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THE ASSESSMENT OF FINANCIAL RISK

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L'EVALUATION DU RIQUE FINANCIER

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RESUME

Cet article décrit un cadre général de mesure et d'évaluation du risque financier et montre comment il peut Ctre utilisé pour résoudre une large gamme de problèmes pratiques dans des domaines donnant lieu à des considérations de risque.

L'approche fondamentale est la suivante : le risque financier est une fonction à la fois de la probabilitt?de divers résultats indhirables et de la sévérité des conséquences financières découlant de ces résultants indhirables ; en faisant l'hypothèse qu'un décideur peut associer les conséquences des différents résultats à des termes numériques, cette approche de base conduit à une mesure du risque qui est la somme ou l'intégrale du produit d'une fonction de pondération du risque par la probabilité concernée. On suppose ensuite qu'un décideur choisit entre différents profils financiers, en se fondant sur deux mesures d'équivalence, à savoir cette mesure du risque et une valeur espérée généralisée.

Après avoir montré comment ce cadre général peut Ctre applique à trois domaines parfaitement séparés - à savoir : le choix des investissements, la determination du montant optimal de réassurance et le Paradoxe de Saint Pétersbourg -, on montre que cette solution est plus satisfaisante dans chacun de ces domaines d'application, que celle fondée sur la théorie de l'utilité.

Le cadre général est ensuite appliqué à trois autres domaines : la majoration des primes en assurance - vie, la majoration des primes en assurance non - vie et la fixation des prix des options. On montre en particulier que le modèle de fixation des prix des options de Black - Scholes peut Ctre dérivé sans faire l'hypothèse de la constitution d'une couverture parfaite.

THE ASSESSMENT OF FINANCIAL RISK

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SUMMARY

This paper describes a general framework for the measurement and assessment of financial risk and shows how it can be used to solve a wide range of practical problems in areas where considerations of risk arise.

The basic approach is that financial risk is a function involving both the likelihood of various unsatisfactory outcomes and the severity of the financial consequences arising from these unsatisfactory outcomes. On the assumption that a decision - maker can relate the consequences of different outcomes in numerical terms, this basic approach leads to a measure of risk which is the summation or integral of the product of a risk weighting function and the relevant probability. It is then assumed that a decision - maker will choose between different financial profiles on the basis of two measures of equivalence, namely this risk measure and a generalised expected value.

After demonstrating how the general framework can be applied to three quite separate areas, namely investment selection, the determination of the optimal amount of reinsurance and the St. Petersburg Paradox, it is shown that the solution in each of these application areas is more satisfactory than the corresponding solution based on utility theory.

The general framework is then applied to three further areas - premium loadings in life assurance, premium loadings in non - life insurance, and option pricing. In particular, it is shown that the Black - Scholes option pricing model can be derived without having to assume that a perfect hedge can be constructed.

INTRODUCTION

1 · 1 · There are two important and quite separate bodies of mathematical work that have been developed in the general area of financial risk : Capital Market Theory which is based on variance of short term return as the measure of risk, and what may be called the classical theory of risk as applied to non-life insurance, where the criterion of risk is that the proba lity of ruin is not to exceed some specified small amount. This paper describes a more general framework for the measurement and assessment of financial risk and shows how it can be used to solve a wide range of practical problems in areas where considerations of risk arise.

1 · 2 · When the formation of the AFIR Section within the IAA was being proposed, it was suggested that such a Section should only be set up if it could be shown that "the actuarial approach" could make a significant contribution in the area of financial risk. It will be shown that the general framework is indeed actuarial in nature in that the key principles are extensions of methods used in other areas of actuarial mathematics.

1 · 3 · Before describing the general framework, we shall specify three quite different types of problem that must be capable of being solved before the framcwork can be regarded as being sufficiently comprehensive.

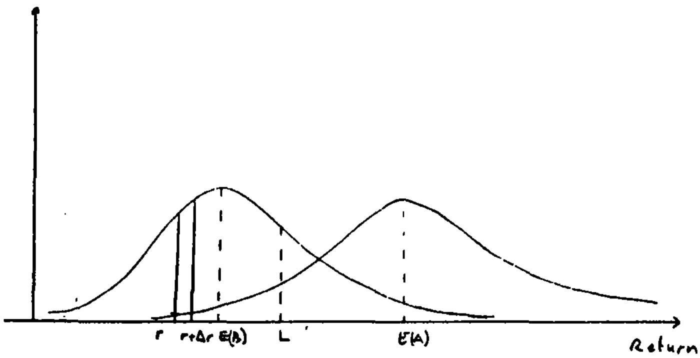
1 · 4 · The first' area relates to choosing between investment profiles of different securities or of different portfolios of securities. Given two different probability distributions, we require a method of determining which will be preferred by an investor whose circumstances and preferences are known. The pioneering paper of Markowitz (1952) introduced variance as the measure of investment risk. However, where portfolios with different expected returns are being compared, the variance of return alone cannot be used ; some "utility function" of expected return and variance is required. In mean - variance analysis (e.g. Markowitz (1987)). this utility approach is formalised in terms of the axioms put forward by Von Neumann and Morgenstem (1944).

1 · 5 · The second area relates to the practical management of an insurance company using a utility concept, or something equivalent, to formalise the consistency requirements that underlie various subjective judgments. Many actuaries have explored this type of methodology, but perhaps the clearest exposition of the general approach is that of Borch (1961). In particular, he explains that the optimisation criterion employed in the classical theory uses only two properties of the profit distribution - its mean and the area of that part to the left of the origin. He describes how the Von Neumann and Morgenstem axioms can be used to formulate a utility theory description of the situation and he then 'applies this to determine the optimal proportion of business to be reinsured in a simplified example. In an earlier paper, Borch (1960) discusses a generalised utility function U (E,V) which allows an insurance company to choose between profit profiles with different expected returns and variances of return. Bühlmann (1970) sets out the Von Neumann and Morgenstem utility approach in a more formalised manner as a possible stability criterion for the management of an insurance company.

1 · 6 · It is interesting to note that the Von Neumann and Morgenstern utility theory approach has been applied to these two apparently quite different areas. The key original work on utility was Bernoulli (1738) in which he demonstrated a method of tackling a problem which could not be solved using the traditional theory of probability, namely the St. Petersburg Paradox. We shall show how a solution to this problem can also be obtained using the general framework developed in the following section.

2 · THE GENERAL FRAMEWORK

2 · 1 · The basic approach, which was first described in Clarkson & Plymen (1988), is that financial risk is a function involving both the likelihood of various "unsatisfactory outcomes" and the severity of the financial consequences arising from these unsatisfactory outcomes. Suppose that two investments A and B are being compared, with $E(A) > E(B)$, and that the density function of A is smaller than the density function of B for all values of return less than some threshold L, as shown in Figure 1.



Fire1

If unsatisfactory outcomes are defined as those below L, then clearly investment A has lower risk than investment B, since the element of risk corresponding to any small element of return r to $r + \Delta r$, where $r < L$, is smaller for A than f a B

2 · 2 · This is, of course, only an ordering of risk in a very special case. For the general case, we require to construct, from first principles, a measure of risk using as few assumptions as possible. In Clarkson (1989), this construction is carried out using as the starting point an example based on the accident risk involved in various sports.

2 · 3 · The first step in this example is to apply a numerical scale to the severity of injury and to define the general nature of injury of a particular severity. We use severities 1, 2, 3 and 4 for minor injury, moderate injury. serious injury, and very serious injury respectively. Injury of severity 5 results in permanent incapacity and injuries of severity 6 and higher result in death.

2 · 4 · The second step is to specify for each sport the likelihood of the occurrence of each type of accident and the probabilities that these result in injury of a particular severity. In the case of windsurfing, an accident involves losing control and falling off the board ; there is very little risk of injury. A suitable model might be to say that the probability of injury of severity X corresponds to an outcome of X from a Poisson distribution with parameter 0.01. We also require to specify the likely number of "accidents" during a day's windsurfing, say 5. For hang-gliding, where an accident involves equipment failure a a stall when high above the ground, the likelihood of an accident occurring is very much smaller, but the consequences are clearly far more severe, usually involving very serious injury or death. Accordingly, we use in this case a Poisson parameter of 7 and a daily frequency of 1 in 5,000.

2 - 5 - The third and most important step is to postulate that an individual can relate the consequences of different severities of injury in numerical terms, e.g. the consequences of moderate injury are Y times as serious as those of minor injury. Suppose that in the present case an individual attaches a factor of ten to each increment in severity, so that he applies a weighting factor of 1 to severity 1, 10 to severity 2, 100 to severity 3, etc.

2 - 6 - We can now define a risk measure R as follows :

$$R = p(A) \sum_{X=1}^{\infty} W(X)p(X),$$

where p(A) is the probability that an accident occurs

p(X) is the probability that an accident results in injury of severity X,

and W(X) is the weighting factor for injury of severity X.

In the case of windsurfing we have :

$$R = 5 \times [1 \times 0.01 + 10 \times \frac{0.01^2}{2} + 100 \times \frac{0.01^3}{6} + 1\ 000 \times \frac{0.01^4}{24} + \dots]$$

$$= 0.052$$

Similarly f a hang - gliding we have :

$$R = \frac{1}{5{,}000} \times [1 \times 7 + 10 \times \frac{7^2}{2} + 100 \times \frac{7^3}{6} + 1\ 000 \times \frac{7^4}{24} + \dots]$$

$$= 14.260$$

In various situations where risk is involved, an individual will not wish to be involved in a situation where he perceives the risk to be above a certain threshold. This threshold is often highly intuitive, but will generally be related to some "standard" situation with which he is familiar. In the sports example, he may be aware of the pattern of outcomes for a "medium risk" sport with Poisson parameter 0.5 and be prepared to participate in that Sport subject to the probability of serious injury or worse being less than 1 in 1,000. This gives a threshold risk value of $R_0 = 0.480$, so that windsurfing is less risky than this standard whereas hang - gliding is more risky.

2 - 7 - In choosing between different sports, an individual will first of all exclude those with a higher level of risk than his risk threshold. A reasonable criterion f a the final choice, in the absence of other constraints such as cost or availability, is to select from the remaining group the sport with the highest "enjoyment value". Where two sports have the same "enjoyment value", the one with the lower risk will be preferred

2 - 8 - Clearly the most important step in this approach is to postulate that an individual can quantify his beliefs about risk to the extent that he can specify for any two outcomes A_1 and A_2 that the risk of A_1 is $X(A_1 ; A_2)$ times the risk of A_2. This is equivalent to saying that he is indifferent in risk terms, as between outcome A_1 with probability ΔP and outcome A_2 with probability $X(A_1 ; A_2)\, \Delta P$. If we take any outcome A_0 and assign to it an arbitrary positive value $R(A_0)$, we can then define, for all outcomes A_r, the risk weighting function :

$$W(A_r) = R(A_0) \,.\, X(A_r ; A_0).$$

2 - 9 - In extending this approach to financial situations where the probability density function of outcome r at some future time is $p(r)$, we note the following :

i) The severity of adverse fmancial consequences increases as r decreases.

ii) There will be a value L of r above which no adverse consequences arise.

We can then define the measure of financial risk as :

$$R = \int_{-\infty}^{L} W(L-r)p(r)dr, \qquad \text{.................. (2.1)}$$

where $W(s)$ is a risk weighting function of the type described in Section 2.8. Clearly $W(s) > 0$ and $W'(s) > 0$, and it is shown in Clarkson (1989) that we can in general assume that $W\ (s) > 0$.

2 - 10 - The equivalent of the "enjoyment value" in the case of financial *outcomes* is clearly the expected value :

$$E(r) = \int_{-\infty}^{\infty} r\, p(r)dr. \qquad \text{.................. (2.2)}$$

The corresponding decision - making criterion is then :

"Maximise the expected return subject to the risk value R not exceeding a threshold value R_0".

This implies that, in the E - R diagram, the indifference curves are vertical straight lines below the risk threshold R_0.

2 - 11 - Suppose, for example, that there are 6 possible outcomes A, B, C, D, E and F with expected value and risk value co - ordinates (E,R) of (3, 2), (4, 4), (5, 2.5), (6, 2.8), (6.2.9) and (7, 3.5) respectively and that the threshold value of risk is 3. Then B and F are rejected on account of the risk value being higher than the threshold value ; A and C are rejected since D and E have higher expected value ; finally D is chosen in preference to E as it has a lower risk value than E.

2 · 12 · However, it can easily be shown that these decision rules are not sufficiently general. Suppose that two other outcomes, G and H, are introduced with (E,R) co - ordinates of (5.9, 0.1) and (10, 3.1) respectively. The risk value of G is very significantly lower than that of D, whereas there is only a very small reduction in the expected value. Depending on the circumstances and preferences of the decision - maker, G may be chosen in preference to D. Similarly, if the decision - maker does not reject D on risk grounds, he might regard the slightly higher risk value of H as acceptable in view of the very much higher expected value of this outcome. To obtain a more general set of decision rules, the indifference curves must have positive gradient rather than being vertical, and the risk threshold must be allowed to increase as the expected value increases. Also, the existence of the risk threshold will in general result in the gradient of the indifference curves decreasing as risk increases.

2 · 13 · This more general set of decision rules corresponds to the pattern of indifference curves shown in Figure 2 ; for small values of risk the curves are asymptotic to vertical straight lines, the curves then decrease in gradient as risk increases and are asymptotic to the curve which represents the risk threshold for varying expected value.

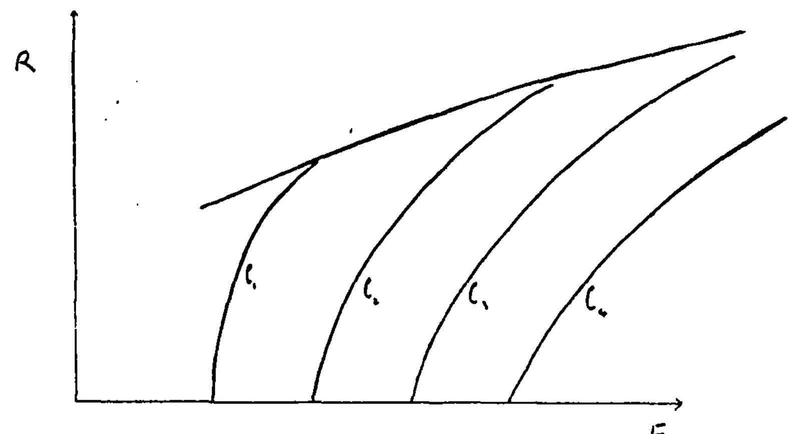


Figure 2

2 · 14 · The final generalisation which is required relates to the expected value when very high returns are possible. If an individual's current wealth is X, then he might regard possible future wealth of 20X as being not much different from possible future wealth of 10X. He will therefore prefer future wealth of 10X with probability 2ΔP to future wealth of 20X with probability ΔP. We can achieve the necessary generalisation by defining the expected value E as :

$$E = \int_{-\infty}^{\infty} u(x)p(x)dx$$

where p (x) is the density function for outcome x,

$$u(x) = \begin{cases} x & x \leq M \\ z(x) & x > M, \end{cases}$$

M is a value of outcome higher than L,

$z(x)$ and $z'(x)$ are positive,

and $z''(x)$ is negative.

For example, if an individual has wealth of £25,000, he might assess expected values on the basis of M = 50,000 and :

$$z(x) = 250{,}000 - 200{,}000\, e^{-\frac{x - 50{,}000}{200{,}000}}$$

For most practical purposes it can be assumed that M is infinite.

2 - 15 - We can now summarise the general framework as follows :

"A decision - maker will choose between different financial profiles on the basis of two measures of equivalence, the expected value E defined by (2.2) and the risk value R defined by (2.1), using indifference curves in the E - R diagram having general properties of the type shown in Figure 2."

2 - 16 - In applying this general framework to practical examples it is necessary to specify L, W(s), and the equations of the relevant indifference curves in the E - R diagram.

2 - 17 - In the case of L, it is generally appropriate to use the value of the expected wealth of the decision - maker on the basis of his present financial circumstances.

2 - 18 - A highly satisfactory formulation for W(s) is the one - parameter family :

$W(s) = s^a$, $a > 1$.

It is clear that a can be interpreted as a measure of a particular aspect of the risk aversion of the decision - maker, since he is indifferent in risk terms as between a shortfall of s with probability $X\Delta P$ and a shortfall of s with probability ΔP if :

$x = 2^a$.

This relationship provides a direct method of determining a.

2 - 19 - This formulation also simplifies the computational aspects immensely. If we have a density function p(r) which - without loss of generality - can be assumed to have expected value zero and unit variance, we can change the location and/or dispersion to obtain distributions of outcomes with similar "shape" but with expected value μ and standard deviation σ. It can readily be shown that, in an obvious notation,

$$R_L^a(\mu, \sigma) = \sigma^a R^a_{\frac{L-\mu}{\sigma}}(0,1),$$

so that, for each value of a, the one single - entry table for R as a function of L is all that is required.

2 · 20 · Finally, we note that, when a = 2 and L is equal to the expected value, the measure of risk is identical to the semi - variance. Since Markowitz regarded semi - variance as the fundamental measure of risk but then introduced variance as a proxy f a it to simplify the computational aspects of the portfolio selection problem, "portfolio theory" can be regarded as a simplified special case of the general framework described in Section 2.15. Also, this suggests that a = 2 specifies an "average" degree of aversion to risk We can therefore regard values of 1.5, 2, 2.5 and 3 as representing low, medium, high, and very high degrees of aversion to risk respectively.

2 · 21 · Values of $R^a{}_L(0,1)$ can be calculated very easily using approximate integration once the density function of return is known, and specimen values in the case of a normal distribution are set out below :

L \ a =	1.5	2	2.5	3
-1	0.07567	0.07534	0.08056	0.09129
-0.5	0.19520	0.20964	0.24040	0.29077
0	0.43002	0.50000	0.61663	0.79788
0.5	0.82445	1.04036	1.38223	1.91577
1	1.40460	1.92466	2.75550	4.09129

2 - 22 · Similar arguments suggest that a suitable formulation for indifference curves in a particular neighbourhood of the E - R diagram is :

$$E_0 = E - kR^b,\ k > 0,\ b > 1,$$

where k and b can be interpreted as risk aversion parameters and E_0 is the value of outcome corresponding to zero risk For a particular value of risk, the parameter k determines the gradient of the indifference curve. If the risk value is doubled, the excess of expected mum over E_0 increases by a factor of 2^b.

2 · 23 · Suppose that, for a particular level of risk, a decision - maker requires an addition of 5% to the zero risk value of expected return If the risk value is doubled, an excess return of 15%, which corresponds to b = 1.58, might then be appropriate. Accordingly, it seems reasonable to use 1.25, 1.5, 1.75 and 2 as the values of b which correspond to low, medium, high and very high degrees of this element of aversion to risk. For convenience, values of 2 and 1.5 f a a and b respectively are used in most of the examples which follow, but, as shown in Section 3, it is a trivial matter to recalculate the results for other values of these parameters.

3 - INVESTMENT SELECTION

3 - 1 · An individual's wealth can be represented by a single investment where the return to some future date follows a normal distribution with expected value 5 and variance 2. If he can exchange this investment for another which is normal with expected value 6, what is the maximum variance that he should accept on this new investment ?

3 • 2 • Suppose that his attitude to risk can be described by a = 2 and b = 1.5, and that he is indifferent between his existing investment and one with certain value 4. The risk on his present investment is

$$\begin{aligned} R_1 &= R^2_5\,(5,2) \\ &= 2.R^2_0\,(0,1) \\ &= 1. \end{aligned}$$

The relevant E - R indifference curve is

$$4 = E - k.R^{1.5},$$

where E = 5 and R = 1, so that k = 1. The risk R_2 on the maximum variance investment with E = 6 is given by

$$4 = 6 - R_2^{1.5},$$

so that $R_2 = 1.5874$.

Since $R_2 = R^2_5\,(6,\sigma)$

$$= \sigma^2\, R^2_{-1}\,(0,1) = 1.5874$$

we require to find o such that

$\sigma^2\, R^2(0,1) = 1.5874.$

Using the values of 0.25240 and 0.25705 for $R^2_{-0.4}\ (0,1)$ and $R^2_{-0.39}\ (0,1)$ respectively gives, by linear interpolation, $\frac{1}{\sigma} = 0.3991$ and $\sigma^2 = 6.28$. Hence the maximum variance for the normal distribution investment with expected value 6 is 6.28.

3 • 3 • Repeating these calculations for different values of a and b gives the following table :

Maximum variance

b \ a =	1.5	2	2.5	3
1.25	11.41	6.72	4.61	3.47
1.5	10.62	6.28	4.37	3.32
1.75	9.95	5.98	4.21	3.22
2	9.47	5.77	4.09	3.15

4 - INSURANCE APPLICATIONS

4 - 1 - In the example given by Borch (1961) an insurance company writes a portfolio of business for a net premium of P plus a loading of $\lambda_1 P$, and it is desired to find the optimal proportion r to reinsure on the basis that the company has to pay the net premium rP of the ceded quota, plus a loading $\lambda_2 rP$.

The probability that the total amount of claims being ma& under the contracts in the portfolio does not exceed x is :

$F(x) = 1 - e^{-x}$

4 - 2 - The net premium is

$$P = \int_0^\infty x\,dF(x)$$
$$= 1,$$

the expected profit E is $\lambda_1 - \lambda_2 r$, and the risk threshold of nil profit is achieved when $x = a$, where :

$$a = \frac{1 + \lambda_1 - r(1 + \lambda_2)}{1 - r}.$$

The risk R is :

$$R = \int_a^\infty \{(1-r)x-(1-r)a\}^2 e^{-x}\,dx$$
$$= (1-r)^2 \int_a^\infty (x-a)^2 e^{-x}\,dx$$
$$= 2(1-r)\,e^{-a}.$$

4 - 3 - Assume now that the insurance company assesses risk and expected return an the basis of indifference curves $E_0 = E - kR^{1.5}$. For varying values of r, the values of E and R trace out a curve in the E - R diagram which is concave upwards as shown in Figure 3. For a particular value of k, the optimal value of r corresponds to the point an indifference curve touches this curve.

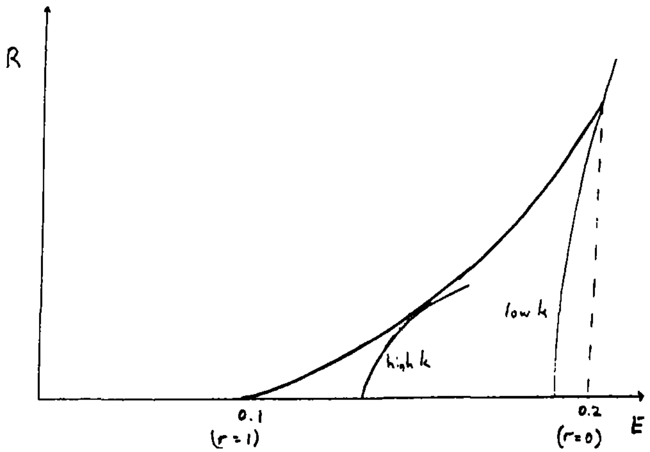


Figure 3

For the values of λ_1 and λ_2 used by Borch (namely 0.2 and 0.1 respectively) and b = 1.5, the optimal values of r f a various values of k are shown in the table below.

k	0.68	1	2	4	8	16	32
r	0	0.17	0.40	0.56	0.67	0.76	0.81

4 · 4 · The above example describes the insurance situation from the point of view of the insurance company. The general framework can also be applied to the insurance situation from the point of view of an individual wishing to insure against a random loss. Suppose that an individual with current wealth of Y is exposed to the risk of a loss of X with probability q. His expected wealth is Y - qX, since he has probability 1 - q of remaining at wealth Y and probability q of having wealth Y - X should the loss occur. Suppose that he can pay an insurance premium $(1+\lambda)P$, where P is the net premium qX and $\lambda > 0$, to insure against the loss. Then using a risk threshold of the expected wealth Y - qX, the individual has the choice of paying the premium and having certain wealth of $Y - (1+\lambda)P$ with risk $(\lambda qX)^a$, or of remaining at expected wealth Y - qX with risk $q(1-q)^a X^a$. The maximum insurance premium he should pay corresponds to the intersection between the risk curve for varying λ and the indifference curve passing through the point $(Y - qX, q(1-q)^a X^a)$.

5 · THE ST. PETERSBURG PARADOX

5 · 1 · Suppose that A tosses a coin until heads appear. If it shows heads on the first throw, he receives 1 from B, if it does not show heads until the second throw, he receives 2 from B, if it docs not show heads until the third throw, he receives 4 from B, and so on, with A receiving 2^{N-1} from B if it first shows heads on the Nth toss. How much should A pay B to enter this game?

On the basis of classical probability theory alone, the expected value of the payment to A is

$$\frac{1}{2} + \frac{2}{2^2} + \frac{2^2}{2^3} + \frac{2^3}{2^4} + \dots$$

$$\text{i.e.} \qquad \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

which is infinite. Thus A should pay an infinite amount to B to enter the game, whereas common sense suggests that no reasonable person would pay B more than a modest amount. The paradox is then that probability theory is apparently directly opposed to the dictates of common sense.

5 · 2 · We shall assume that A has certain present wealth of £25,000 and assesses expected values on the basis described in Section 2.13, i.e. with decreasing marginal value of wealth above £50,000 and a wealth horizon of £250,000 corresponding to

$$u(x)' = \begin{cases} x & x \leq 50{,}000 \\ 250{,}000 - 200{,}000\, e^{-\frac{x-50{,}000}{200{,}000}} & x > 50{,}000 \end{cases}$$

Daniel Bernoulli gave his solution for A having present wealth of 10, 100 or 1,000; we shall therefore calculate A's payment to B on the basis of £2,500, £250 and £25 as the amount to be paid if heads appear on the fust toss.

5 · 3 · We first of all calculate the expected values ignoring risk. If A pays 2,500 x M_1 to B for possible payments of 2,500, 5,000, 10,000 etc, then we require to find the value of M_1 such that:

$$\sum_{r=1}^{\infty} 2^{-r}.u(25{,}000 - 2{,}500\, M_1 + 2^{r-1}.2{,}500) = 25{,}000.$$

Similar equations define M_2 and M_3, the corresponding multiples for possible first payments of £250 and £25 respectively.

Clearly an iterative approach is necessary. Using a first estimate of 4 for M_1 and noting that M_2 and M_3 are approximately equal to $M_1 + 0.5 \log_e 10$ and $M_1 + \log_e 10$ respectively, it can easily be shown that $M_1 = 4.002$, $M_2 = 5.628$ and $M_3 = 7.240$. Hence, if risk is ignored, A should pay B multiples of 4.002, 5.628 and 7.240 of the respective possible first payments.

5 · 4 · Suppose now that A assesses risk and expected returns in a similar fashion to the investor described in Section 3, i.e. with a = 2, b = 1.5 and an additional 25% of expected wealth being required to offset a standardised risk value of 0.04. In the case of the possible fust payment being £25, the risk value is

$$R = \left(\frac{M_1-1}{10}\right)^2.2^{-1} + \left(\frac{M_1-2}{10}\right)^2.2^{-2} + \left(\frac{M_1-4}{10}\right)^2.2^{-3} + \ldots \text{ to as many terms as required}$$

In the E - R diagram, the relevant indifference curve is the one which passes through (25,000, 0) and (31,250, 0.04). Hence

$25{,}000 = 31{,}250 - k.(0.04)^{1.5}$,

i.e. $k = 781{,}250$.

We now require to find the value of M_1 such that the excess of the expected wealth over £25,000 and the risk value correspond to a point an this indifference curve. This gives

$$\sum_{r=1}^{\infty} 2^{-r}.\,u(25{,}000-2{,}500\, M_1 + 2^{r-1}\ 2{,}500) - 25{,}000 = 781{,}250\, R^{1.5}.$$

Similar equations define M_2 and M3. The (unique) values obtained are $M_1 = 2.980$, $M_2 = 5.480$ and $M_3 = 7.240$.

5 · 5 · These results, together with the corresponding values obtained by Bernoulli, are summarised in the table below:

Wealth as multiple of unit value	Ignoring risk	Allowing for risk	Bernoulli
10	4.00	2.98	3
100	5.63	5.48	$4\frac{1}{3}$
1,000	7.24	7.24	6

It must be stressed that the values allowing for risk are significantly affected by the assumptions regarding A's attitude to risk ; a gambler with a very low degree of aversion to risk might regard it as rational to pay any value up to that calculated above ignoring risk

6 - COMPARISONS WITH UTILITY THEORY

6 - 1 - The three basic problems solved using the general framework described in Section 2 can also be tackled using utility theory; brief comments on the respective utility *theory* solutions are given below.

6 - 2 - Example 1.1 of the U.S. actuarial textbook "Actuarial Mathematics" involves a decision - maker with a utility function of $u(x) = -e^{-5x}$ choosing between two investments having a normal distribution, one X with mean 5 and variance 2, and the other Y with mean 6 and variance 2.5. It is shown that X will be chosen since it ha. the higher value of expected utility, and that the decision - maker would be indifferent between the two had Y had a variance of 2.4. The solution comments that "Since the decision - maker is risk averse, the fact that the distribution of Y is more diffuse than the distribution of X is weighted heavily against the distribution of Y in assessing its desirability." In the solution in Section 3 the variance on Y would have to be very significantly higher than 2.4 before an investor would be indifferent between X and Y.

6 - 3 - In the utility theory solution to the reinsurance problem discussed in Section 4, Borch uses a quadratic utility function of the form

$$u(x) = -ax^2 + x + b,$$

where the parameter a, which can be taken as a measure of the company's risk aversion, is less than 0.417 to ensure that $u(x)$ is increasing over the whole range considered. Borch shows that, for a = 1/3, the optimal proportion r to reinsure is 0.86. However, if r is recalculated for different values of a in the admissible range of 0 to 0.417, it can be seen that r varies with a in a highly irregular manner :

a	0	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.417
r	0	0.51	0.68	0.76	0.82	0.84	0.87	0.89	0.89

This suggests that the utility theory solution is much less realistic than the solution given in Section 4 where variation of r with the risk aversion parameter k is much more regular.

6 · 4 · The solution to the St Petersburg Paradox in Section 5 shows that various factors regarding an individual's attitude to wealth have to be taken into account, while Bernoulli's solution, which employs a logarithmic utility function, is independent of all these factors. Bernoulli also applied his logarithmic utility function to calculate the maximum insurance premium P that a merchant, with other assets of Z, should pay to insure against the possible loss (with probability q) of a ship valued at X. Using X = 10,000 and q = 0.05, Bernoulli shows that the P is 800 when Z = 5,043. Using the general framework approach to this problem as set out in Section 4.4, a very much higher value of P would result. This suggests that Bernoulli's logarithmic utility function implies an unrealistically low degree of aversion to risk.

6 · 5 · Clearly there are very significant differences between the solutions obtained in Sections 3, 4 and 5 and the corresponding solutions using utility theory. It is therefore desirable to investigate why these differences arise before demonstrating other problems which can be solved using the general framework.

6 · 6 · Within a suitably small neighbourhood of the E - R diagram it can be assumed that the indifference curves are linear, with a gradient, say, of m. Then the optimal situation is the one which maximises the function

$$U = \int_{-\infty}^{\infty} xp(r)dx - m\int_{-\infty}^{L} W(L-x)p(r)dx$$
$$= \int_{-\infty}^{\infty} u(x)p(r)dx,$$

where u(x) is linear above the decision - maker's current wealth and concave downwards below that value. The utility theory solution could therefore be valid provided, firstly, that the utility function has the properties of u(x) described above and, secondly, that the E - R indifference curves are straight lines all with the same gradient in the region being considered.

6 · 7 · The first assumption is rarely valid, since an individual's utility function is generally regarded as being independent of his current wealth. We can deduce from this the following unsatisfactory features of utility theory in practical applications.

(i) Since the link between the individual's attitude to risk and the shape of the curve below his present wealth is broken, it is very difficult to relate his risk preferences to the characteristics of the curve. Section 6.2 gives an example of this.

(ii) Since the curvature of the utility function will in general change slowly over its whole range rather than becoming zero at and above the individual's current wealth, the degree of aversion to risk will normally be understated. The comments in Section 6.4 about a logarithmic utility function demonstrate this.

(iii) The situation in (ii) is even more unsatisfactory in the case of a quadratic utility function, where the curvature increases with wealth. It is well known, for instance, that the maximum insurance premium for a particular loss distribution increases with the wealth of the decision - maker if a quadratic utility function is used. This result, which is clearly unreasonable, is perhaps the most obvious indication that the foundations of utility theory are quite unsound. Section 6.3 also highlights the unsatisfactory nature of results derived from a quadratic utility function.

6 · 8 · Since the second assumption is invalid if outcomes with significantly different values of risk are being compared, it is unsound to postulate that the same utility function $u(x)$ can be used to compare all possible outcomes. This implies that, in the context of financial risk. there is no theoretical justification for the existence of a measure of utility to satisfying the Von Neumann and Morgenstem (VNM) axioms. Indeed, it is easy to demonstrate that no measure of utility satisfying these axioms can be equivalent to the decision - making process described in Section 2 · 15 - Consider two different outcomes A and B on the same indifference curve in Figure 2, with expected values and risk values of E_A and E_B, and R_A and R_B, respectively. Let C be the outcome with expected value $0.5(E_A + E_B)$ and risk value $0.5(R_A + R_B)$. Then since all indifference curves are concave downwards, outcome C lies to the right of the indifference curve on which A and B lie and hence is preferred to both A and B. However, by Axioms (3.B.a) and (3.B.b) of the VNM set of axioms, C has the same utility value as A and B. (Von Neumann and Morgenstern comment that these particular axioms "would be hard to abandon".)

6 · 9 · In the light of the above arguments, we conclude that utility theory based on the VNM axioms has no practical or theoretical justification in the context of financial risk.

7 - LIFE ASSURANCE

7 · 1 · A life office writes a block of 100 term assurance policies each with a sum assured of 100, a term of one year, and a probability of 0.01 that the assured life dies during the term. The office charges a loading of λ times the net premium. If the office assesses risk on the basis of

$$i_0 = E - kR^{1.5} , \qquad \text{......................} \quad (7.1)$$

where i_0 is the "risk - free" deposit rate and E is the return on capital employed, we wish to calculate λ for different values of the risk parameter k. Assuming the lives are independent, the probability that not more than two claims arise in 0.921, and the office regards 100 as the "capital employed* for this block of business.

7 · 2 · If the net premium is calculated on the basis that claims are paid at the end of the year and the interest rate is i_0, the gross premium for each contract is

$$P = \frac{1 + \lambda}{1 + i} .$$

Then the expected value after one year is $100\lambda + 100(1 + i_0)$ and the value per unit capital employed, if n claims *occur,* is

$$E = i_0 - (n - 1 - \lambda)$$

The risk value is

$$R = \sum_{n=1}^{\infty} p(n)(n - 1 - \lambda)^2,$$

where $p(n)$ is the probability that n claims arise. This gives

$$R = 0.62396 - 0.73206\lambda + 0.26424\lambda^2. \qquad \text{.................. (7.2)}$$

Substituting in (7.1) gives

$$k = \frac{\lambda}{(0.62396 - 0.73206\lambda + 0.26424\lambda^2)^{1.5}}.$$

We then obtain the following table

k	0	0.1	0.2	0.4	0.6	0.8	1
λ	0	0.045	0.085	0.150	0.203	0.248	0.287

8 - NON - LIFE INSURANCE

8 - 1 - An insurance.company writes a block of 100 non - life policies for a period of one year. For each policy, the probability that a claim arises is 0.05, the expected value of a claim is 20, and the probability that the value of a claim does not exceed X is 20F(X) where :

$$F(X) = 1 - e^{-X}.$$

As in Section 7, the insurance company charges a loading of λ times the net premium and assesses risk on the basis of equation (7.1), where i_0 is the "risk - free" rate and E is the return on capital employed. Again we wish to calculate λ for different values of the risk parameter k.

8 - 2 - Suppose that the company regards 100 as the "capital employed for this block of business and that the number of claims follows a Poisson process with parameter 5. The gross premium P for each contract is

$$P = \frac{1 + \lambda}{1 + i_0}.$$

Thus the expected value after one year is $100 + 100(1 + i_0)$ and the value per unit capital employed if the total value of claims is X is

$$E = i_0 + \lambda - (X - 1).$$

The risk value is

$$R = \int_{1+\lambda}^{\infty} [X - (1+\lambda)]^2 \, p(X)d(X) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.1)$$

where p(X) is the density function for total claims of 100X.

8 · 3 · Since p(X) is a highly complex function of X, it is impossible to evaluate equation (8.1) in analytic terms. However, if we simulate on a random or stochastic basis both the number of claims and the amount of each claim, then the expression

$$R(N) = \frac{1}{N} \sum_{n=1}^{N} \{Y(n) - (1+\lambda)\}^2$$

where Y(n) is the greater of $1+\lambda$ or the total claim value X(n) on the nth simulation, has a limiting value equal to the value of the integral in equation (8.1) as N tends to infinity.

8 · 4 · Many readily available computer programs could be used for the stochastic simulation, but a table of random numbers is all that is required We can use four digits to specify the number of claims and then successive pairs of digits to specify the percentile points' in the distribution of F(X). Since the probabilities of no claims, not more than one claim and not more than two claims are 0.0067, 0.0404, and 0.1265 respectively, we interpret numbers in the range 0001 to 0067 as no claims, numbers in the range 0068 to 0404 as one claim, numbers in the range 0405 to 1265 as two claims, and so on. We also interpret the numbers 00, 01, 02 ... 98 and 99 as F(X) being equal to 0.005, 0.015, 0.25, ... 0.985 and 0.995 respectively. Thus 53 corresponds to F(X) = 0.535 and hence a claim value X of 0.766.

8 · 5 · The table of random numbers used begins with the digits:

8856 5327 5933 3572 6747 7734.

The first four digits 8856 indicate 8 claims, and the following 8 pairs of digits correspond to claim values of 0.766, 0.322, 0.904, 0.408, 0.439, 1.291, 1.124 and 0.644 respectively, so that the first total claims value X(l) is 5.898. Continuing in this manner leads to the following values of R(200) for various values of A.

λ	0.1	0.2	0.3	0.4
R(200)	0.327	0.280	0.239	0.204

Random errors arising from the estimation process will in general be negligible in comparison to the estimation errors involved in specifying F(X) and the probability of a claim.

8 · 6 · Substitution of the above values of R(200) in equation (7.1) gives the following values of k for the specified values of λ.

λ	0.1	0.2	0.3	0.4
k	0.535	1.350	2.568	4.341

From these values we obtain by interpolation the following table :

k	0	0.5	0.1	1.5	2	2.5	3
λ	0	0.09	0.16	0.22	0.26	0.29	0.33

It is interesting to note the striking similarity between this non-life insurance example and the life insurance example in Section 7. The only minor difference is that in the non - life case the risk value is estimated by stochastic simulation whereas in the life case it is calculated analytically.

9 - OPTION PRICING

9 - 1 - Suppose that an investor with risk parameters a=2 and b=1.5 has a holding of ordinary shares and writes a call option on this holding with an exercise price equal to K times the current share price. What is the minimum premium that he should accept for writing this option ?

9 - 2 - We assume that the share price follows a lognormal distribution with σ, the standard deviation of the logarithmic stock return per unit time, of 0.3 and that the (geometric) mean stock return per unit time is $\log_e 1.12$. We can assume without loss of generality that the current share price is 1. It can be shown (see, for example, Jarrow and Rudd (1983)) that to a purchaser of a call option its expected value on expiry is

$$D = (1 + i_1)^t \, N(h) - K\, N(h - \sigma\sqrt{t})$$

where $h = \{\log_e (1 + i_1)^t - \log_e K + 0.5\,\sigma^2 t\} \div \sigma\sqrt{t}$.

t is the life of the option,

and $\log_e (1 + i_1)$ is the (geometric) mean stock return per unit time.

9 - 3 - When K = 1 and t = 1, D = 0.1952. To simplify the analysis, we assume that risk values can be calculated on the basis that the share price at the end of the option period follows a normal distribution with expected value 1.12 and standard deviation 0.3. If the investor holds the ordinary shares. the expected value after one year is 1.12 and the risk value is $R^2_{1.12}(1.12, 0.3)$. If he holds deposits at the "risk - free" rate of, say, 8%, the expected value is 1.08 and the risk value is $(0.04)^2$ If he writes the option for a premium of C and places the premium on deposit at the "risk - free" rate, the expected value is 1.12 + 1.08C - D, and the risk value is $R^2_{1.12}(1.12 + 1.08C, 0.3)$.

All three of these outcomes must lie on the same indifference curve :

$$E_0 = E - kR^{1.5} .$$

Using $R^2_0(0,1) = 0.5$ and putting $Y = 3.6$, the solution is obtained from:

$$R^2_{-Y}(0.1) = 0.5 - 12.0555D + 3.6166 \; Y,$$

giving C = **0.1566,** which **corresponds** to an expected return of **9.39%.** Repeating the calculations f a K = **0.9 and K = 1.1** gives minimum premiums of **0.2125** and **0.1128** respectively, **and** expected **returns** of **8.88%** and **9.92%** respectively.

9 **·** 4 **·** Denoting the "risk - free" rate, the expected **return** on the ordinary shares, and the expected return when the option is written by i_0, i_1 **and** i_2 respectively, the general expression for the premium for a call option of life t is

$$C = \frac{D + (1 + i_2)^t - (1 + i_1)^t}{(1 + i_0)^t}$$

This can be expressed in the form:

$$C = \frac{D^{i_0} + (D - D^{i_0}) + (1 + i_2)^t - (1 + i_1)^t}{(1 + i_0)^t} ,$$

where D^{i_0} is calculated using i_0 rather than i as the **expected** annual return on the shares. Then, since an increase in the expected value of the **ordinary** shares on **expiry** of the option increases the value of D, we can put

$$D - D^{i_0} = k_1 \left\{(1 + i_1)^t - (1 + i_0)^t\right\},$$

where $0 < k_1 < 1$ and k_1 decreases as K increases.

Similarly, we can put

$$(1 + i_1)^t - (1 + i_2)^t = k_2 \left\{(1 + i_1)^t - (1 + i_0)^t\right\},$$

where $0 < k_2 < 1$ and k_2 decreases as K increases.

Finally, the following table shows that, for **differing** values of K, k_1 and k_2 are very nearly equal in the case of the one year call option discussed above.

K	D	D^{i_0}	i_2	k_1	k_2
0.8	0.3385	0.3031	0.0847	0.885	0.882
0.9	0.2607	0.2290	0.0888	0.792	0.780
1	0.1952	0.1680	0.0939	0.680	0.652
1.1	0.1426	0.1202	0.0992	0.560	0.520
1.2	0.1019	0.0841	0.1042	0.445	0.395

9 - 5 - This suggests that the following simplified formula will provide an accurate approximation for the premium:

$$C = \frac{D^{\dot{w}_0}}{(1 + i_0)^t}.$$

This, of course, is the option pricing formula first derived by Black and Scholes (1973). The table below shows that in the case of the one year call option the approximation is very accurate.

K	Calculated value	Black-Scholes value
0.8	0.2807	0.2806
0.9	0.2125	0.2120
1	0.1566	0.1556
1.1	0.1128	0.1113
1.2	0.0797	0.0779

In the case of options with lives of three months or less the values will for all practical purposes be identical.

10 - CONCLUDING REMARKS

10 - 1 - The classical risk theory optimisation criterion, namely maximise the return subject to the probability of ruin not exceeding some chosen small value, is equivalent to the situation described in Sections 2.10 and 2.11 with the risk value equal to the value of the profit distribution to the left of the origin As described in Section 1.5, Borch suggests that the classical model should be generalised to take into account all properties of the profit distribution. The risk measure described in this paper allows this generalisation to be made by taking account of all points in the profit distribution below the chosen risk threshold.

10 - 2 - The classical theory of non - life insurance involves heavy mathematics, which makes it inaccessible to many actuaries. For instance, when Plackett (1971) gave a lecture on risk theory to the Faculty of Actuaries, the major part of the lecture and of the background note dealt with the distribution of the total amount of claims. As shown in Section 8, an elementary simulation process is all that is required when the general framework described in Section 2 is applied to non - life insurance.

10 - 3 - Many of the basic mathematical tools in the modem theory of finance, and in particular mean - variance analysis and option pricing, again involve heavy mathematics. Using the general framework described in Section 2, numerous practical problems can be solved using only elementary mathematics. Furthermore, the Black-Scholes option pricing model can be derived without having to make the usual somewhat unrealistic assumption that a "perfect hedge" can be constructed.

10 - 4 - The original Markowitz formulation of Portfolio theory can be regarded as a special case of the general framework provided that the variance relates to the end - period return. However, in Modern Portfolio Theory it is customary to use short term price volatility as a proxy for this variance, which seriously weakens the theoretical validity of the whole approach. Furthermore, mean - variance analysis makes extensive use of quadratic utility functions, which, as described in Section 6.7, exhibit various highly unsatisfactory properties.

10 - 5 - Similar criticisms of Modem Portfolio Theory are made by Akant and Schielke (1982), and the alternative measure of risk they suggest is a special case of the risk measure defined in Section 2.9 with $W(s) = s$. They also depart from the conventional assumption that utilities are assigned to absolute states of wealth achieved by the decisions taken and instead analyse choices in terms of changes of wealth relative to a neutral reference point. This is equivalent to taking the risk threshold L as the current wealth of the decision - maker as described in Section 2.17. Finally, Akant and Schielke also draw attention to the unsatisfactory properties of quadratic utility curves.

10 - 6 - The two key features that distinguish traditional actuarial mathematics from methods used by schools of finance and the accounting profession are the use of compound interest and the use of probabilities derived from a life table to describe uncertain future events. If the actuarial rate of interest is i, unit amount in t years' time is equivalent to a present value of $(1+i)^{-t}$, and actuarial functions such as annuity values are therefore measures of equivalence involving the summation or integration of the product of this present value and the relevant probability. The risk measure described in this paper is very similar, the equivalence between different outcomes is described by the risk weighting function, and the value is the summation or integral of the product of this function and the relevant probability.

10 - 7 - In an editorial article entitled "Actuaries of the Third Kind ? ", Bühlmann (1987) traces the history of actuarial science from its birth in the 17th century, when it was exclusively devoted to problems of life assurance, to the present. "Actuaries of the Second Kind succeeded in getting their methods applied to non - life insurance also, and "Actuaries of the Third Kind" are now in the process of creating a new scientific philosophy for handling investment problems.

10 - 8 - Suppose that we adopt the following much more extensive definition of actuarial science :

"Practical financial management based on measures of equivalence which relate to uncertain future events".

Then the general approach to financial risk described in this paper is actuarial in nature and Can be used by actuaries of all three kinds to obtain practical solutions to a very wide range of financial problems involving risk.

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