^2 \exp\left[\frac{(1+3\varphi)\nu^2}{1-\varphi}\right]$.

For convergence, as $t \to \infty$, we require $Qc < 1$ and $Q^2cw < 1$.

Then,
$$\lim_{t\to\infty} Var\,F(t) = e^{-3\bar{d}}\,\frac{2R^2Qc^2w}{(1-Qc)(1-Q^2cw)} + \frac{e^{-4\bar{d}}}{(1-Q\,cw)} - \frac{R^2c^2}{(1-Qc)^2}\,e^{-2\bar{d}} \qquad (29)$$

Then, formulae for $\lim_{t\to\infty} Var\,C(t) = \frac{Var\,F(t)}{(\ddot{a}_{\overline{M}|})^2}$ may be obtained.

The next step would be to investigate the existence of an "optimal" M* as in section 2.4. (for the case of random rates of return). This wak is currently in progress and it may be possible to report further results at the AFIR Colloquium.

4. CONCLUSIONS

The variability of contributions (C) and fund levels (F) resulting from random (real) rates of return has been studied mathematically. The funding methods considered are the Aggregate Method and those Individual Methods that prescribe the normal cost to be adjusted by the difference between the actuarial liability and the current fund, divided by the present value of an annuity for a term of "M" years. A simple demographic/financial model permits the derivation of formulae for the first two moments of F and C, when earned (real) rates of return from an independent identically distributed sequence of random variables. The way these moments depend on M has then analysed, with the help of a numerical example. The approach has been extended to include the case of an autoregressive model for the earned (real) rate of return. Expressions for the first two moments of C (t) and F (t) have been obtained and reported and their detailed properties are currently under investigation.

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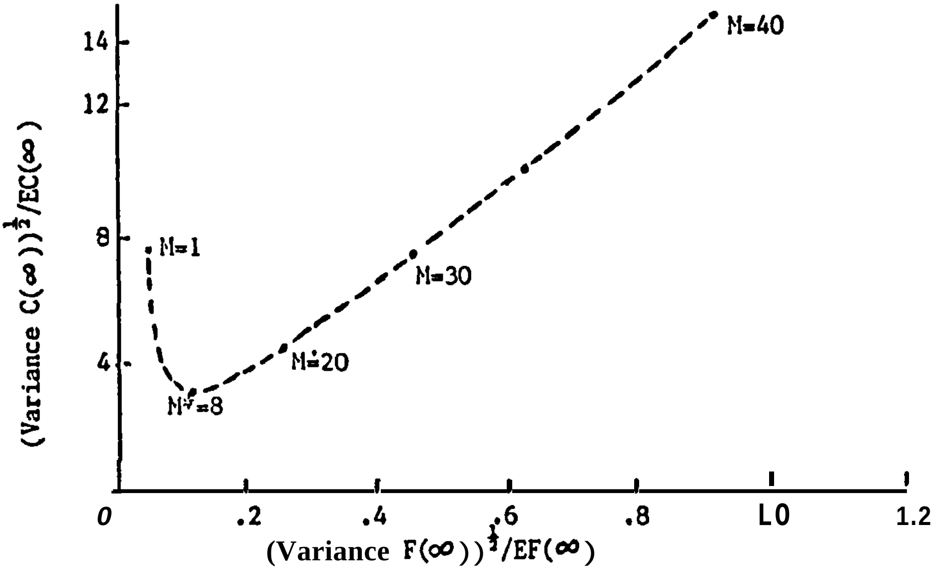
RELATIVE STANDARD DEVIATIONS OF F (T) AND C (T), AS T → ∞



RELATIVE STANDARD DEVIATIONS OF F (T) AND C (T), AS T - > ∞

[i = .01, σ = .05]

Funding Method	$(\text{Var } F(\infty))^{1/2}$ / AL	$(\text{Var } C(\infty))^{1/2}$ / NC
EAN M = 1	5.0 %	154 %
M = 5	8.3	52.9
M = 10	11.7	37.9
M = 20	16.8	28.7
M = 40	25.3	23.7
M = 60	33.4	22.9
M = 80	41.9	23.5
Aggregate (= EAN with M = 17)	15.3	30.6

Table 1

Relative Standard Deviations

of F ( ) and C ( )