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WARRANT VALUATION : A BINOMIAL APPROACH

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EVALUATION D'UN BON DE SOUSCRIPTION : UNE APPROCHE BINOMIALE

WARRANT VALUATION AND EQUITY VOLATILITY

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ABSTRACT

This paper shows how equity volatility is affected by the issue of equity warrants. In general, observed volatility in share yields is lower than volatility of returns on corporate assets. Moreover, this volatility is unstable and changes with the value of the company.

A method is proposed for simultaneously determining the volatility of equity and the equilibrium price f a warrants, which depends on this volatility. Dilution resulting from warrants is also shown to take effect progressively during the life of the warrants.

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1. INTRODUCTION

Warrants are usually treated as call options on the equity of the firm; to account for their dilution effect, the call value is adjusted by a factor equal to $1/(1+q)$, where q is the dilution factor.[1] This approach is correct only if the options are calculated for an identical firm which does not have warrants. Unfortunately, such an identical firm does not exist in reality. If, however, one tries to estimate the value of the call option for the equity of the firm with warrants, a major problem is encountered, namely: the distribution of the rates of returns of the equity is nonstationary as well as its estimated volatility.

In this paper we show how the volatility of equity is affected by the issue of warrants. In general, the observed volatility of equity will be smaller than the volatility of the rate of return distribution for the firm's assets. In addition it will be unstable, and vary with changes in the value of the firm.

A way to estimate the volatility of the firm, and to track the volatility of equity at each point in time will be suggested These results will be used to solve simultaneously for the volatility of equity and the value of the warrants (which is based on the volatility). In addition, we show that the diluting effect on the stock price occurs smoothly over the life of the warrant.

2. Assumptions:

An all equity firm (denoted by an asterisk), has initial assets at time 0 with a current value of V^*_0. The market value of assets follows a binomial distribution, and at time 1 it is expected to be either uV^*_0 or dV^*_0, where u-1 and d-1 are, respectively, the rates of

1. See for example, Galai and Schneller (1978).

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increase and decrease in value in any period. It is further assumed, without loss of generality, that the firm has a two-period horizon. Hence the assets values at time 1 are:

$$V^*_1 = \begin{cases} uV^*_0 = V^*_{11} \text{ with probability } g \\ dV^*_0 = V^*_{12} \text{ with probability } 1-g \end{cases} \quad (1)$$

and at time 2 the values are:

$$V^*_2 = \begin{cases} u^2V^*_0 = V^*_{21} \text{ with probability } g^2 \\ udV^*_0 = V^*_{22} \text{ with probability } 2g(1-g) \\ d^2V^*_0 = V^*_{23} \text{ with probability } (1-g)^2 \end{cases} \quad (2)$$

where V^*_{ij} is the value of the firm at time i in state of nature j.

The all equity firm has N^* shares outstanding at time 0, and each share is worth S^*_0 so that $N^*S^*_0 = V^*_0$.

Now, assume that an identical firm issues n warrants with maturity at time 2, which, when exercised, are converted into n new shares at the exercise price of \$X per share. The proceeds from selling the warrants are assumed to be invested in a scale expansion of the firm, thus not affecting the rates of return distribution on total assets. Hence, the firm with the warrants has an initial value of

$$V_0 = NS_0 + nW_0 \quad (3)$$

where So is the share price, Wo is the price per warrant, and N is the number of original shares. Under these assumptions, the rates of return in any period on the assets of the firm with warrants are also expected to be, like for the all equity firm, either u (with probability g) or d (with probability 1-g).

The problem is, then, to determine the equilibrium values Wo and So, provided the pure equity firm characteristics S^*_0, N^*, u and d, are known. However, pricing the warrants and the equity becomes more complicated than pricing conventional call options of stocks, since, when the warrants are exercised, there is a cash flow into the corporation, and simultaneously, the claim of the initial shareholders is diluted

3. The model

let us denote by So and Wo the current values of the stock and warrants respectively, and by S_{tj} and W_{tj} their values at time t (t = 1, 2) in state of nature j (j = 1, 2, 3). At maturity (t = 2), the value of equity. S_{2j} (j = 1, 2, 3), is given by :

$$S_{2j} = \text{Min}\left[\frac{V_{2j}}{N}, \frac{V_{2j} + nX}{N+n}\right] \quad (4)$$

The first term, V_{2j}/N, is the value of a share in the event the warrants are not exercised, which in this case is equal to the value of the firm divided by the N original shares. The second term, $(V_{2j} + nX) / (N+n)$, is the value of a diluted share, assuming all n warrants are exercised.[2] It is assumed that it is optimal to exercise the warrants at maturity in state 2 where

$$\frac{V_{2j} + nX}{N + n} > X$$

which is equivalent to the condition that the undiluted value per share is greater than the striking price

$$\frac{V_{2j}}{N} > X.$$

Taking the minimal value in (4) emphasizes the dilution effect.

In order to find the equity and warrant values of the firm at time 1, riskless portfolios, composed of these financial claims, and the shares of the pure equity firm, are formed. By eliminating arbitrage opportunities, the equilibrium values are found for the firm with the warrants.

3.1 The Value of Equity

For the f i i with warrants, the value of a share at time 1, in state 1 is:

$$S_{11} = PV(S_{21}, S_{22})$$

S_{11} is the present value (PV) at time 1 of the two possible future cash flows. Similarly, the value of a share at time 1, state 2, S_{12}, is given by

$$S_{12} = PV(S_{22}, S_{23})$$

Since at time 1 exercising the warrants is uncertain, the present value function must take it into consideration. This will be done by constructing a riskless portfolio, containing one share long of the firm with warrants, and a share(s) short of the pure equity firm.

To derive S_{11}, the portfolio $S_{11} + \alpha S^*_{11}$ is constructed, where $S^*_{11} = V^*_{11}/N^*$ is the value of a share in the pure equity firm. This portfolio yields at t = 2 the following cash flow:

$$S_{21} + \alpha S^*_{21} \text{ with probability } g$$

$$S_{22} + \alpha S^*_{22} \text{ with probability } (1-g)$$

2 We assumed the warrants are European and can only be exercised at maturity. The issue of early exercise of American-type warrants, developed by Emanuel (1983) and Constantinides (1984), is dividend policy driven, and as such is not discussed in our paper.

where $S_{2j}^* = V_{2j}^*/N^*$. This portfolio is riskless when[3]

$$S_{21} + \alpha S^*_{21} = S_{22} + \alpha S^*_{22}$$

which is the case when

$$\alpha = -(S_{21} - S_{22})/(S^*_{21} - S^*_{22}) \tag{5}$$

Note also that $S^*_{21} - S^*_{22} = uV_0^*\,(u-d)/N^*$. Since the investment in the portfolio is riskless, it should yield the risk-free interest rate, hence:

$$(S_{11} + \alpha S^*_{11})R = S_{21} + \alpha S^*_{21} \tag{6}$$

where R is one plus the risk-free interest rate. By substituting (5) in (6) we get

$$S_{11} = \frac{1}{R}\left[\frac{S_{21}\,(R-d) + S_{22}\,(u-R)}{(u-d)}\right] \tag{7}$$

In this formulation S_{11} is the present value of a weighted sum of the two relevant known future values S_{21} and S_{22}, with weights $(R-d)/(u-d)$ and $(u-R)/(u-d)$, respectively. These weights sum to 1, and, in mathematical terms, serve as probabilities. Therefore the term in the bracket can be regarded as an expected future value for the share at time 1, in state 1. The discounting is done at the risk-free rate since the risk of the share can be fully hedged by having a position in the shares of the pure equity firm. It is interesting to note that in valuing S_{11}, the scale parameters of the pure equity firm, as measured by N* and V^*_0, do not enter the model.

In a similar way S_{12} can be derived as:

$$S_{12} = \frac{1}{R}\left[\frac{S_{22}\,(R-d) + S_{23}\,(u-R)}{(u-d)}\right] \tag{8}$$

Also, the current value of a share at time 0 is given by

$$S_0 = \frac{1}{R}\left[\frac{S_{11}\,(R-d) + S_{12}\,(u-R)}{(u-d)}\right] \tag{9}$$

$$= \frac{1}{R^2}\left[\frac{S_{21}\,(R-d)^2 + 2S_{22}\,(u-R)(R-d) + S_{23}\,(u-R)^2}{(u-d)^2}\right]$$

3 See Cox, Ross and Rubinstein *(1979)*. They derive the general formula for calls and puts under the assumption of a stable and stationary binomial distribution.

If we denote $q = \frac{(R-d)}{(u-d)}$ and $(1-q) = \frac{(u-R)}{(u-d)}$ and substitute in (9).
we obtain

$$S_0 = \frac{1}{R^2}\left[S_{21}q^2 + 2S_{22}q(1-q) + S_{23}(1-q)^2\right] \tag{9}$$

At this stage the model should be applied to a more specific case. Let us assume that the warrants are exercised at time 2 only if V_{21} or V_{22} are realized; when V23 is realized, the warrant is worthless[4] In this scenario, the value of a share at time 2 is given by:

$$S_{2j} = \begin{cases} \frac{V_{21} + nX}{(N+n)} = \frac{u^2V_0 + nX}{(N+n)} & g^2 \\ \frac{V_{22} + nX}{(N+n)} = \frac{udV_0 + nX}{(N+n)} & 2g(1-g) \\ \frac{V_{23}}{N} = \frac{d^2V_0}{N} & (1-g)^2 \end{cases}$$

Substituting the values of S_{23} (j = 1, 2, 3) in equations (7), (8) and (9), the values of a share at time 1 and time 0 are given by

$$S_{11} = \frac{uV_0}{N+n} + \frac{nX}{(N+n)R} \tag{7'}$$

$$S_{12} = \frac{1}{R}\left[\frac{udV_0 + nX}{(N+n)}\frac{(R-d)}{(u-d)} + \frac{d^2V_0}{N}\frac{(u-R)}{(u-d)}\right]$$

$$= \frac{1}{R(N+n)}\left[dV_0\left(R + \frac{nd}{N}(1-q)\right) + nXq\right] \tag{8'}$$

and

$$S_0 = \frac{V_0}{(Ntn)} + \frac{V_0d^2(1-q)^2n}{R^2(N+n)N} + \frac{nXq(2-q)}{R^2(N+n)} \tag{9'}$$

The value of equity is utility-independent and is a function of known parameters. It is not a function of the probabilities of the stock going up or down In (9') the current value of a share is presented as the value of a fully diluted stock, $V_0/(N+n)$, plus the expected present value of the diluted share in case the value of the firm declines and the warrants are not exercised, plus another term reflecting the present value of the probable cash-flow nX if warrants are exercised.

3 3 The Value of Warrants

Following a similar approach, the value of the warrant can be derived for each state of nature at each time period.

4 The "trees" f a the firm, equity and warrants are described in Appendix 1. the "trees" contain also the rate of increase, u_{tj} and u'_{tj}, and rates of decrease, 4j and d'_{tj} for equity and warrants respectively (t = 1, 2; j = 1.2.3)

At time 2 ($j = 1, 2, 3$) :

$$W_{2j} = \text{Max}\{S_{2j} - X, 0\} \tag{10}$$

At time 1 ($j = 1, 2$)

$$W_{1j} = \frac{1}{R}\left[\frac{W_{2j}(R-d) + W_{2,j+1}\,(u-R)}{(u-d)}\right] \tag{11}$$

And the current value Wo is given by

$$W_o = \frac{1}{R}\left[\frac{W_{11}(R-d) + W_{12}\,(u-R)}{(u-d)}\right] \tag{12}$$

For the specific case in which the warrants are exercised at time 2 in states 1 and 2 but not in 3, the values of the warrants are given by

$$W_{2j} = \begin{cases} W_{21} = S_{21} - X = \dfrac{u^2V_o - NX}{(N+n)} \\ W_{22} = S_{22} - X = \dfrac{udV_o - NX}{(N+n)} \\ W_{23} = 0 \end{cases} \tag{10'}$$

and

$$W_{1j} = \begin{cases} W_{11} = \dfrac{uV_o}{(N+n)} - \dfrac{NX}{R(N+n)} \\ W_{12} = \dfrac{1}{R}\left[\dfrac{udV_o - NX}{(N+n)}\right]\dfrac{(R-d)}{(u-d)} \end{cases} \tag{11'}$$

and the current value is

$$W_o = \frac{(R-d)}{R^2(N+n)(u-d)}\left[(RuV_o - NX) + (udV_o - NX)\frac{(u-R)}{(u-d)}\right]$$

$$= \frac{(R-d)}{R^2(N+n)(u-d)}\left[uV_o\left[R + d\,\frac{(u-R)}{(u-d)}\right] - NX\left[1 + \frac{(u-R)}{(u-d)}\right]\right] \tag{12'}$$

$$= \frac{q}{R^2(N+n)}\left[uV_o\,(R+d(1-q)) - NX(2-q)\right]$$

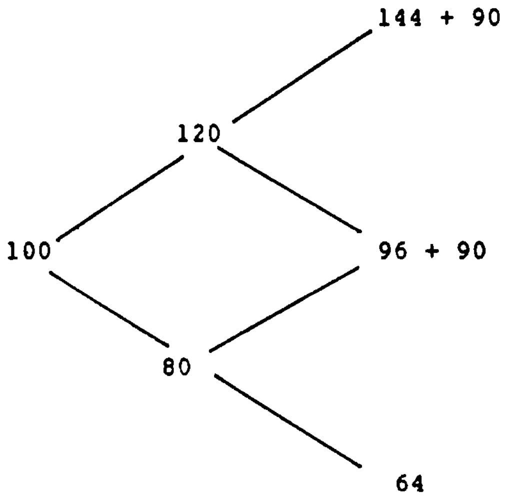
In this formulation, the current value of a warrant is a function of known parameters. The effect of the dilution is integrated in the formula by dividing the expected future value of the warrant by the post-exercise number of shares, $N + n$, and by subtracting in the expected future value the probable payment of NX for exercising the warrants.

4. A Numerical Example

Assume:

$V_0 = 100$
$u = 1.2$
$d = 0.8$
$R = 1.1$
$N = n = 1$
$X = 90$

The values for the firm are given by the following tree:



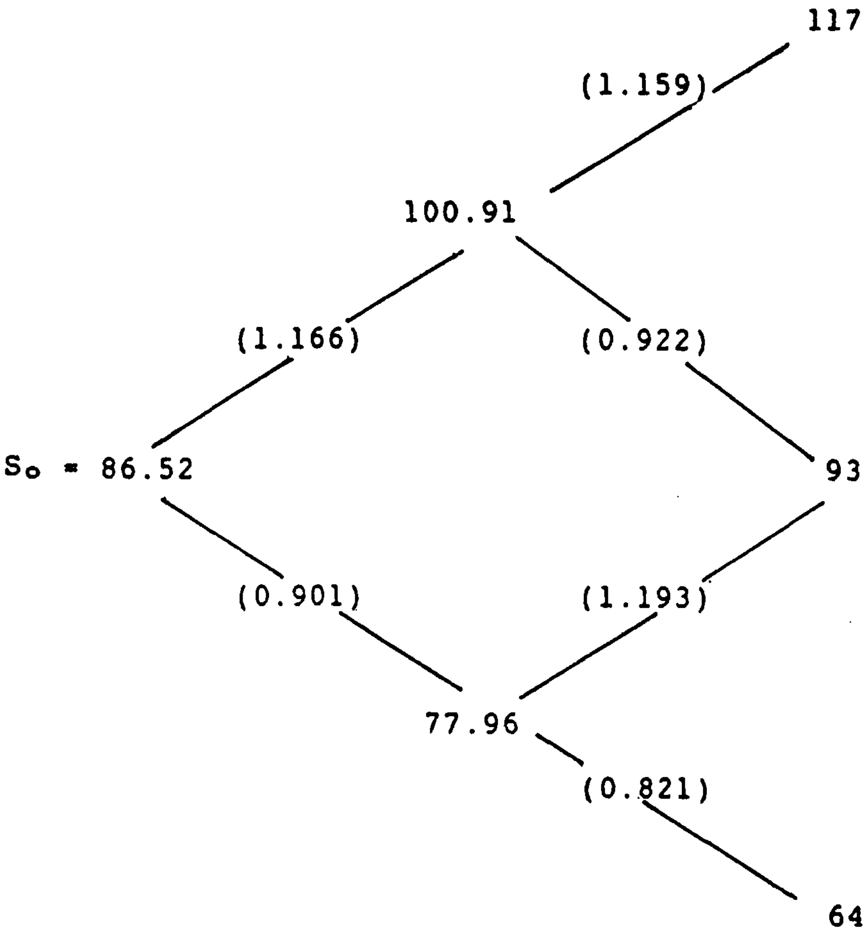
For the equity the values are given by

$$S_{23} = \begin{cases} \dfrac{144 + 90}{2} = 117 & j = 1 \\ \dfrac{96 + 90}{2} = 93 & j = 2 \\ 0 & j = 3 \end{cases}$$

$$S_{13} = \begin{cases} 100.91 & j = 1 \\ 77.96 & j = 2 \end{cases}$$

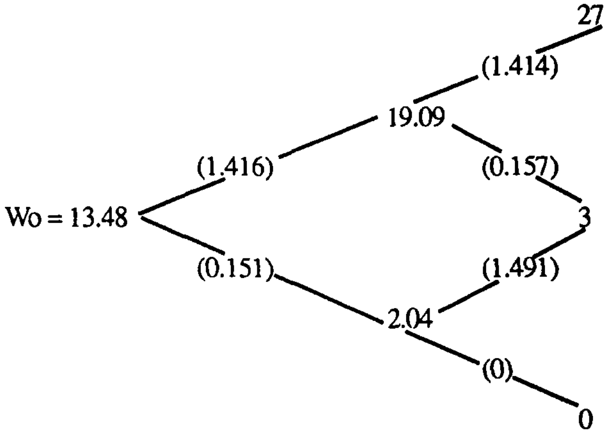
and $S_0 = 86.52$.

The "tree" for equity will then be



The change ratios from period to period are indicated in parentheses. It should be noted that, first, the rate of increase for the equity value is less than the rate of increase for the asset value of the firm, and second, the distribution of the equity value is *nonstationary*, i.e. the rate of increase, if the market initially goes up, is 15.9%, but 19.3% if the market initially goes down. This nonstationarity has important implications for measuring the volatility based on stock prices.

For the warrants the values are given by the following tree:



For the warrant, the volatility is larger than for the equity of the firm since the range between the rate of increase and decrease at each lime period is bigger. Again, the distribution is nonstationary.

It is easy to verify in the numerical example, that the value of a simple call option (with k = 90) on a share of a pure equity firm with V*o = 100, u = 1.20 and d = 0.80, is the same as the value of the warrant multiplied by a constant reflecting the dilution factor. Such a call has a present value of 26.98, which is twice the present value of the warrant, and the multiplication factor reflects the number of shares after the warrants are exercised?

It is to be noted also, that the value of a call option on a share of the firm with warrants is the same as the warrant. This can easily be verified by comparing the values at time 2 of the call and the warrant, both with a striking price of 90.

5. The Volatility of Equity

One of the conclusions of the numerical example is that the distribution of the equity value of a firm with warrants is nonstationary; it is different from period to period, and also, for a given time period it depends on the realized state-of-nature. In addition, it is shown that the volatility of the equity value is less than the volatility of the asset value.

In this section we formally prove these results for the binomial distribution. Far the firm with no warrants it is assumed, as before, that at each state, the asset value can either grow by a factor u or decrease in value by d

For the firm's equity, we use the following notations:

$$u_0 = \frac{S_{11}}{S_0}, \quad d_0 = \frac{S_{12}}{S_0}$$

and $$u_{11} = \frac{S_{21}}{S_{11}}, \; u_{12} = \frac{S_{22}}{S_{12}}, \text{ and } d_{11} = \frac{S_{22}}{S_{11}} \quad d_{12} = \frac{S_{23}}{S_{12}}$$

5 See Galai and Schneller (1978) where it is shown that

$$c = \frac{1}{1+q} v$$

where q is the ratio of new shares to old shares and measures the dilution of *existing* equity.

For the warrants the notations arc :

$$u_0' = \frac{V_{11}}{V_0}, \quad d_0' = \frac{V_{10}}{V_0}$$

and

$$u_{11}' = \frac{V_{21}}{V_{11}}, \; u_{12}' = \frac{V_{22}}{V_{12}}, \; \text{and } d_{11}' = \frac{V_{22}}{V_{11}}, \; d_{12}' = \frac{V_{23}}{V_{12}}$$

As a mesure of volatility, the ratio between the "up" and "down" factors u_{ti}/d_{ti} is used A smaller ratio means less dispersion and, hence, smaller volatility.

To prove that $d_{11} \geq d$ and $u_{11} \leq u$ we start with the definition of u_{11} as

$$u_{11} \equiv \frac{S_{21}}{S_{11}}$$

Replacing S_{11} by its valuation equation, we obtain

$$u_{11} = \frac{S_{21}}{\frac{1}{R}\left[S_{21}\frac{(R-d)}{(u-d)} + S_{22}\frac{(u-R)}{(u-d)}\right]} =$$

and since $S_{21} \equiv u_{11}\, S_{11}$ and $S_{22} \equiv d_{11}\, S_{11}$, then

$$= \frac{u_{11}R}{u_{11}\frac{(R-d)}{(u-d)} + d_{11}\frac{(u-R)}{(u-d)}}$$

By rearranging the expression for u_{11}, we get

$$R = u_{11}\frac{(R-d)}{(u-d)} + d_{11}\frac{(u-R)}{(u-d)}$$

But since

$$R = u\frac{(R-d)}{(u-d)} + d\frac{(u-R)}{(u-d)}$$

it must be that if $d_{11} \geq d$ then also $u_{11} \leq u$. The same approach can be used to show that if $d_{12} \geq d$ then $u_{12} \leq u$.

For the specific case where warrants are exercised if V_{21} or V_{22} is realized, it can be shown that $d_{11} \geq d$ and d_{12} 2 d. As a matter of fact, $S_{22} = d_{11}\, S_{11}$, and by substituting for S_{22} and S_{11}, we get

$$\frac{udV_o + nx}{(N+n)} = d_{11}\left[\frac{uV_o}{(N+n} + \frac{nx}{(N+n)R}\right]$$

However $d_{11} \leq R$ and therefore, in order to maintain the above equality, $d_{11} \geq d$. From the last expression, d_{11} can be expressed in terms of the known parameters as

$$d_{11} = \frac{udV_o + nX}{uV_o + nX/R} \tag{13}$$

It is interesting to note that $d_{11}/n \geq 0$ and therefore if potentially additional new shares are issued through warrants, the volatility of equity tends to decline.

For the secondnd state at time 1, we use the ratio of $u_{12} \equiv S_{22}/S_{12}$ and $d_{12} \square S_{23}/S_{12}$:

$$\frac{U_{12}}{d_{12}} = \frac{S_{22}}{S_{23}} = \frac{\dfrac{udV_o + nx}{(N+n)}}{\dfrac{d^2V_o}{N}}$$

and compare it to the ratio u/d. It can be shown that $u_{12}/d_{12} \leq u/d$ if $udVo > NXo$, and the latter inequality is a result of the assumption that $W_{22} > 0$(see (10'). In a similar way it can also be shown that u_0/d_0 is less than u/d. Moreover, the volatility is different at each state since $u_{11}/d_{11} \neq u_{12}/d_{12} \neq u_0/d_0$.

The equity of the firm issuing warrants is less risky (as measured by the ratio u/d, for example) than that of a pure equity firm with identical business risk. This is due to the bond component of the warrants, which can be considered as an asset of the firm. A warrant can be replicated in a dynamic framework by issuing additional shares and investing the proceeds in government bonds. This tends to reduce the risk of the assets of the firm and, hence of its equity.

An important conclusion of the above analysis is that a time series of rates of return on the equity of a firm with warrants cannot be used to estimate the volatility. The estimator will be meaningless due to the nonstationarity of the distribution for equity; and the nonstationarity problem will be more serious as a function of the dilution factor.

Based on this analysis it is expected that the volatility estimated from the time series of rates of return on the cquity will be a downward biased estimator for the volatility of the firm's assets. If this estimator is used in calculating the value of warrants, as is usually the case in our experience, the warrants will be undervalued.

6. Conclusions

This paper showed the major pitfalls in evaluating warrants. The problems stem from the fact that warrants have an impact on the capital structure of the firm, and hence on the process generating stock prices. This latter process is not expected to be stationary.

A simple binomial distribution for the value of the firm's assets is used to support the claims raised in the paper, though the claims are general and apply to alternative distributions.

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