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PRICE AND RISK OF VARIABLE RATE BONDS : AN APPLICATION OF THE COX, INGERSOLL, ROSS MODEL TO ITALIAN TREASURY CERTIFICATES

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PRIX ET RISQUE DES OBLIGATIONS A TAUX VARIABLE : UNE APPLICATION DU MODELE DE COX, INGERSOLL ET ROSS AUX CERTIFICATS DE CRÉDIT DU TRESOR ITALIEN

RESUME

"Prix et risque des obligations à taux variable : une application du modèle de Cox, Ingersoll et Ross aux certificats de crédit du Trésor italien"

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L'article présente une analyse d'une classe particulière de titres liés à un indice, les certificats de crédit sur le Trésor italien (CCT : Certificati di Credito del Tesoro).

Pour étudier correctement les CCT, et plus généralement tous les titres liés à un indice, il convient d'utiliser des modèles stochastiques. Dans notre analyse des CCT, nous utilisons le cadre de base du modèle à variable d'état unique de Cox, Ingersoll et Ross. Le prix d'équilibre et le risque de base des cash - flows qui dépendent du taux d'intérêt sont dérivés en introduisant le concept d'obligation stochastique à coupon nul. Le risque total de taux d'intérêt de ces actifs est considéré comme résultant de la somme de deux composantes en général de signes opposés. Ce résultat pour être utilisé pour étudier l'efficacité d'immunisation du système d'indexation.

La procédure d'estimation - selon la méthode proposée par Brown et Dybvig - est fondée sur les cotations des bons du Trésor italiens (BTP : Buoni del Tresoro Poliennali) à la bourse de Milan.

SUMMARY

"Price & risk of variable rate bonds : an application of the Cox, Ingersoll, Ross Model to Italian Treasury credit certificates"

This paper presents an analysis of a particular class of index - linked security, the italian Treasury Credit Certificates (Certificati di Credito del Tesoro, CCT).

A correct study of CCTs (and, more generally, of any index - linked security) must be performed by using stochastic models.

In our analysis of CCTs we use the basic framework of the single - state variable Cox, Ingersoll, Ross model. The equilibrium price and the basis risk of interest dependent cash - flows are derived by introducing the concept of stochastic zero - coupon bond. The total interest rate risk of these assets is recognized as resulting from the sum of two components in general having opposite signs. This finding can be used to study the immunizing efficiency of the indexation design.

The estimation procedure (following the method proposed by Brown and Dybvig) is based on the prices of the italian Treasury bonds (Buoni del Tesoro Poliennali, BTP), quoted on Milan Stock Exchange.

PRICE AND RISK OF VARIABLE RATE BONDS : AN APPLICATION OF THE COX, INGERSOLL, ROSS MODEL TO ITALIAN TREASURY CREDIT CERTIFICATES

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1 - INTRODUCTION

Several types of variable rate bonds are currently traded on the Italian securities market. In these bonds, coupon payments are linked to the current level of a predetermined market index.

This paper presents an analysis of a particular (but important [1]) class of index-linked securities, the Treasury Credit Certificates (Certificati di Credito del Tesoro, CCT). CCTs are medium or long-term Government securities in which coupon payments adjust according to changes in short - term interest rates. The coupon values are determined as a function of the issue price of six or twelve - month Treasury Bills. The time interval between each observation of the indexing price and the corresponding payment is such that only the next two coupons, at most, are known at any given time. In this sense the stochastic nature of interest rates is central to the analysis of these instruments.

From the Treasury's point of view, CCTs are an instrument for increasing the average maturity of the public debt (typically, their maturity is between 4 and 10 years, while only three, six and twelve-month Treasury - Bills are issued on the Italian market). From the investor's view-point, the indexation design has the advantage of stabilizing the price of the contract, by immunizing it against changes in interest rates.

Usually, the risk carried by these securities is evaluated by assuming as certain a particular hypothesis regarding the evolution of interest rates and by calculating the duration of the bond according Macaulay's traditional definition. In this way, the distinguishing feature of variable rate bonds - their stochastic coupon payment schedule is lost and CCTs are treated as if they provide known cash - flows. The immunizing effect of the indexation design is so "sterilized" and the riskiness of a CCT is roughly equal to that of a fixed rate Treasury bond with the same maturity. The result of this deterministic approach is not satisfactory, since holding a CCT is felt to be the equivalent to a roll - over strategy of short-term loans and therefore should imply an essentially lower risk level.

Therefore, a correct analysis of CCTs (and, more generally, of any index-linked security) must be performed by using stochastic models, which can be very complicated, depending on several state variables. When we are dealing with CCTs, however, we can obtain meaningful results by simplified methods, since the nature of the index variable makes it possible to carry out the analysis by using a (simple and testable) single - factor stochastic model for the term structure of interest rates.

Attempts to valuate floating rate securities in the framework of traditional immunization theory have been made by Chance [1983] and Morgan [1986], who perform the analysis under the assumption that the term structure of interest rates changes only by additive shifts. As pointed out by Ingersoll, Skelton and Weil [1978], "parallel" movements of the yield curve cannot occur in equilibrium in a competitive market. Independently whether these assumptions are empirically consistent, approaches of this kind do not correctly recognize the role of expectations in modelling uncertainty, since the probability distributions of the random variables involved are not considered. In these semideterministic models logical difficulties arise in the analysis of the (interest rate) risk characteristics of bonds with unknown cash - flows, as shown by the discrepancies in calculating durations that &rive from the use of an "average life" definition (Chance) rather than an "elasticity" definition (Morgan).

These difficulties are overcome by the approach proposed by Cox, Ingersoll and Ross (CIR) (1980) for the valuation of variable rate loan contracts. This approach utilizes a one-factor version of the stochastic general equilibrium model for asset pricing developed by the same Authors(1978) and is consistent with rational expectations and maximizing behaviour.

In our analysis of CCTs we use the basic framework of the single state-variable CIR model. The equilibrium price and the basis risk of interest dependent cash-flows are derived by introducing the concept of stochastic zero-coupon bond, as defined by De Felice and Moriconi [1989]. The total interest rate risk of these assets is recognized as resulting from the sum of two components in general having opposite signs. This finding can be used to study the immunizing efficiency of the indexation design.

2 - SINGLE FACTOR DIFFUSION MODELS

Typically, single factor diffusion models for the analysis of default-free bond markets are based on the following main assumptions :

a) the single variable that determines the state of the economy at rime t is the spot rate of interest r (t), that is, the yield available on a discount bond which will mature in the next instant of rime [2];

b) the spot rate follows a continuous Markov process, which is described by the Ito stochastic differential equation :

(1) $$dr = f(r,t)\,dt + g(r,t)\,dz,$$

where f and g^2 are the instantaneous drift and variance, respectively, and Z is a standard Brownian motion.

c) the market is perfect and frictionless : trading in all assets takes place continuously; there are no taxes or transaction costs; riskless arbitrage opportunities are precluded; the investors have homogeneous expectations (all investors share the same ∞ dimensional distribution function relative to the underlying stochastic process $\{r(t)\}$).

Under these assumptions, the price at time t, P (t), of any asset traded on the market is a function of the state variable r(t), of the time and of the contractual parameters vector c :

(2) $$P(t) = P(r, t; c).$$

The dependence of price function on the spot rate (the state variable) justifies these assets can be referred to as "interest rate sensitive" (IRS) financial claims.

By the Ito differential rule, it follows that the price P(t) satisfies the stochastic differential equation :

$$\frac{dP}{P} = \mu dt - \sigma dZ$$

where the infinitesimal parameters $\mu = \mu(r, t; c)$ and $\sigma = \sigma(r, t; c)$ are given by :

$$(3) \qquad \mu = \frac{P_t}{P} + f\frac{P_r}{P} + \frac{1}{2}g^2\frac{P_{rr}}{P},$$

$$(4) \qquad \sigma = -g\frac{P_r}{P}.$$

By invoking familiar hedging arguments, we obtain the noarbitrage condition :

$$(5) \qquad \frac{\mu(r,t;c) - rP(r,t;c)}{\sigma(r,t;c)} = q(r,t),$$

which prescribes that the ratio $(\mu - rP)/\sigma$ is independent of c, i.e. it is the same for all IRS assets. In this sense the function q (r, t) characterizes the market at time t. Since the left - hand quantity in equation (5) can be interpreted as the increase in expected instantaneous rate of return per additional unit of risk, the quantity q(r, t) is usually called the market price of risk.

Substituting (3) and (4) into (5) we obtain the general valuation equation :

$$(6) \qquad P_t + (f+qg)P_r + \frac{1}{2}g^2P_{rr} - rP = 0.$$

This partial differential equation, together with appropriate boundary conditions, can be used to price any IRS asset.

In particular, the price at time t, v (r, t ; s), of a zero - coupon bond maturing with unit value at time s ($t \geq s$) will be obtained under the boundary condition : v (r, s ; s) = 1.The function v (r, t ; s) determines the term structure of interest rates prevailing at time t [3]. From a different point of view, v (r, t ; s) can be interpreted as the equilibrium discount factor which determines the present value of a deterministic cash - flow to be paid at time s.

By assumption c), the price W of a coupon bond that entitles the holder to the payments $x_1, x_2, ..., x_m$ to be received on the future dates $t_1, t_2, ..., t_m$, will be (value - additivity principle) :

$$(7) \qquad W(r,t;x) = \sum_{k=1}^{m} x_k v(r,t;t_k),$$

where $x\ (t_1, t_2, \ldots, t_m, x_1, x_2, \ldots, x_m)$.

In the one factor diffusion models we just outlined a natural measure [4] of basis risk for any IRS assets P (t) is given by :

$$(8) \qquad \Omega(r,t;c) = -\frac{P_r(r,t;c)}{P(r,t;c)} .$$

In many cases it is possible to express the basis risk in units of time by defining a stochastic duration D(r, t ; c) for the asset P (t) as the maturity of a pure discount bond with the same risk :

$$(9) \qquad -\frac{v_r(r,t;t+D)}{v(r,t;t+D)} = \Omega(r,t;c) .$$

Usually, it is possible to express directly the stochastic duration of P as :

$$(10) \qquad D = \phi^{-1}(\Omega) - t,$$

where ϕ^{-1} (.) is the inverse fonction, with respect to variable

3 - STOCHASTIC ZERO - COUPON BONDS

The concept of an ordinary (deterministic) zero-coupon bond can be generalized by defining stochastic zero-coupon bonds, which are the basic tool to value a broad class of IRS financial claims with stochastic cash-flows.

A stochastic zero-coupon bond maturing at time s is a default free security that promises to pay, at time s, a cash-flow X, which is fixed at time Tε [t. s] as a function F (r (T)) of the current spot rate level (at time T). Both time T and the form of the function F (.) are determined at time t. We shall denote the price at time t of a stochastic zero-coupon bond that pays X at time s as P (r, t ; T, s, X). The quantity X is a r(T)-measurable random variable.

If T = t the cash - flow is deterministic (and the bond is an ordinary pure discount bond). When T = s the definition of stochastic zero-coupon bond can be expressively used to model an European option on interest rate [5].

In the case t < T < s, X can be considered as an index - linked coupon, s - T = λ being the lag between the time the coupon is paid and the time T in which it is fixed (fixing time). The indexation &vice is specified by the function F(r(T)).

We define the ex-ante value of X as :

$$(11) \qquad z(r,t;T,s,X) = \frac{P(r,t;T,s,X)}{v(r,t;s)} .$$

Since vz = P, the ex-ante value of the random payment X (to be delivered at time s) is the deterministic cash-flow to be paid at time s which has the same present value as X.

It should be noted that only if $T = t$ the ex-ante value z is independent of $r(t)$, since we have : $z = Xv(t,s) / v(t,s) / v(t,s) = X$ (deterministic cash-flow). When it is $T > t$, the ex-ante value depends on the current level of the *spot* rate.

By definition (11) the basis risk $\Omega(r, t; T, s, X)$ of a stochastic zero-coupon bond can be expressed as :

$$\Omega = -\frac{1}{zv}\frac{\partial}{\partial r}(zv) = \Phi + \Gamma ,$$

where :

$$(12) \qquad \Phi(r,t;T,s,X) = -\frac{v_r(r,t;s)}{v(r,t;s)}$$

and:

$$(13) \qquad \Gamma(r,t;T,s,X) = -\frac{z_r(r,t;T,s,X)}{z(r,t;T,s,X)}$$

Hence, by the definition of the ex-ante value, it is possible to separate the risk inherent to a stochastic cash-flow into the *sum* of two components : the "discount risk" Φ, which measures the discount factor sensitivity to state variable variations, and the "coupon risk", Γ, which describes the corresponding volatility of the random cash-flow.

Without specifying the parameters f and g of the spot rate process and the market price of the risk q, an integral representation for the IRS asset price can be &rived by well-known martingale arguments. For a stochastic zero-coupon bond, we obtain :

$$(14) \qquad P(r,t;T,s,X) = E_t[F(r(T))\exp(-S(t,s))] ,$$

where E_t is the expectation operator conditional upon the state variable at time t and $S(t,s)$ is the stochastic integral :

$$S(t,s) = \int_t^s r(u)du + \frac{1}{2}\int_t^s q^2(r(u),u)du - \int_t^s q(r(u),u)dZ(u) .$$

The exponential factor in (14) can be interpreted as the stochastic discount factor which must be used in the model to calculate the present value of the stochastic cash-flow $X = F(r(T))$, to be paid at time s.

In the case of an ordinary discount bond paying a unit of money at time s, the price integral representation is immediately obtained by condition $F(r(t)) = 1$; equation (14) then becomes :

$$(15) \qquad P(r,t;t,s,1) = v(r,t;s) = E_t[\exp-(S(t,s))].$$

4 - MODELLING VARIABLE RATE BONDS

Let us consider, at time t, a stochastic zero-coupon bond which pays, at time s, the index-linked amount:

$$I = F(r(T)) = C\ i(T,\mu),$$

where C is a positive real number and the indexation variable, i (T, p), is the yield rate on an *ordinary* **zero-coupon bond** issued **at time** T and maturing at time T + μ :

$$(16) \qquad i(T,\mu) = \frac{1}{v(r,T;T+\mu)} - 1, \qquad T\epsilon[t,s],\ \mu>0.$$

By equation **(14)**, the integral expression for the price of the index-linked cash-flow I is given by:

$$P(r,t;T,s,I) = C\{E_t[\frac{e^{-S(t,s)}}{v(r,t;T+\mu)}]-E_t[e^{-S(t,s)}]\},$$

and then:

$$(17) \qquad P(r,t;T,s,I) = C\{\beta(r,t;T,s,\mu)-v(r,t;s)\},$$

where β is the conditional expectation:

$$(18) \qquad \beta(r,t;T,s,\mu) = E_t[\frac{e^{-S(t,s)}}{v(r,T;T+\mu)}].$$

For **economic** consistence, the random variable 1 / v (r, T ; T + μ) **is** never less **than** one ; hence it is : β (r, t ; T, s, μ) ≥ E_t [exp (- s (t, s))] = v (r, t ; s). **This** result ensures that the price of the stochastic cash-flow I is never negative.

Let us consider a stochastic **zero-coupon** bond which, at time t, promises to pay the sum K = C + I, at time s. Since C is deterministic (known at time t), the price of this bond will be:

$$(19) \qquad P(r,t;T,s,K) = C\ v(r,t;s) + P(r,t;T,s,I) = C\ \beta(r,t;T,s,\mu).$$

If it is T > t, the basis risk of the bond is:

$$(20) \qquad \Omega(r,t;T,s,K) = -\frac{\beta_r(r,t;T,s,K)}{\beta(t;T,s,K)},$$

corresponding to the stochastic duration:

$$D = \phi^{-1}(-\frac{\beta_r}{\beta}) - t.$$

(Obviously, in the deterministic case (T = t) we have Ω (K) = ϕ (r,.t ; s) and D = s - t.)
The interpretation of β (r, t ; T s, μ) is straightforward It represents the equilibrium price, at time t, of a default-free security which guarantees, at time s, one unit of money plus an unknown interest payment equal to the return on a discount bond issued at time T and maturing with unit value at time T + μ.

By well-known properties of conditional expectation, it is possible to derive the alternative representation for β :

$$\beta(r,t;T,s,\mu) = E_t\left[\frac{v(r,T;T+\lambda)}{v(r,T;T+\mu)}\, e^{-S(t,T)}\right], \tag{21}$$

where A is the indexation time lag previously defined [6]. It will immediately be noted that if $\lambda = \mu$, then :

$$\beta(r,t;T,s,\lambda) = v(r,t;T);$$

therefore :

$$P(r,t;T,s,K) = C\ v(r,t;T),$$

with basis risk :

$$\Omega(r,t;T,s,K) = \phi(r,t;T)$$

and stochastic duration given by :

$$D = s-t-\mu = T-t.$$

Hence we can deduce that when indexation lag is equal to the indexation maturity μ, the stochastic zero-coupon bond with maturity s-t and terminal value K = C + I is equivalent, both in price (value) and duration (risk) to a deterministic discount bond with terminal value C and maturity s - t - μ.

Let us now consider, at time t, a coupon bond with face value C, which entitles the holder to the payments I_1, I_2, ..., I_m + C, to be received on the dates :

$$t_k = t+k\theta, \quad k=1,2,...,m,$$

where θ (8> 0) is the constant time interval between subsequent payments.

The basic framework for modelling Italian Treasury Credit Certificates is obtained by assuming that the coupon payments are determined as :

$$I_k = \begin{cases} C\left[\dfrac{1}{v(T_k,T_k+\mu)} - 1 + \sigma\right] & \text{if} \quad T_k > t, \\[2ex] C\left[\dfrac{1}{v(t,t+\mu)} - 1 + \sigma\right] & \text{if} \quad T_k \le t, \end{cases} \tag{22}$$

where:

$$T_k = t_k - \lambda \ , \ k=1,2,\ldots,m,$$

and $\mu>0$, $\lambda>0$, $\sigma\geq 0$.

According this rule, each coupon is linked to the price of Treasury-Bills of maturity μ, observed λ units of time before each payment; the coupon value I_k is determined by the yield rate $i(T_k, \mu)$ plus a margin σ ("spread").

This floating rate bond can be indicated by the vector:

$$X = (t_1, t_2, \ldots, t_m, I_1, I_2, \ldots, I_m+C)$$

and can be considered as a portfolio of stochastic zero-coupon bonds. The price function is given by:

$$(23) \quad W(r,t;X) = \sum_{k=1}^{m} P(r,t;T_k,t_k,I_k) + C\, v(r,t;t_m) =$$

$$= C\,\{ \sum_{k=1}^{m} [\beta(r,t;T_k,t_k,\mu)-(1-\sigma)v(r,t;t_k)] + v(r,t;t_m)\}.$$

By using the ex-ante value [7] of stochastic coupon I_k:

$$(24) \quad z(r,t;T_k,t_k,I_k) = \frac{P(r,t;T_k,t_k,I_k)}{v(r,t;t_k)} \ , \ k=1,2,\ldots,m,$$

the bond price can be expressed as:

$$(25) \quad W(r,t;X) = \sum_{k=1}^{m} v(r,t;t_k)\ z(r,t;T_k,t_k,I_k) + C\, v(r,t;t_m).$$

In deriving the basis risk $\Omega(r,t;X)$ of the floating rate bond X, we must recall that if $T_k \leq t$ the ex-ante coupon is independent of r. Hence (by using shortened notation) we obtain:

$$(26) \quad \Omega(r,t;X) = -\frac{W_r(r,t;X)}{W(r,t;X)} =$$

$$= -\frac{1}{W}\,(\sum_{k=1}^{m} v_r(t_k)z(t_k) + Cv_r(t_m) + \sum_{T_k>t} v(t_k)z_r(t_k)) .$$

The (total) basis risk $\Omega(r,t;X)$ can be expressed as the discount risk $\Phi(r,t;X)$ plus the coupon risk $\Gamma(r,t;X)$:

$$\Omega(r,t;X) = \Phi(r,t;X) + \Gamma(r,t;X),$$

where

$$(27) \qquad \Phi(r,t;X) = -\frac{1}{W}\left[\sum_{k=1}^{m} \phi(t_k)v(t_k)z(t_k) + \phi(t_m)v(t_m)C\right]$$

and :

$$(28) \qquad \Gamma(r,t;X) = -\frac{1}{W}\sum_{T_k>t} v(t_k)z(t_k) \ .$$

When $\lambda = \mu = \theta$ and $a = 0$ [8], we have, for each k :

$$\beta(r,t;T_k,t_k,\lambda) = v(r,t;T_k) = v(r,t;t_{k-1}),$$

where t_0 = t. Then the ex-ante *coupon* is :

$$(29) \quad z(r,t;T_k,t_k,I_k) = C\left[\frac{v(r,t;t_{k-1})}{v(r,t;t_k)} - 1\right] = C\, j(t;t_{k-1},t_k),$$

where j(t ; T, s) denotes the forward rate at time t, for the period between T and s.

By equation (23) the price is :

$$W(r,t;X) = C;$$

so this variable rate bond is priced at par, as a fixed rate bond with "forward coupons", that is with interest payments corresponding to the forward interest rates implied by the term structure at time t and operating in the intervals between subsequent coupon dates.

For the basis risk (by using the expression (29) in (27) and (28)) we obtain :

$$\Omega(r,t;X) = \phi(r,t;t_1),$$

which corresponds to the stochastic duration :

$$D = t_1 - t = \theta.$$

5 - VALUING VARIABLE RATE BONDS BY THE COX, INGERSOLL AND ROSS MODEL

To derive analytical solution and numerical results for variable rate bonds price and risk, we require a specific model for the state variable and the market price of risk. We use the single factor version of the Cox, Ingersoll and Ross (CIR) model [9].

The infinitesimal parameters of the diffusion process (r (t)) are of the form :

$$(30) \qquad f(r,t) = \alpha(\gamma - r), \quad \alpha, \gamma > 0 \ ,$$

$$(31) \qquad g(r,t) = \rho\sqrt{r}, \qquad \rho > 0 \ ;$$

moreover it is assumed that [10]:

$$(32) \quad q(r,t) = \pi\sqrt{r}/\rho \quad , \pi \text{ constant.}$$

Under these assumptions, the fundamental valuation equation (6) takes on the form :

$$(33) \quad P_t + [\alpha\gamma - (\alpha-\pi)r]P_r + \frac{1}{2}\rho^2 r P_{rr} - rP = 0.$$

The expression of v (r, t ; s) is obtained by solving this equation under the boundary condition v (r, s ; s) = 1. We can write the solution as :

$$(34) \quad v(r,t;s) = A(t,s)\ e^{-rB(t,s)} ,$$

where :

$$A(t,s) = \left(\frac{2d\ e^{(\alpha-\pi+d)(s-t)/2}}{(\alpha-\pi+d)[e^{d(s-t)}-1] + 2d}\right)^{(2\alpha\gamma/\sigma^2)}$$

$$B(t,s) = \frac{2[e^{d(s-t)} - 1]}{(\alpha-\pi+d)[e^{d(s-t)}-1] + 2d}$$

and

$$d = \{(\alpha-\pi)^2 + 2\rho^2\}^{1/2}.$$

When time to maturity $\tau = s - t$ tends to infinity, yield to maturity h (t, s) approaches a constant h_∞ that is independent of the current kvel of the spot rate : $h_\infty = 2a/(d + a - \pi)$.

Also the basis risk of a deterministic zero-coupon bond is independent of r (t) :

$$(35) \quad \phi(r,t;s) = -\frac{v_r(r,t;s)}{v(r,t;s)} = B(t,s).$$

The explicit for of the β function can be obtained by solving equation (33) under the boundary condition :

$$(36) \quad \beta(r,T;T,s,\mu) = \frac{v(r,T;s)}{v(r,T;T+\mu)} ,$$

which is derived by expression (21) with t = T. The solution is given by [11]:

$$(37) \quad \beta(r,t;T,s,\mu) = M(t;T,s,\mu)\ e^{-rN(t;T,s,\mu)} ,$$

Where:

$$N(t;s) = \frac{2(e^{d(T-t)} - 1) - \dfrac{B(T,s)}{B(T,T+\mu)}[(a-r-d)(e^{d(T-t)}-1) - 2d]}{(\alpha-\pi+d)(e^{d(T-t)}-1) + 2d + \rho^2\dfrac{B(T,s)}{B(T,T+\mu)}[e^{d(T-t)}-1]} ,$$

$$N(t;s) = \frac{A(T,s)}{A(T,T+\mu)} \cdot \left[\frac{2d\, e^{(\alpha-\pi+d)(T-t)/2}}{(\alpha-\pi+d)\,[e^{d(T-t)}-1]+2d+\varrho^2 \dfrac{B(T,s)}{B(T,T+\mu)}\,[e^{d(T-t)}-1]} \right]^{2\alpha\gamma/\sigma^2} .$$

It can be immediately seen that the basis risk of the stochastic zero-coupon bond which pays the amount K = C + I at time s (see expression (20)) is given by :

$$\Omega(r,t;T,s,K) = N(t;T,s,\mu)$$

(independent of r (t)). Moreover, the inequality can be shown :

$$N(t;T,s,\mu) \leq B(t;s),$$

which reduces to an equality only if T = t.

By using solution (34) and (37) we can derive analytical expressions for price, risk and ex-ante values of variable rate bonds, as defined in section 4.

Numerical results are obtained when the parameters of the CIR model are specified.

6 - ESTIMATION PROCEDURE

The estimation procedure follows the method proposed by Brown and Dybvig [1986], which performs a best fitting of the one-factor CIR model to cross-sections of default-free coupon bond prices. According Brown and Dybvig, we express the functions A(t,s) and B(t,s) under the alternative form :

$$A(t,s) = \left(\frac{d\, e}{d_1\,[e^{d(s-t)}-1] + d}\right)^{\nu}$$

$$B(t,s) = \frac{e^{d(s-t)} - 1}{d_1\,[e^{d(s-t)}-1] + d}$$

where :

$$d_1 = (\alpha-\pi+d)/2,$$

$$\nu = 2\alpha\gamma/\varrho^2 .$$

The model price W (t) of any default-free coupon bond with deterministic cash-flows will be expressed as a function of the parameters d, d_1, ν and r. All these parameters can be estimated on the basis of data on the prices W *(t) of Treasury bonds, trading at time t.

By assuming that the deviations between W (t) and W* (t) are i.i.d. zero-mean normal random variables, it is possible to obtain maximum likelihood estimates of d, d_1, υ and r by nonlinear regression procedures applied to the prices of bonds of different maturities quoted at time t.

Even if the estimated value of ρ can be obtained from d, d1 and υ by the relation $\rho^2 = 2 (d\, d_1 - d_1^{\,2})$, it is possible to solve only for the parameters a - a and a Υ, rather than far a, Υ and π separately. Nevertheless the results of the estimation procedure can be used to price any IRS financial claim, since the valuation equation (33) can be expressed in term of d, d_1 and u. In particular, it will be possible to perform computations of the β values by modifying equation (37) according to the Brown and Dybvig notation.

The estimation is based on the prices of Treasury bonds (Buoni del Tesoro Poliennali, BTP) quoted on the secondary market at the same date [12]. We consider prices from Milan Stock Exchange on 24 July 1989 ; for details see Table 1.

7 - RESULTS

The results of non-linear regression procedure f a the estimation of the parameters d, d_1, υ and the spot rater are as follows:

$d = 0{,}2575$

$d_1 = 0{,}2504$

$\upsilon = 18{,}9416$

$r = 0{,}1014$

The resulting values of ρ^2 (the underlying process variance) and of "long term rate" h_∞ are:

$\rho^2 = 0{,}0035$

$h_\infty = 0{,}1344$

The yield curve corresponding to the estimated parameters is drawn in Figure 1 ; the structure of semiannual spot interest rates and the implied forward rates are listed in Table 2.

The estimated parameters can be used to perform the analysis of a typical Treasury credit certificate with variable coupon as if issued on the estimation date (29 july 1989). We consider a tax-exempt CCT with ten-year maturity ; the annual coupon is calculated on the basis of the average yield on twelve month Treasury-Bills at the auctions held in the two months ending one month before the day in which entitlement begins. The coupon value is obtained by multiplying the redemption value (100 lire) by this average rate of interest.

TABLE 1

	maturity	coupon rate %	tax rate %	issue price (lire)	market price (lire)
1	01-01-90	12.50	0.00	98.75	100.65
2	01-01-90	9.25	6.25	98.75	99.10
3	01-02-90	12.50	0.00	98.75	101.10
4	01-02-90	9.25	6.25	98.75	98.85
5	01-03-90	12.50	0.00	99.00	101.10
6	01-03-90	9.15	6.25	98.75	98.75
7	01-03-90	10.50	12.50	98.75	98.95
8	15-03-90	10.50	12.50	98.85	98.95
9	01-04-90	12.00	0.00	99.50	100.70
10	01-04-90	9.15	6.25	98.75	98.35
11	01-04-90	10.50	12.50	99.25	98.80
12	15-04-90	10.50	12.50	98.85	98.90
13	01-05-90	10.50	0.00	99.25	99.70
14	01-05-90	9.15	6.25	98.75	98.45
15	01-05-90	10.50	12.50	99.00	98.75
16	18-05-90	10.50	12.50	99.00	98.75
17	01-06-90	10.00	0.00	99.75	99.60
18	01-06-90	9.15	6.25	98.75	98.15
19	16-06-90	10.50	12.50	99.00	98.60
20	01-07-90	9.50	0.00	99.00	99.00
21	01-07-90	10.50	6.25	99.75	99.35
22	01-07-90	11.00	12.50	99.15	98.85
23	01-08-90	9.50	0.00	99.00	98.70
24	01-08-90	10.50	6.25	99.00	98.85
25	01-08-90	11.00	12.50	99.15	98.80
26	01-09-90	9.25	0.00	99.50	98.25
27	01-09-90	11.25	12.50	99.00	98.80
28	01-09-90	11.50	12.50	99.10	98.95
29	01-10-90	9.25	6.25	98.50	97.40
30	01-10-90	11.50	12.50	99.50	99.20
31	01-10-90	11.50	12.50	99.10	98.85
32	01-11-90	9.25	6.25	98.75	97.35
33	01-12-90	9.25	6.25	98.75	97.55
34	01-03-91	12.50	0.00	99.00	102.30
35	01-11-91	11.50	12.50	99'.10	97.35
36	21-12-91	11.50	12.50	99.25	97.65
37	01-01-92	9.25	6.25	98.75	93.50
38	01-02-92	9.25	6.25	98.75	93.40
39	01-02-92	11.00	12.50	99.00	95.40
40	01-03-92	9.15	6.25	98.75	92.95
41	01-04-92	9.15	6.25	98.75	93.90
42	01-04-92	11.00	12.50	98.00	95.20
43	01-05-92	9.15	6.25	98.75	93.85
44	01-05-92	11.00	12.50	97.50	95.00
45	01-06-92	9.15	6.25	98.75	95.50
46	01-07-92	10.50	6.25	99.75	99.35
47	01-07-92	11.50	12.50	98.35	96.60
48	01-08-92	11.50	12.50	97.40	96.15
49	01-09-92	12.50	12.50	98.80	97.65
50	01-10-92	12.50	12.50	98.80	97.65
51	01-11-93	12.50	12.50	99.10	97.05
52	17-11-93	12.50	12.50	100.20	96.95
53	01-01-94	12.50	12.50	99.80	97.50

FIGURE 1

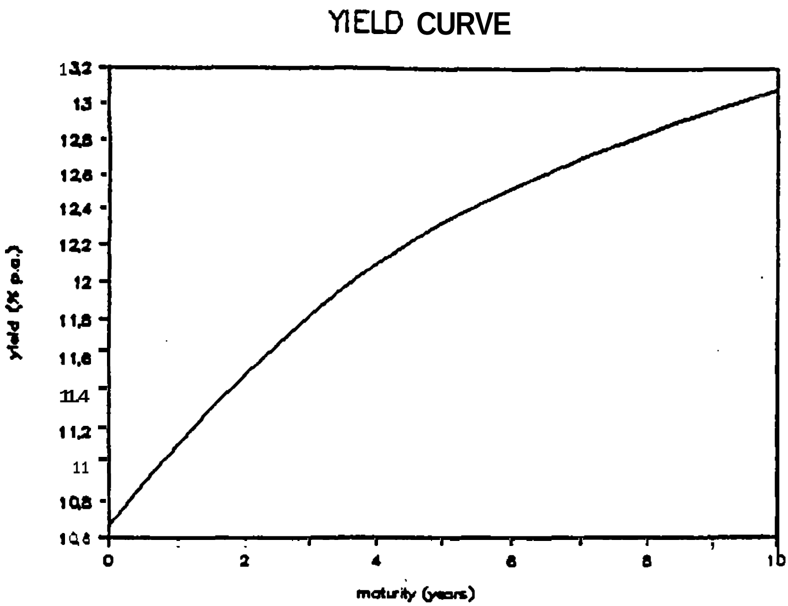

TABLE 2

maturity (years)	spot rate (%)	forward rate (%)
0.5	5.3127	5.3127
1.0	5.4156	5.5185
1.5	5.5098	5.6984
2.0	5.5958	5.8543
2.5	5.6747	5.9909
3.0	5.7471	6.1100
3.5	5.8137	6.2140
4.0	5.8750	6.3048
4.5	5.9314	6.3841
5.0	5.9836	6.4541
5.5	6.0318	6.5150
6.0	6.0764	6.5683
6.5	6.1177	6.6144
7.0	6.1561	6.6564
7.5	6.1917	6.6921
8.0	6.2248	6.7231
8.5	6.2557	6.7512
9.0	6.2845	6.7748
9.5	6.3114	6.7965
10.0	6.3366	6.8159

This indexation device is quite realistic; by adding a spread of 50 or 75 basis points, bonds actually traded on the Italian market can be identified. The definition of the indexation variable as an average yield does not essentially complicate the valuation formulas, derived in section 4.

The main features of the bond are shown in Table 3. The cash-flow schedule is analyzed by *calculating* for each payment the model present-value, the ex-ante value and the risk components.

TABLE 3

COUPON.

payment date	P	z	Ω	Φ	Γ
24/07/90	10.0068	11.1200	0.8777	0.8777	0.0000
24/07/91	9.3223	11.5907	-5.3353	1.5687	-6.9040
24/07/92	8.7112	12.1812	-3.0643	2.1119	-5.1762
24/07/93	8.0022	12.6345	-1.3815	2.5385	-3.9201
24/07/94	7.2603	12.9820	-0.1161	2.8735	-2.9896
24/07/95	6.5284	13.2504	0.8457	3.1367	-2.2910
24/07/96	5.8307	13.4570	1.5817	3.3435	-1.7619
24/07/97	5.1815	13.6169	2.1480	3.5064	-1.3584
24/07/98	4.5869	13.7396	2.5855	3.6349	-1.0494
24-07-99	4.0487	13.8346	2.9249	3.7367	-0.8118

REDEMPTION VAWE

payment date	P	z	Ω	Φ	Γ
24-07-99	29.2649	100.0000	3.6809	3.6809	0.0000

BOND

P(X) = 98.7440   D(X) = 0.8745

Ω(X) = 0.7874   Φ(X) = 2.8421   Γ(X) = -2.0548

The immunizing effect of the indexation is illustrated by the coupon risk which is negative f a **non-deterministic** cash-flows.

It might be interesting to **compare** this floating rate bond with "hypothetical" **bonds,** with the same payment dates and face value, but with **fixed** coupons. In Table 4, we **make** the comparison with four **different fixed-rate** bonds.

TABLE 4

		FIXED RATE				VARIABLE RATE
payment date	forward rate	forvard coupon	par coupon	ex-ante coupon	flat coupon	coupon
24/07/90	0.1112	11.1244	12.7567	11.1200	11.1200	11.1200
24/07/91	0.1189	11.8863	12.7567	11.5907	11.1200	I_2
24/07/92	0.1247	12.4670	12.7567	12.1812	11.1200	I_3
24/07/93	0.1291	12.9106	12.7567	12.6345	11.1200	I_4
24/07/94	0.1325	13.2503	12.7567	12.9820	11.1200	I_5
24/07/95	0.1351	13.5112	12.7567	13.2504	11.1200	I_6
24/07/96	0.1371	13.7111	12.7567	13.4570	11.1200	I_7
24/07/97	0.1387	13.8651	12.7567	13.6169	11.1200	I_8
24/07/98	0.1398	13.9834	12.7567	13.7396	11.1200	I_9
24-07-99	0.1408	14.0756	12.7567	13.8346	11.1200	I_{10}
	Price	100.0000	100.0000	98.7440	90.9246	98.7440
	Risk	2.8341	2.7833	2.8364	2.8204	0.7874
	Duration	4.8811	4.7131	4.8888	4.8350	0.8745

The cash-flow schedule of the **"ex-ante-coupon** bond (having deterministic interest payments equal to **the** ex-ante coupons) differs from the "forward-coupon bond" because the condition $\lambda = \mu = \theta$,we **considered** in section 4, is not completely matched. The ex-ante **coupon** is systematically lower than **the corresponding** forward coup, **as** *can* be seen from the fact that the model price is below par. The model price of the **"flat-**coupon bond (defined with coupons constantly equal to the first deterministic **coupon** of the variable rate bond) is **significantly** low, **as** it should be because the estimated forward rates dominate the flat level of 11.12%. The remarkable difference between **fixed** rate **bond and variable** rate bond durations shows clearly **the immunization** lost due **to** deterministic coupon streams.

In Table 5 **BTP's actual** prices are compared with estimated model **prices** (which are obtained by using equation (7) and by subtracting **accumulated** interests). **(see Table** 5) The mean absolute pricing error is around 0.3321 lire (compared with a typical market **price** of **100 lire)** and the standard **deviation** of the **error** is 0.5531.

The ability of the CIR model to fit the BTP actual market prices is fairly **good;** nevertheless, also in **this case "by itself, goodness** of fit in **prices** does not appear to provide a good **test** of **whether the** model has any **economic** content, or whether... it is merely a complicated method of **curve** fitting" **[Brown, Schaefer,** 1988, p. 20] [13].

TABLE 5

	maturity	market price (lire) (A)	model price (lire) (B)	difference (A-B)	stochastic duration (years)
1	01-01-90	100.6500	100.8235	-0.1735	0.438
2	01-01-90	99.1000	99.1327	-0.0327	0.439
3	01-02-90	101.1000	100.8421	0.2579	0.467
4	01-02-90	98.8500	98.8854	-0.0354	0.483
5	01-03-90	101.1000	101.0493	0.0507	0.545
6	01-03-90	98.7500	98.7413	0.0087	0.561
7	01-03-90	98.9500	99.0151	-0.0651	0.558
8	15-03-90	98.9500	98.9666	-0.0166	0.597
9	01-04-90	100.7000	100.8166	-0.1166	0.633
10	01-04-90	98.3500	98.5377	-0.1877	0.647
11	01-04-90	98.8000	98.9160	-0.1160	0.645
12	15-04-90	98.9000	98.8102	0.0898	0.684
13	01-05-90	99.7000	99.8484	-0.1484	0.725
14	01-05-90	98.4500	98.3879	0.0621	0.733
15	01-05-90	98.7500	98.7849	-0.0349	0.730
16	18-05-90	98.7500	98.7145	0.0355	0.779
17	01-06-90	99.6000	99.3955	0.2045	0.817
18	01-06-90	98.1500	98.1902	-0.0402	0.822
19	16-06-90	98.6000	98.5727	0.0273	0.864
20	01-07-90	99.0000	98.9239	0.0761	0.908
21	01-07-90	99.3500	99.2106	0.1394	0.907
22	01-07-90	98.8500	98.9370	-0.0870	0.907
23	01-08-90	98.7000	98.7104	-0.0104	0.903
24	01-08-90	98.8500	98.9776	-0.1276	0.899
25	01-08-90	98.8000	98.7326	0.0674	0.901
26	01-09-90	98.2500	98.3258	-0.0758	0.993
27	01-09-90	98.8000	98.8169	-0.0169	0.987
28	01-09-90	98.9500	99.0500	-0.1000	0.985
29	01-10-90	97.4000	97.4698	-0.0698	1.086
30	01-10-90	99.2000	99.0068	0.1932	1.073
31	01-10-90	98.8500	98.9627	-0.1127	1.073
32	01-11-90	97.3500	97.2973	0.0527	1.178
33	01-12-90	97.5500	97.1191	0.4309	1.268
34	01-03-91	102.3000	102.0989	0.2011	1.339
35	01-11-91	97.3500	97.5670	-0.2170	1.943
36	21-12-91	97.6500	97.4099	0.2401	2.112
37	01-01-92	93.5000	94.4817	-0.9817	2.181
38	01-02-92	93.4000	94.1889	-0.7889	2.036
39	01-02-92	95.4000	96.1975	-0.7975	2.000
40	01-03-92	92.9500	93.8215	-0.8715	2.133
41	01-04-92	93.9000	93.5832	0.3168	2.233
42	01-04-92	95.2000	95.8285	-0.6285	2.194
43	01-05-92	93.8500	93.3985	0.4515	2.334
44	01-05-92	95.0000	95.6643	-0.6643	2.297
45	01-06-92	95.5000	93.1723	2.3277	2.440
46	01-07-92	99.3500	96.1467	3.2033	2.507
47	01-07-92	96.6000	96.5455	0.0545	2.500
48	01-08-92	96.1500	96.2375	-0.0875	2.279
49	01-09-92	97.6500	98.4762	-0.8262	2.343
50	01-10-92	97.6500	98.3949	-0.7449	2.447
51	01-11-93	97.0500	97.3049	-0.2549	3.059
52	17-11-93	96.9500	97.3499	-0.3999	3.123
53	01-01-94	97.5000	97.2213	0.2787	3.308

The previous argument is weakened when the parameters of the CIR model are used to perform evaluations on a segment of market different from the one in which they have been estimated. We consider a particular class of CCTs with five-year maturity at issue and semiannual coupons [14]. On 24 July 1989, twelve different issues of this type of CCT were traded on the market. In Table 6 details are given of the market characteristics of these bonds, as well as model prices and stochastic durations. The data was taken from the Milan Stock Exchange and from the screen-based market (Mercato Telematico). The prices were either the trading prices or the average between the bid and asking price quotations where trading prices were not available.

TABLE 6

	maturity	first coupon (lire)	second coupon (lire)	issue price (lire)	market price (lire) (A)	model price (lire) (B)	difference (A-B)	stochastic duration (years)
1	01-05-93	6.40		99.2500	98.2500	98.7212	-0.4712	0.4419
2	01-06-93	6.70		99.2500	98.9500	98.9314	0.0186	0.5244
3	01-07-93	6.60		99.2500	99.0000	98.9601	0.0399	0.6072
6	01-08-93	6.10	6.35	99.2500	98.1000	98.7914	-0.6914	0.6546
5	01-09-93	6.10		99.1000	97.5500	98.5750	-1.0250	0.6202
6	01-10-93	6.10		99.1000	98.0000	98.4985	-0.4985	0.4775
7	01-11-93	6.40		99.1000	98.0500	98.5163	-0.4663	0.4497
8	01-01-91	6.60		99.1000	98.0800	98.7589	-0.6788	0.6149
	01-02-94	6.25	6.35	99.1000	97.7300	98.5958	-0.8658	0.6609
	03-94	6.25		98.0000	97.2500	98.3960	-1.1460	0.6239
	. 03-96	6.50		97.7500	97.2400	98.3920	-1.1520	0.5467
12	01-05-94	6.50		97.5000	97.8500	98.3611	-0.5111	0.4793

The model appears to overestimate the actual prices (the mean absolute error is around 0.6554 lire, with standard error near 0.3785). This overpricing might confirm the hypothesis that the market considers the twelve-month Treasury-Bills auction yield non-consistent with the prevailing term structure. One might even argue that this evidence is due to investors propension to overestimate the CCT basis risk (as would be shown by using the standard weighted average maturity means of calculating duration) [15].

The values of stochastic duration are rather close to six months (as they should be).

Since Treasury Bills are issued twice a month (in the middle and at the end of each month), the coupon value is determined by four price observations and become fully known abut seven months before the coupon is paid. For this reason, two of the listed bonds have two deterministic coupon payments ; these bonds exhibit the highest values of duration, because the higher the number of fixed-rate payments, the lower is the indexation device efficiency. Auction price observations also cause dramatic changes in the expectation of coupon payments and therefore in the indexation device efficiency. The non-monotonic behaviour of duration values with respect to time to maturity (apparent in Table 6) reflects this effect.

FOOTNOTES

1. Today CCTs are the most popular security on the Italian financial market. At the end of 1988 Treasury credit certificates in circulation amounted to 346.3 trillion lire, equal to 43% of the total stock of Government securities.

2. If h (t, s) is the yield to maturity of a zero-coupon bond maturing at time s ($t \leq s$), then:

$$r(t) = \lim_{s \to t} h(t,s).$$

3. By considering v (r, t ; s) as a function of s, we can define the yield curve :

$$h(r,t;s) = -\log v(r,t;s)/(s-t);$$

hence, by solving the valuation equation with *boundary* condition v (s ; s) = 1 we obtain the equilibrium term structure of interest rates at time t, which is completely specified when the current value **r** (t) of *the spot* rate is observed.

4. By equation (4), on a cross-section of bonds this measure is proportional to the standard deviation of returns.

5. For example, we can consider an European call option written on a bond, with exercise price K and expiration time s. The function F (.) will be determined by the terminal condition:

$$X=\max\{P^*-K,0\},$$

where $P^* = P(r(s), s; c)$ is the value of the underlying bond at time s.

6. The equivalence of expressions (18) and (21) is proven in Castellani, De Felice, Moriconi [1988], pp. 6-7.

7. Barone and Cesari [1986] perform an analysis of CCTs by assuming that, when $\mu = \theta$, the coupon payment at time t_k is determined as the forward rate j (t ; T_k, t_k) for the period from T_k to t_k. In *our* notations they use the assumption :

$$\beta(r,t;T_k,t_k,\theta)=v(r,t;T_k), \text{ which leads to:}$$

$$z(r,t;T_k,t_k,I_k) = C[\frac{v(r,t;T_k)}{v(r,t;t_k)} - 1] = C\, j(t;T_k,t_k) .$$

8. This case is explicitely considered by Cox, Ingersoll, Ross [1980], pp. 393-4.

9. The same model was used by the Bank of Italy [1989] to estimate the term structure of interest rates prevailing on the Italian market between December 1987 and March 1989.

10. As pointed out by Richard [1978] this assumption ensures that the risk adjustment is never unimportant nor dominating with respect to the drift.

11. The proof is given by Castellani [1988].

12 BTPs are medium-tern fixed rate Government securities with semi-annual coupons. They are negotiated on the secondary market at a price to which must be added accrued interest. This is calculated on a basis of 360 days per year (each calendar month to be considered thirty days), from and including the day from which interest is to accrue, up to and including the date of the transaction.

Until 1986, income from Government securities was tax exempt. Between September 1986 and August 1987 this income was liable to taxation, which hit interest coupons and the difference between the face value (100 lire) and issue price, at the rate of 6.25 per cent. After August 1987, the tax rate was increased to a rate of 12.50 per cent. Taxes are withheld at source.

13. Barone, Cuoco and Zautsik [1989] perform a comparison between yield curves derived on the Italian market from the CIR model and those obtained by using cubic splines.

14. The coupon is calculated on the basis of a semi-annual rate of interest equivalent to the average yield on twelve-month Treasury-bills at the auctions held in the two months ending one month before the day in which entitlement begins. A 50 basis point spread is added. These bonds are subject to the withholding tax at the rate of 12.50% (CCTs are taxed in the same way as Treasury bonds).

We computed the conditional expectations of the semi-annual equivalent rate by performing suitable "linearizations". By a preliminary analysis based on comparison with numerical solutions of the valuation equation, these approximations seems not to affect the results appreciably.

15. The durations of CCTs in Table 6 published in financial newspapers are longer than 3 years.

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