

CONTRIBUTION N° 18

THE MEASUREMENT OF INVESTEMENT RISK

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MESURE
DU RISQUE
D'INVESTISSEMENT

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RESUME

L'objectif est de **présenter** une **théorie générale** fondée sur une **nouvelle définition** du risque **d'investissement** et de **comparer** le cadre **théorique** ainsi proposé à l'approche de **Markowitz**.

On explique d'abord **pourquoi** l'approche **fondée** sur la **théorie** du **portefeuille**, qui utilise la **variabilité** à court terme de la rentabilité comme **mesure** du risque, n'est pas **satisfaisante**. Une discussion des **risques impliqués** par certains sports **dangereux**, met en **évidence** plusieurs **principes** importants et **conduit** à l'**axiome fondamental**, selon lequel le risque d'investissement **est fonction** à la fois de la **probabilité** d'une **rentabilité inférieure** à un certain **seuil** et de la gravité **des conséquences financières** de cette faible rentabilité. La **valeur** du risque peut alors **être exprimée** comme la **somme** ou l'**intégrale** du **produit** d'une fonction de **pondération** du risque par la **probabilité** concernée. **Après avoir formulé** d'autres axiomes concernant le comportement de l'investisseur, on applique le cadre **théorique résultant** à divers exemples **pratiques**. On montre ensuite que la formulation de **Markowitz** de la **théorie** du portefeuille peut **être considérée** comme un cas **particulier simplifié** de ce **cadre théorique**.

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1. INTRODUCTION

1.1. In the paper "Improving the Performance of Equity Portfolios" by **Clarkson** and **Plymen** [1] (presented to the the Institute of Actuaries on 25th **April** 1988) the authors concluded that **Modern** Portfolio Theory methods made no contribution whatever to improving **the** performance of equity portfolios and suggested that attention should be paid instead to the application of fundamental analysis, which - if **carried** out by skilled and experienced analysts - should lead to higher expected returns. The only practical application of techniques related to Modern Portfolio **Theory** **appeared to be** in the area of Index Funds, where it is desired to track the performance of a chosen index as closely as possible.

The authors also **criticised** the use of variance of return as the universally accepted theoretical measure of risk and suggested that - except possibly in the very short term - variance of return did not accord with what practical investors perceived risk to be. The authors suggested instead that investment risk should be some measure involving the likelihoods of lower than expected values of long-term return. Accordingly, the **main** practical conclusion of the authors was that if more attention was paid to achieving a higher expected **return** then the risk "**would** look after itself".

1.2. Although that paper is essentially practical in nature, in Section 7 there is the following theoretical example of two investments A and B where every "reasonable" **investor** would prefer A to B regardless of the fact that investment A has four times the variance of investment B :

"**Suppose** for example, that we have two shares A and B, where **the** returns to a particular future date depend on certain scenarios X_1, X_2, \dots, X_N . For each scenario X_i , the return on share A (which is always positive) is twice the return on share B. **Since** the return on share A is always greater than the return on share B, any reasonable investor will regard share A as "less risky" than share B regardless of the respective variances of **return**."

If the **return** can be expressed in terms of continuous density functions $p_A(r)$ and then

$$p_A(r) = \frac{1}{2} p_B\left(\frac{r}{2}\right)$$

and the situation is as shown in Figure 1.

THE MEASUREMENT OF INVESTMENT RISK

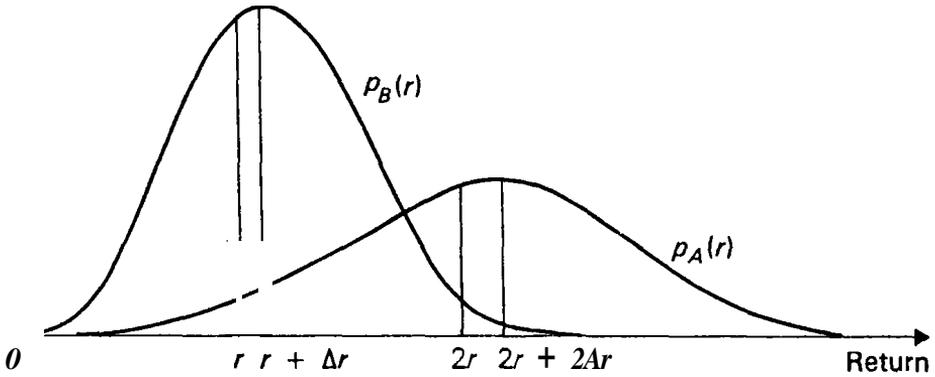


FIGURE 1

If for a small element ΔY in the set of possible scenarios the return on B lies between r and $r + \Delta r$, the corresponding return on A for that same scenario lies between $2r$ and $2r + 2\Delta r$. Accordingly, the return on A is - in all circumstances - twice the return on B (and hence greater than the return on B since this is always positive), and any reasonable investor will choose investment A in preference to investment B. The variance of return on investment A is four times that of investment B (see Appendix 1.1. for the proof), but this is clearly irrelevant when choosing between investment A and investment B.

13. It might be argued that this example is highly artificial in that the return on one investment stochastically dominates the return on the other. However, if it is assumed only that the density functions $p_A(r)$ and $p_B(r)$ are related by :

$$p_A(r) = \frac{1}{2} p_B\left(\frac{r}{2}\right)$$

a similar conclusion can be drawn.

We note that for any positive value of return L , the probability of the return on investment A falling below L is always lower than the probability of the return on investment B falling below L (see Appendix 1.2.). This holds for all positive values of L . Accordingly, on the premise that investment risk relates to the consequences of the return being lower than some threshold value, any reasonable investor will again regard the risk on investment A as being lower than that on investment B.

14. This modified example, which does not rely on stochastic dominance, provides a strong counter-example to the axiom used by Markowitz [2] and all subsequent proponents of MPT, namely that all investors choose between investments solely on the basis of expected return and variance of return. In any theoretical system built up from

basic axioms, the existence of a **counter-example** such as this which **contradicts** real life **behaviour** must imply that there is an axiom of the system which is inconsistent with the real **world**. It can therefore be argued that the MPT **definition** is not **sufficiently** general and that the **Markowitz** approach to portfolio selection is - at best - only a special **case** of a more general portfolio selection framework which employs a realistic measure of investment risk. **Furthermore**, the entire literature of **portfolio** theory then represents - again at best - a special case where certain **restrictions on** generality apply, and hence the "conclusions" of portfolio theory models can be applied in practice only if these **restrictions on** generality are roughly in **accord** with real life behaviour.

1.5. **There** is, I believe, a strong parallel between the derivation of a general theory of investment risk and my earlier derivation of a general model for the gilt-edged market [3]. Using only the very general axiom that arbitrage would remove all "blatant anomalies" in the price **structure**, I showed that the price structure **P** had to **be** consistent with the relationships :

$$\frac{\partial}{\partial i} \left(\frac{1}{P} \right) < 0 \dots \quad (1)$$

$$\frac{\partial^2}{\partial i^2} \left(\frac{1}{P} \right) \leq 0 \dots \quad (2)$$

where **P** is a **smooth** function **P(n, g)** of **term** to maturity **n** and **coupon g**, and **i** is the running yield $\frac{g}{P}$

In all previous models it had been assumed that inequality (2) **could** be replaced by the much more restrictive relationship :

$$\frac{\partial^2}{\partial i^2} \left(\frac{1}{P} \right) = 0.$$

I found that strict inequality in (2) did indeed give a better representation of **the** gilt-edged market, and so the resulting model was a distinct improvement in terms of goodness of fit and hence also in practical applications such as the identification of cheap and dear stocks.

The model used by the Bank of England [4] had been based on rational expectations of net redemption yields and hence, like all other models then in use, assumed equality in (2). In the light of my results, the Bank of England [5] modified their **model** to allow strict inequality and again a better fit resulted

The important point is that the earlier models were special cases of my more general **model and** were quite **satisfactory** in practice because the restrictions on generality that they contained were not seriously inconsistent with **the** real world.

1.6. If, as suggested above, the **Markowitz** approach to risk is not sufficiently general, it is clearly highly desirable to develop a more general theory of investment risk by

returning to first principles and using only those axioms which can be **shown** to be of universal validity. This paper describes how a general theory of this type can be developed and then examines in detail whether the **Markowitz** approach can indeed be regarded as a special **case of** this more general framework.

2. THEORETICAL DEVELOPMENT

21. It is generally **accepted** that the two **main** aspects of the **probability distribution** of return **that** may be of interest to investors are the location and the dispersion. Provided that the **density function** of return $p(r)$ is known, the obvious measure of location to **use** is the expected value of **return** defined by :

$$E(r) = \int_{-\infty}^{\infty} r p(r) dr.$$

However, it is often unrealistic to assume that $p(r)$ is known ; the available **information** can in practice be of such poor quality that only a vague idea of the shape of $p(r)$ can be **obtained**.

22. The other feature, the dispersion, is not so easy to define. Investors will generally prefer the lowest dispersion (i.e. the least "uncertainty") when they have to choose from a family of symmetric distributions with the same expected value as **portrayed** in Figure 2.

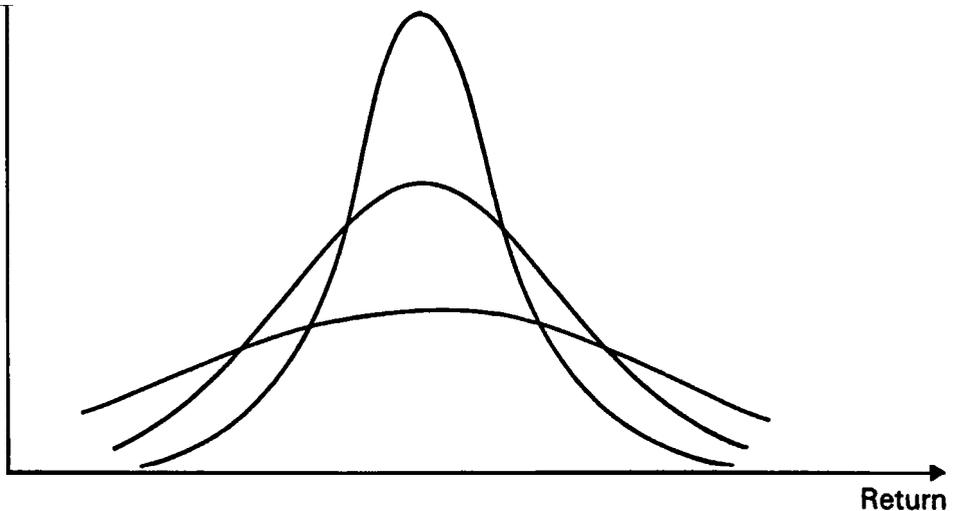


FIGURE 2

In this case this is equivalent to selecting the **distribution** with the lowest variance.

23. **Variance** of return, however, is not **the** whole story **as far as** dispersion is **concerned**. For **two** distributions A and B with **the** same expected value and variance, distribution A

could be highly skewed towards high values of **return** while distribution B was highly skewed towards low values of **return**, as shown in Figure 3.

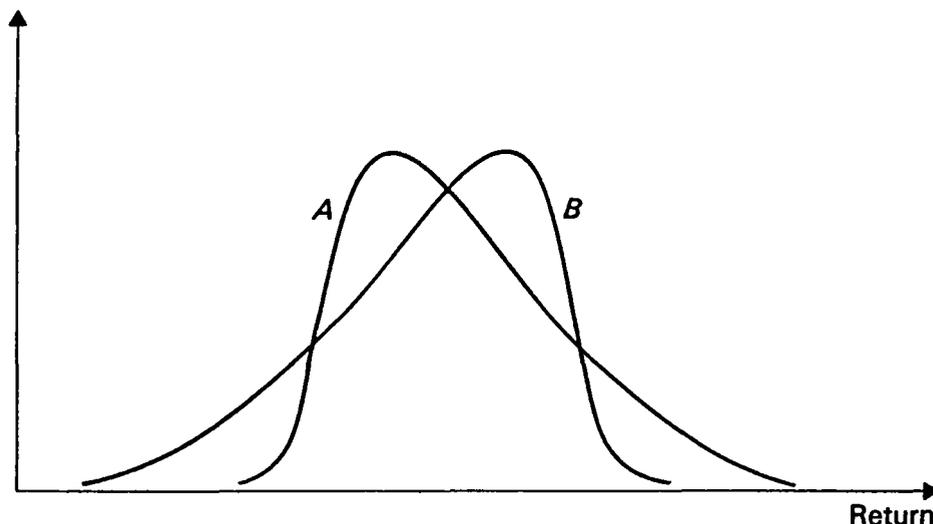


FIGURE 3

Most investors would prefer distribution **A**, since there is a much smaller probability of low values of return arising. In other words, it is the dispersion on the downside that is generally perceived as comprising investment risk ; a higher dispersion as a result of skewness on the upside does not equate to higher risk. It is therefore apparent that variance of **return** alone is not a satisfactory measure of investment risk.

24. Risk is an intuitive concept which is present in all areas of human activity rather than **being** restricted to the business or investment world, and an individual's choice **as** between various possible courses of action will very often be dependent on his perception of the risk attached to each **course** of action

25. Consider, for example, the following five sporting activities, each of which could form the central part of an adventure holiday :

hang-gliding

rapid river canoeing

ski-ing (**on** prepared pistes)

ski-mountaineering (**i.e.** mainly off-piste descents)

wind-surfing.

Despite being unaware **of** any **precise** underlying probability **distributions**, most people will immediately say that hang-gliding is the most **risky** and wind-surfing is the least risky. How are these risk assessments arrived at, and how should we rank the other activities?

26. The obvious approach is to consider the likelihoods and consequences of an "adverse occurrence" in each sport. In the case of **hang-gliding**, this **involves** a stall or equipment failure when high above the ground and would result in **almost** certain death ; in the **case** of wind-surfing, it involves no more than losing control, falling into the water, and having to clamber back onto the board. With **piste** ski-ing, the **main** risk is the likelihood (decreasing with the experience and competence of the skier) **of** a bad fall or **collision** which could result in **anything** from **minor** bruises to **compound fractures** or **serious** head injuries. In ski mountaineering there are various additional risks : **descents** on unpisted slopes are much more **difficult** ; medical help could take hours to arrive in the event of an accident ; there is also a high avalanche risk which involves a very real **possibility of** death. With rapid river canoeing, the main hazard is a capsize in rapids. While this is potentially more dangerous than **falling** off a wind-surfer, injuries such as broken limbs are much less common than a ski-ing but the risk **of** death (generally by drowning) is higher.

27. Given this background, we *can* produce the following table of the likelihoods and consequences of adverse **occurrences** in each sport :

Sport	Adverse occurrence		Likelihood of adverse occurrence resulting in :	
	Nature	Likelihood	Serious injury	Death
Hang-gliding	Plunge to ground	very low	very high	very high
Rapid river canoeing	Capsize in rapids	moderate	low	low
Ski-ing	Bad fall	moderate	moderate	very low
Ski-mountaineering	Avalanche	low	high	high
	Bad fall	moderate	moderate	low
Wind-surfing	Falling off	very high	negligible	negligible

The probability of death during a particular interval of time is the product of the probability **that** an adverse occurrence arises and the **probability** that this type of adverse occurrence results in death. Most people would rank the various probabilities of death in the following increasing order of magnitude :

Lowest : **Wind-surfing**
ski-ing
 rapid river canoeing
 ski-mountaineering

Highest : hang-gliding

Finally, most people will give the same **ranking** for the "overall risk" of each sport.

28. If we take a group of people who all agree on this overall ranking, some will be more risk-averse than others as evidenced by where their "cut-off point" of riskiness corresponds to the perceived risks of the various sports. Most **able-bodied** individuals would, for instance, be prepared to accept the risks involved in ski-ing, but **only** the most adventurous **would** be prepared to participate in hang-gliding.

29. There is clearly a risk (however small) of serious injury or death in all **day-to-day** activities. Most people are prepared to fly on normal passenger aircraft for a holiday and

business purposes, and virtually everyone is prepared to travel by car, although it is evident that there is a non-zero risk of death in each **case**. **These** risks are perceived as being below the "cut-off point" of **riskiness** that they are **prepared** to accept.

2.10. Although **the** sports example described above is highly subjective in nature, it **highlights** important **features** of the intuitive approach to risk :

- (1) The perceived degree of risk relates both to the severity of the consequences of an adverse occurrence and also to **the** probability **of** these consequences arising.
- (2) **The** perceived degree of risk is highly dependent on the **probabilities** of the most serious possible consequences.
- (3) In assessing whether a particular activity is acceptable in terms of the attendant risk, an individual will compare the perceived risk with some threshold of riskiness based on his personal **preferences** and experience.
- (4) Individuals may differ in the degree of risk they are **prepared** to accept.

2.11. Before applying this intuitive approach to **the** investment case, it is **convenient** to show how the sports example can be formalised in mathematical terms.

Given that an adverse occurrence arises, the consequences can be classified on a numerical scale as :

0	No injury
1	Minor injury
2	Moderate injury
3	Severy injury
4	Very severe injury
5	Permanent incapacity
6 and higher	Death

A satisfactory class of probability distributions is the **Poison** distribution (a limiting 'case of the binomial distribution) where the parameter (which represents both the expected value and the variance) **increases** with **the likely** severity of the consequences. A method of "weighting" for the different degrees of severity is also required. A reasonable choice here is **1** for minor injury, **10** for moderate injury, **100** for severe injury, **etc.**, so that each next higher degree of severity increases the **contribution** to the risk by a factor of **10**.

For ski-mountaineering, using $\lambda = 3$ for the avalanche risk and $\lambda = 0.2$ for the bad fall risk, the risk values per occurrence are calculated as follows :

Severity	Avalanche			Bad fall		
	Probability	Weight	Contribution to risk	Probability	Weight	Contribution to risk
0	·04978707	0	·000	·81873059	0	·000
1	·14936121	1	·149	·16374612	1	·164
2	·22404181	10	2·240	·01637461	10	·164
3	·22404181	100	22·404	·00109164	100	·109
4	·16803136	1,000	168.031	·00005458	1,000	·055
5	·10081881	10,000	1,008·188	~00000267	10,000	·022
6 and higher	·08391793	100,000	8,391·793	~00000007	100,000	·007
			9,592·805			·521

We **now** require estimates of the likely number of occurrences **per day**. Assuming **one** chance in a **thousand** of an avalanche, and two bad falls, **per day**, the **final** value of risk is:

$$9,592.805 \times 10^{-3} + 2 \times .521, \quad \text{i.e. } 10.635.$$

For the other sports, the following values of λ and daily rate of occurrence are used :

	λ	Daily occurrences
Hang-gliding	7	1 in 5,000
Rapid rivet canoeing	0.5	1 in 5
Ski-ing	0.1	2
Wind-s ng	0.01	5

These values give the following values of risk, in increasing order of magnitude :

Sport	Risk
Wind-s ng	0.052
Ski-ing	0.310
Rapid river canoeing	1.380
Ski-mountaineering	10.635
Hang-gliding	14.260

To **complete the** example we require some method of calculating a "**cut-off**" value of risk. An obvious approach here is to specify that the probability of an accident of a **certain** severity (or worse) should not **exceed** some **chosen small** quantity. A reasonable choice is "not more than one chance in a thousand of serious injury or worse". To translate this into a value of risk we require to **know** the relative probabilities of the various levels of severity, and a satisfactory assumption here is that they follow the same pattern as for a "medium **risk**" sport (i.e. rapid river canoeing), which is in **accordance** with a Poisson **distribution** with $\lambda = 0.5$. For a single occurrence, the **situation** is :

Severity	Probability	Weight	Contribution to risk
0	.60653065	0	.000
1 urfi	.30326533	1	.303
2	.07581633	10	.758
3	.01263605	100	1.264
4	.00157951	1,000	1.580
5	.00015795	10,000	1.580
6 and higher urfi	.00001418	100,000	1.418
			<u>6.903</u>

Since the probability of severity 3 or greater is 0.01439, the cut-off level of risk is $6.903 + 14.39$, i.e. 0.480. An individual who specifies this cut-off level of risk would therefore only consider participating in wind-surfing and ski-ing ; the other three sports are too risky for him. In the absence of other constraints (such as cost or availability) he will choose between wind-surfing and ski-ing on the basis of whichever he regards as the more enjoyable. Also, if he attached the same "enjoyment" value to both, it would be logical for him to choose wind-surfing on account of its lower risk. The final situation is portrayed (using arbitrary "enjoyment" values) in Figure 4 using a logarithmic scale for the risk values.

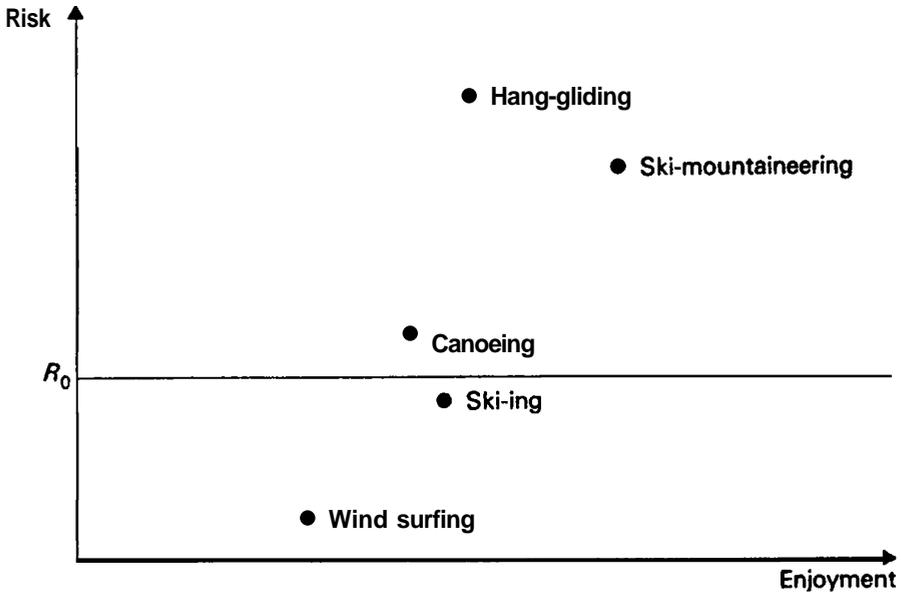


FIGURE 4

2.12. We are now in a position to formulate the following general axioms regarding risk.

- (1) Risk is a function both of the probabilities of possible adverse outcomes and also of the severity of the consequences of these adverse outcomes.
- (2) If the probabilities of the possible adverse outcomes are known, risk can be expressed as a non-negative measure :

$$R = \int W(s)p(s) ds \quad \text{or} \quad \sum W(s_i)p(s_i)$$

where $p(s)$ (or $p(s_i)$) relates to consequences of severity s (or s_i) and $W(s)$ is a positive function of s which increases with s .

- (3) For a given value of the measure of "enjoyment" E and individual will prefer the course of action with the lowest risk.
- (4) Each individual has a threshold of risk R_0 and will not pursue a come of

action which involves a value of risk higher than R_0 .

- (5) An individual will choose between all possible courses of action by **maximising** E subject to the risk not exceeding R_0 .
- (6) **Different** individuals may differ in their degree of aversion to risk by using different **functions $W(s)$ and/or different** risk thresholds R_0 .

2.13. Before **extending** the analysis of risk to the investment case it may **bc** useful to point out that **the** development so far is based on three key steps, each of which is consistent with the principle of universal validity described in paragraph 1.6 :

- (i) Risk is primarily a function of the probabilities attaching to various adverse consequences.
- (ii) The risk weighting function W is essentially an indifference function which reflects an individual's perceptions of the relative seriousness of various adverse consequences.
- (iii) The risk of a particular situation can be compared with the risk of a "standard" situation.

On this basis, the theory of risk measurement is an extension of probability theory.

2.14. In translating these general axioms into provisional axioms that apply to investment, we note the following :

- (i) Since investment returns are essentially **continuous** in nature, an integral measure rather than a summation is appropriate.
- (ii) **If $p(r)$ is the density function of return of the investment being considered,** the severity of the adverse consequences **increases** as r decreases.
- (iii) There will be a value L or r above which there are no adverse **consequences**.
- (iv) The obvious measure of "enjoyment" is the expected return E (r).

2.15. The provisional axioms of **investment** risk are thus :

- (1) Investment risk is a function both of the probability of the return being below a certain threshold and also of the severity of the financial consequences of these values of return.
- (2) If the density function of return, $p(r)$, is known, risk can be expressed as a non-negative measure :

$$R = \int_{-\infty}^L W(L-r)p(r)dr$$

where **$W(s)$ is a positive** function of s which increases with s and L is the value of return above which no adverse **consequences** arise.

- (3) For investments with the same expected return, an investor will prefer the

investment with lowest risk.

- (4) Each investor has a threshold of risk R_0 and **will** not make an **investment** which involves a value of risk higher than R_0 .
- (5) An **investor** will choose between all possible investments by **maximising** the expected return subject to R not exceeding R_0 .
- (6) Different **investors** may differ in their degree of aversion to risk by using **different** functions $W(s)$ **and/or** different thresholds or risk R_0 .

2.16. Axiom 2 is clearly a formalisation of the approach described in Section 7 of **Clarkson** and **Plymen**. Also, when the **probability distribution** is "moved to the right" by an amount X, the new **distribution** $p_X(r)$ is related to the original **distribution** $p(r)$ by :

$$p_X(r) = p(r - X) \quad \text{with } X > 0,$$

and the investment risk on the new distribution is lower than the investment risk on the original distribution (see **Appendix** 2.1 for the proof).

This justifies the main practical conclusion set out in **Clarkson** and **Plymen**, namely "if we look after the expected **return**, the risk will look after itself".

2.17. Form axiom 2 we can deduce two other elementary (but still useful) results without any further information about the risk weighting function W.

For two investments **A** and **B**, the risk on **A** is less than the risk on **B** if the respective density functions of return are such **that**

$$P_A(r) < P_B(r)$$

for all values of r less than L.

For two investments **A** and **B**, the risk on **A** is less than the risk on **B** if the respective density functions of returns for values below L follow the general pattern **shown** in Figure 5 and Area 1 is less than or equal to Area 2.

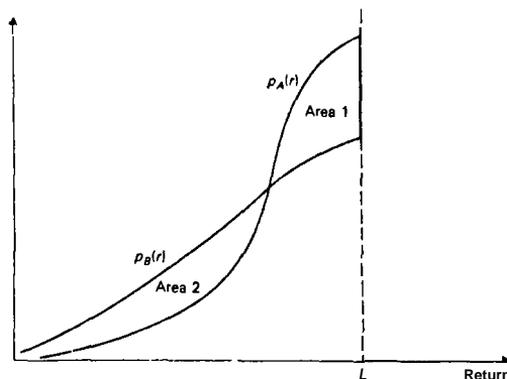


FIGURE 5

See Appendix 2.2. for the **proof**.

2.18. **Axioms 3, 4 and 5 describe** how, having calculated the expected **return E** and the risk R , an investor will choose which investment (if any) is best suited to his requirements. If the attainable **E - R combinations** can **be** represented as a continuous region, there are three **cases**, depending on how the value of R_0 compares with R_{min} , the **minimum** value of risk in the region, and with $R(E_{max})$, the value of risk **corresponding** to the **maximum** value of E in the region. These three cases, namely :

- Case 1: $R_0 \geq R(E_{max})$
- Case 2: $R_{min} \leq R_0 < R(E_{max})$
- Case 3: $R_0 < R_{min}$,

are illustrated in Figure 6.

In Case 1, the optimum investment is the one with the maximum value of E ; in Case 2, it is at point X where the straight line $R = R_0$ intersects the boundary; in Case 3 there is no suitable investment.

2.19. We now examine by means of an example the properties of the function W . Suppose that two investors X and Y have identical perceptions of risk and very similar financial **circumstances**. Both are considering **investing** in a particular area on a ten year time horizon to repay part of a house mortgage and to provide a lump sum for retirement.

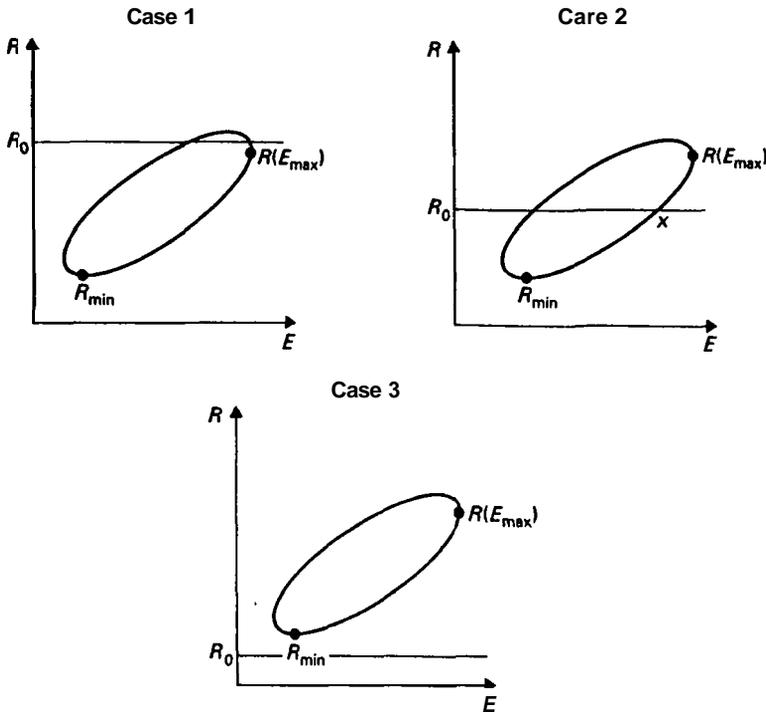


FIGURE 6

X invests £30,000 and Y invests £20,000, and a typical investment in the area in question gives an expected value of 2.5 in ten years' time of a unit initial investment. Y will receive £25,000 from a paid-up non-profit endowment policy **maturing** in ten years' time. **Apart from this**, neither X nor Y has any other investments. X and Y both regard a value of **£75,000** in ten years' time as being a satisfactory outcome, and both **assess** the consequences of various shortfalls from **£75,000** as follows :

Shortfall £	Consequence	Severity
10,000	Moderately lower lump sum	Minor
20,000	Significantly lower lump sum	Moderate
30,000	Very small lump sum	Serious
40,000	No lump sum	Very serious
50,000	£10,000 shortfall on loan ; other savings used to make up shortfall	Exceptionally serious
60,000	£20,000 shortfall on loan ; forced sale of house to repay loan	Catastrophic

Let the risk weighting functions used by X and Y be $W_X(s)$ and $W_Y(s)$ respectively, and let the density function of return be $p(r)$, so that adverse financial consequences arise if $r < 2.5$. For a particular value s of shortfall to X, the value of risk weighting function which relates to the **consequences** of this shortfall is $W_X(2.5 - r_1)$ where :

$$30,000 - r_1 = 75,000 - s.$$

For this value of r_1 , the shortfall to Y is :

$$50,000 - 20,000 - r_1 = \frac{2}{3}s.$$

and this is the shortfall to X when the value of r is r_2 where :

$$r_2 = 2.5 - \frac{2}{3}(2.5 - r_1)$$

$$\text{i.e. } 2.5 - r_2 = \frac{2}{3}(2.5 - r_1)$$

In other words, Y attaches the same consequences to return r_1 as X attaches to return $2.5 - \frac{2}{3}(2.5 - r_1)$.

$$\begin{aligned} \text{Hence } W_Y(2.5 - r_1) &= W_X(2.5 - \{2.5 - \frac{2}{3}(2.5 - r_1)\}) \\ &= W_X(\frac{2}{3}\{2.5 - r_1\}). \end{aligned}$$

Since $\frac{2}{3}$ is the ratio of the amount invested by Y to the amount invested by X, 2.5 is the common threshold of return above which no adverse consequences arise, and $2.5 \cdot r_1$ is the **shortfall** in return, we can deduce the following general result :

Where **different** investors X and Y assess risk in the same manner and employ the same threshold of return, their risk weighting functions $W_X(s)$ and $W_Y(s)$ are related by :

$$W_Y(s) = W_X\left(\frac{s}{k}\right)$$

where k is the ratio of the amount invested by X to the amount invested by Y .

2.20. Suppose now that for a particular investment the density function of return $p(r)$ is modified slightly by "removing" two small units of probability Δp between $r - \Delta r$ and $r + \Delta r$ and "adding" Δp between $r - r_1 - \Delta r$ and $r - r_1$, and between $r + r_1$ and $r + r_1 + \Delta r$, as shown in Figure 7.

The expected value of the distribution remains unchanged.

All investors will regard one quantum increase in the shortfall (from $L - r$ to $L - r + r_1$) as increasing the risk by more than the reduction resulting from one quantum decrease in the shortfall (from $L - r$ to $L - r - r_1$), and hence the modified distribution has higher risk. It is shown in Appendix 2.3 that we can deduce from this the following result :

$$\frac{d^2 W}{ds^2} > 0.$$

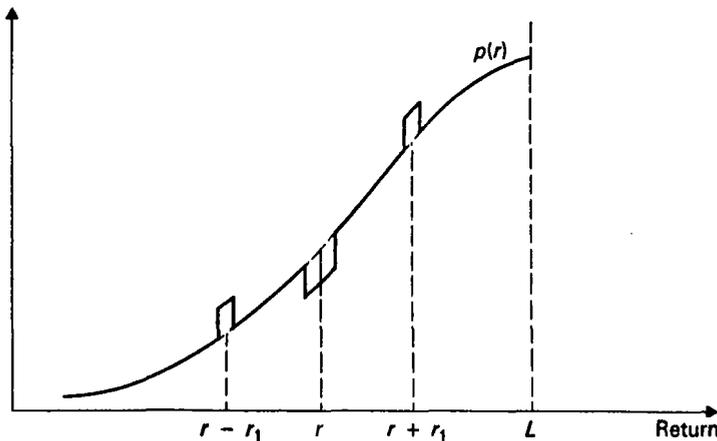


FIGURE 7

2.21. The final property of $W(s)$ that we examine relates to its rate of increase with s . In the sports example, we used a multiplicative factor of 10 for each successive increase in the degree of severity. Some other factor, such as 5 or 20, could also be used ; the actual value depends on how risk-averse the individual is. Again in the case of investment risk, it is up to the individual investor to specify how rapidly his risk weighting function should increase. For the same consequences and severities as set out in section 2.19 above, one investor might regard the consequences of a £20,000 shortfall as being four times as serious as the consequences of a £10,000 shortfall. Another investor, possibly with no family or close relatives to leave his money to, might regard the consequences as being only three times as serious. A third investor who tended to worry very much about the possibility of financial hardship might regard the consequences as being eight times as serious.

Clearly the actual level of $W(s)$ is arbitrary ; it is the **proportionate** rate of increase with s that **matters**.

$$\frac{1}{W} \frac{dW}{ds}$$

2.22. We can now summarise the general properties of the **risk** weighting function $W(s)$.

(i) $W(s) > 0$ and $\frac{dW}{ds} > 0$ from **Axiom 2**.

(ii) $\frac{d^2W}{ds^2} > 0$ from **Section 2.20**.

(iii) From section 2.19 above it can be seen that a scale parameter k is required to relate the various values of **return** to the severity of the **consequences** for a particular investor. **Accordingly**, W is a function of s/k where $k > 0$.

(iv) To allow for a varying degrees of aversion to **risk** as **described** in **section 2.19** above it would be highly convenient to have a one-parameter **family** of functions $W(s, a)$ where a is the parameter describing the degree of aversion to risk. Then $\frac{1}{W} \frac{dW}{ds}$ increases with a .

2.23. The two most obvious simple classes of function satisfying these requirements are :

$$W_a\left(\frac{s}{k}\right) = \left(\frac{s}{k}\right)^a \quad \text{for } a > 1$$

and

$$W_a\left(\frac{s}{k}\right) = e^{as/k} \quad \text{for } a > 0.$$

In the **first case** :

$$\frac{dW}{ds} = a \frac{s^{a-1}}{k^a} > 0,$$

$$\frac{d^2W}{ds^2} = a(a-1) \frac{s^{a-2}}{k^a} > 0,$$

and $\frac{1}{W} \frac{dW}{ds} = \frac{a}{s}$ which increases with a .

In the **second case** :

$$\frac{dW}{ds} = \frac{a}{k} e^{as/k} > 0,$$

$$\frac{d^2W}{ds^2} = \left(\frac{a}{k}\right)^2 e^{as/k} > 0,$$

and $\frac{1}{W} \frac{dW}{ds} = \frac{a}{k}$ which increases with a .

224. Since $\frac{1}{W} \frac{dW}{ds}$ decreases with s in the first case and is constant in the second, it ~~might~~ be thought that these functions are so different that quite different rankings of risk would be obtained. However, the situation is similar to the "curve of deaths" $\mu_x l_x$, which is the product of one function which increases rapidly with x and another which decreases to zero as x increases and results in a well-defined "hump". The product $W(s)p(s)$ is in general similar, as shown in Figure 8.

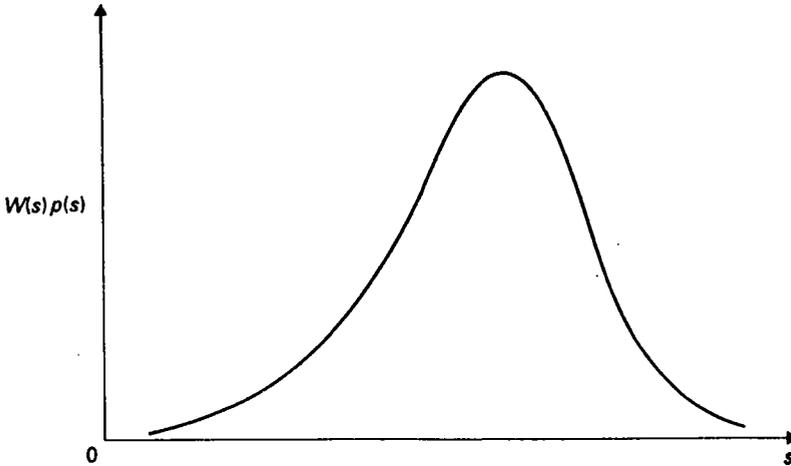


FIGURE 8

For different risk aversion parameters a_1 and a_2 for the first and second classes of function respectively, very similar values of risk, and hence ranking of risk, will be obtained if

$$\frac{a_1}{s} \approx \frac{a_2}{k}$$

in the vicinity of this "hump".

2.25. For the first class of function, there is a very obvious limiting condition as a tends to 1 from above (corresponding to the investor becoming totally indifferent to risk); the risk value tends to the limit

$$R = \int_{-\infty}^L \left(\frac{L-r}{k}\right) p(r) dr = \int_0^{\infty} \left(\frac{s}{k}\right) p(s) ds \quad \text{where } s = L-r.$$

For the second class of function, we note that

$$R = \int_0^{\infty} e^{as/k} p(s) ds \quad \text{where } s = L-r$$

$$\begin{aligned}
&= \int_0^{\infty} \left[1 + \frac{as}{k} + \frac{1}{2!} \left(\frac{as}{k} \right)^2 + \frac{1}{3!} \left(\frac{as}{k} \right)^3 + \dots \right] p(s) ds \\
&= \int_0^{\infty} p(s) ds + a \int_0^{\infty} \left(\frac{s}{k} \right) p(s) ds + \frac{a^2}{2!} \int_0^{\infty} \left(\frac{s}{k} \right)^2 p(s) ds + \frac{a^3}{3!} \int_0^{\infty} \left(\frac{s}{k} \right)^3 p(s) ds + \dots
\end{aligned}$$

The **first** integral is a **constant**; the others are the values of risk on the first basis with the degree **of** risk parameter equal to 1, 2, 3, ... As a tends to zero, the third and subsequent terms can be ignored, and in an obvious notation:

$$R^{\text{II}} = \text{Constant} + a \cdot R^{\text{I}}$$

Thus for very small a , the second class of function gives the same ranking order of risk as does the first class of function for its limiting case.

2.26. Suppose that two investors X and Y assess risk in the same way and are considering different amounts of a particular investment and hence have different scale parameters k_x and k_y . On the **first** basis,

$$\begin{aligned}
R_x &= \int_0^{\infty} \left(\frac{s}{k_x} \right)^a p(s) ds \\
&= \left(\frac{k_y}{k_x} \right) \int_0^{\infty} \left(\frac{s}{k_y} \right)^a p(s) ds \\
&= \left(\frac{k_y}{k_x} \right)^a R_y
\end{aligned}$$

2.27. A further key result can be obtained when the first class of function for W is used.

If we have a density function $p(r)$, which - without loss of generality - can be assumed to have expected value zero and unit variance, we can change the location **and/or** dispersion to obtain other distributions with similar "shape" but different expected value **and/or** variance. Thus the related density function of return with expected value μ and variance σ is

$$p(r, \mu, \sigma) = \frac{1}{\sigma} p\left(\frac{r-\mu}{\sigma}\right).$$

Then, in an obvious notation,

$$R_L^a(\mu, \sigma) = \sigma^a \cdot R_{(L-\mu)/\sigma}^a(0, 1)$$

See Appendix 2.4 for the proof.

For a particular value of a , it is only necessary to calculate the risk on the standardised distribution for various values of L . Approximate integration is usually the easiest way to do this. Then the risk, rather than being a function of the three variables L , μ and a , is essentially a function of one variable, namely the modified threshold

2.28. We can obtain a similar result in the case where a density function $p(r)$ is defined for $r > 0$ and the related investment has distribution function $\frac{1}{\lambda} p\left(\frac{r}{\lambda}\right)$

for $\lambda > 0$. Then, again in an obvious notation,

$$R_L^a(\lambda) = \lambda^a R_{L/\lambda}^a(1).$$

See Appendix 2.5. for the proof.

239. The results set out in the three preceding sections are clearly exceptionally useful in both theoretical and practical work. No such simple relationships arise with the second class of function for W . Accordingly, we choose the first class of function to represent the risk weighting function W unless it is apparent that very special circumstances apply and that the second class of function, or some other class of function, should be used instead.

230. By finding a suitable class of function for W we have shown how measures of risk can be calculated in practice. Before the various axioms can be used to select particular investments we must be able to show how the risk threshold R_0 can be calculated. Again we return to the sports example ; the threshold was obtained there by specifying that the probability of injuries of a certain degree of severity or worse was not greater than some chosen value and that the relative frequencies of the various adverse consequences follow a given probability distribution. In the case of investment risk a similar approach is clearly highly satisfactory ; the threshold can be obtained by specifying that the probability of adverse financial consequences of a certain degree of severity or worse is not to exceed some chosen value and that the relative frequencies of the various adverse consequences follow a given probability distribution.

231. This completes the construction of a general theory of investment risk. By modifying the provisional axioms in section 2.15 in the light of the subsequent discussion and adding further axioms to specify how W and R_0 can be calculated we can express this general theory in terms of eight axioms :

Axiom 1 Investment risk is a function both of the probability of the return being below a certain threshold and also of the severity of the financial consequences arising from these values of return.

Axiom 2 If the density function of return, $p(r)$, is known, investment risk can be expressed as a non-negative measure :

$$R = \int_{-\infty}^L W\left(\frac{L-r}{k}\right) p(r) dr$$

where L is the value of return above which no adverse **consequences** arise;

$W(s)$ is defined for $s > 0$;

$W(s) > 0$;

$$\frac{dW}{ds} > 0$$

$$\frac{d^2W}{ds^2} > 0;$$

k is a **positive** scale **parameter**.

- Axiom 3** For investments with the same expected return, an investor will prefer the investment with the lowest risk
- Axiom 4** Each investor has a threshold of risk R_0 and will not make an investment which involves a value of risk higher than R_0 .
- Axiom 5** An investor will choose between all possible investments by **maximising** the expected **return** subject to the risk not **exceeding** R_0 .
- Axiom 6** Different investors may differ in their degree of aversion to risk by using different functions $W(s)$ and/or different threshold of risk R_0 .
- Axiom 7** For most practical and theoretical work there is very little loss of generality in assuming :

$$W\left(\frac{s}{k}\right) = \left(\frac{s}{k}\right)^a \quad \text{where } a > 1.$$

- Axiom 8** For most practical work it will be possible to calculate an explicit value for R_0 by specifying that the probability of financial consequences of a certain degree of severity or worse is not to exceed some chosen value and that the relative frequencies of the various adverse consequences follow a given probability distribution.

3. PRACTICAL EXAMPLE

3.1 We now give a detailed example (an extension of the example in section 2.19 above) which shows how the general **theory** of risk as defined above in terms of **the** eight **axioms** can be applied to practical situations.

3.2. Investors X, Y and Z have identical **perceptions** of **risk** and very similar financial

circumstances. To provide, in ten years' time, for mortgage repayment and a retirement lump sum they have the choice of investing either in area **A** (which involves various **types** of unit trusts) or in area **B** (which involves discretionary Portfolio management with an element of unlimited liability on futures and **options** contracts). In each area there is a choice between relatively defensive and relatively aggressive **portfolios**.

In area **A** the density function at the **value** of the end of ten years for unit initial investment is **lognormal**, "starting" at zero, with expected value 2.5 and standard deviation of 1.25. This "standard" portfolio involves general U.K. equity unit **trusts**. The most defensive portfolio, with an expected return of 1.875 (75 % of **2.5**), involves convertible stocks and high yielding equities, and the most aggressive portfolio, with an expected return of 3.75 (150 % of **2.5**), involves high growth shares. The equivalent annual rates of return are 6.5 % **p.a.**, 9.6 % **p.a.** and 14.2 % **p.a.** respectively for "defensive", "standard" and "aggressive". Intermediate portfolios can also be chosen. Dividing out by 2.5, the values of the various portfolios at the end of the period have density functions

$$p_{\lambda}^{\lambda}(r) = \frac{1}{\lambda} p_A\left(\frac{r}{\lambda}\right) \quad \text{for } .75 \leq \lambda \leq 1.5$$

where $p_A(r)$ is the lognormal distribution derived from the transformation $y = 2.1169 \log_e x$ which has expected value **1** and standard deviation 0.5. The density functions for the portfolios with $\lambda = 0.75, 1, 1.25$ and 1.5 are illustrated in Figure 9.

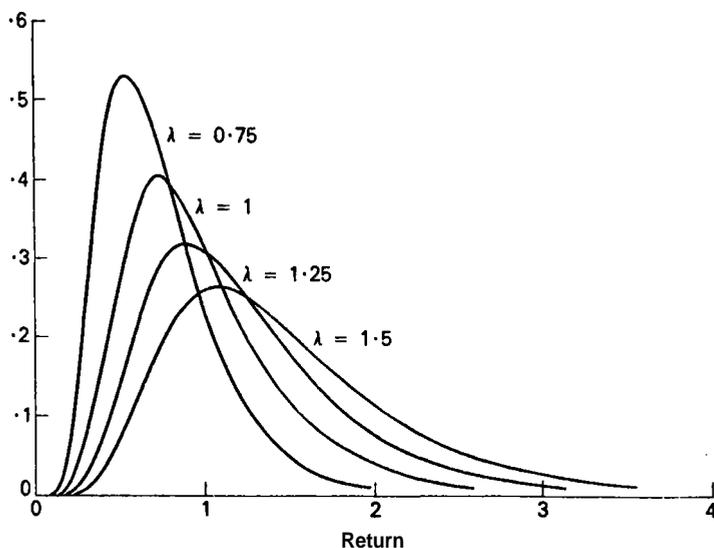


FIGURE 9

In area B, the corresponding distributions (again with the factor 2.5 taken out) are **normal**, ranging from expected value 1 and standard deviation 0.4 to expected value 2 and standard deviation 1. The standard deviation increases linearly with expected return. For the least **risky** portfolio, the expected return is 2.5 standard deviations, so that **the probability** of the **investment** being less than zero in value at the end of the period is 0.62 %. Similarly, the expected return of the riskiest portfolio is 2 standard deviations, giving a probability of 2.28 % of the value **being** less than zero after ten years.

In area B, again dividing out by 2.5, the density functions of the end-period values of the various portfolios are :

$$p_B^\lambda(r) = N(\lambda, \cdot 6\lambda - \cdot 2) \quad \text{for } 1 \leq \lambda \leq 2.$$

The density functions for the portfolios with $\lambda = 1, 1.5$ and 2 are illustrated in Figure 10.

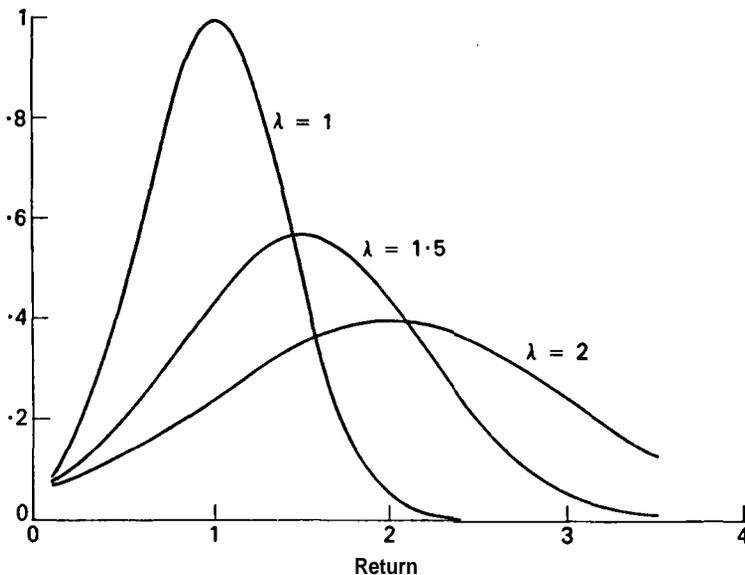


FIGURE 10

X, Y and Z all regard the severity of the consequences of a particular **absolute** amount of shortfall as identical. X invests **£30,000** and Y invests **£20,000**. They **have** no other investments, and both base their financial planning on an expected value of 2.5 times the amount **invested**. Z's circumstances are identical to those of Y except that, in addition to investing **£20,000** in either area A or area B he will receive **£10,000** under a legacy in ten years' time and accordingly measures the shortfall of return from a threshold of **£40,000** rather than **£50,000** in the case of Y.

All three regard a £20,000 shortfall as 4 times as serious as a shortfall of £10,000. Also, all three base their threshold of risk on the criterion that there must not be more than one chance in a hundred of the shortfall exceeding £30,000 when the various categories of shortfall (£10,000 - minor ; £20,000 - moderate ; £30,000 - serious ; £40,000 - very serious ; £50,000 - exceptionally serious ; £60,000 - catastrophic) have relative probabilities corresponding to 0.5, 1, 1.5, 2, 2.5 and 3 standard deviations respectively from the mean on a normal distribution.

To demonstrate the effect of a different degree of aversion to risk, we shall then repeat the analysis with a shortfall of £20,000 being regarded as 8 times as serious as a £10,000 shortfall and with the probability of a shortfall exceeding £30,000 being not greater than one in two hundred and fifty.

3.3. For the risk aversion parameter a , we note that the risk weight function for a shortfall of £ 20,000 is 4 times that for a shortfall of £ 10,000. Hence

$$\left(\frac{20,000}{k}\right)^a = 4 \cdot \left(\frac{10,000}{k}\right)^a,$$

i.e. $2^a = 4$ and hence $a = 2$.

We also note that, without loss of generality, we can put $k_y = 1$ to simplify the calculations.

3.4. To obtain R_0 , we note that the density function for a shortfall of the "standard" type is $b \cdot f(x)$ where b is a constant and $f(x)$ is the standard normal density function. Also,

$$\begin{aligned} .01 &= \text{probability of shortfall} \geq 15 \\ &= b(1 - F(1.5)) \end{aligned}$$

$$\begin{aligned} \text{where } F(x) &\text{ is the standard normal distribution function} \\ &= b(1 - .93319) \end{aligned}$$

$$\text{i.e. } b = \frac{1}{6.681}$$

$$\text{Hence } R_0 = \frac{1}{6.681} \int_{-\infty}^0 x^2 f(x) dx \quad \text{where } f(x) \text{ is the standard normal density function}$$

$$= \frac{.5}{6.681} \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= .07484 \text{ since } \int_{-\infty}^{\infty} x^2 f(x) dx = 1$$

3.5. We calculate first of all, for different values of L , the risk values on $A(1)$ as perceived by Y . These are calculated as :

$$R_L^2(1) = \int_0^L (L-r)^2 p(r) dr$$

where $p(r)$ is the lognormal density function. An extract from tables of values calculated by approximate integration is shown in Appendix 3.1.

To calculate the risk value of $A(\lambda)$ for values of λ other than 1 we use the result set out

in section 2.28, namely :

$$R_1^2(\lambda) = \lambda^2 R_{1/\lambda}^2(1)$$

$$\begin{aligned} \text{For example, } R_1^2(0.8) &= (.8)^2 R_{1.25}^2(1) \\ &= .64 \times .21351 \\ &= .1366. \end{aligned}$$

3.6. We now calculate, for different values of L , the risk values on a normal distribution with zero expected value and unit standard deviation. These are calculated as :

$$R_L^2(0, 1) = \int_{-\infty}^L (L-r)^2 p(r) dr$$

where $p(r)$ is the standard normal density function. Again an extract from tables of values calculated by approximate integration is shown in Appendix 3.1.

To calculate the risk values on $B(\lambda)$ we then use the result set out in section 2.27, namely :

$$R_1^2(\mu, \sigma) = \sigma^2 \cdot R_{(1-\mu)/\sigma}^2(0, 1).$$

For example, $B(1.5)$ has standard deviation 0.7 and hence :

$$\begin{aligned} R_1^2(1.5, .7) &= .49 R_{.714}^2(0, 1) \\ &= .49 \times .13800 \\ &= .0676. \end{aligned}$$

These risk values relate to Y . For X , the scaling factor (using the result set out in section 2.19) is $k_X = 1/1.5$ and so all the risk values for X are 1.5^2 times those for Y . Rather than recalculate these values for X , we can use the same values as for Y and use a different value of R_0 namely

$$R_0^X = \frac{1}{1.5^2} R_0^Y = .03326.$$

3.7. The various risk values are shown in Figure 11.

For all values of expected return in the range 1 to 1.5, A has lower risk than B . For A , risk decreases rapidly as expected return E increases. For B , the risk when $E = 1$ is slightly greater than that for A ; risk then decreases with E to around $E = 1.4$ and then increases with E .

For X , investment in area B is not possible : $B(E)$ has a higher value of risk than R_0^X for all E . Also, for A only values of E greater than 1.33 are possible. Hence X will

choose the investment in area **A** with **maximum** expected **return**, namely **A (1.5)**.

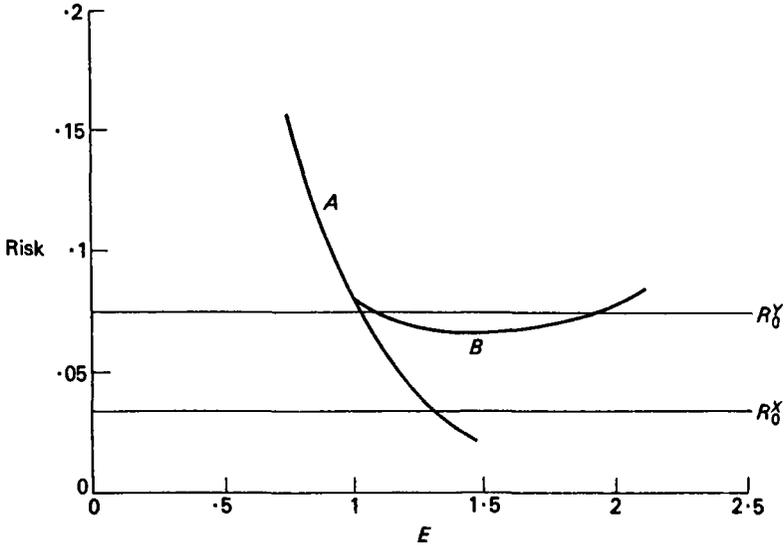


FIGURE 11

For Y, investment in area **A** is possible for E greater than 1.02, and investment in area B is possible for E between 1.08 and 1.98. Y will therefore choose B (1.98), this having the **maximum** expected **return** possible for a risk value not exceeding R_0^Y .

For Z, it is not **possible** to use the risk values already calculated since Z uses a **different** value of risk threshold L, namely 0.8 instead of 1 for X and Y. It is likely, however, that B (2) will be possible for Z, in which case he will choose that **investment**. The value of risk to Z is 0.04776, which is less than $R_0^Z = 0.07484$, and hence Z will choose B (2).

38. For the higher degree of aversion to risk the approach is similar. We then have

$$\left(\frac{20,000}{k}\right)^a = 8 \cdot \left(\frac{10,000}{k}\right)^a,$$

i.e. $2^a = 8$ and $a = 3$.

Similarly

$$R_0^Y = \frac{1}{2.5 \times 6.681} \int_{-a}^0 x^3 f(x) dx \quad \text{where } f(x) \text{ is the standard normal density function}$$

$$= .04777,$$

and $R_0^X = \frac{1}{1.5^3} R_0^Y$

$$= .01415.$$

The corresponding chart for $a = 3$ is shown in Figure 12.

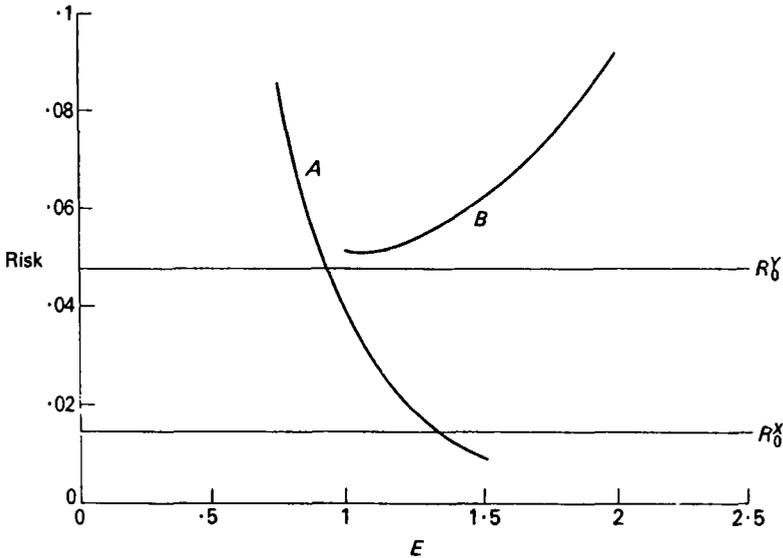


FIGURE 12

Again for investments in area A risk decreases rapidly with E, but this time the risk for B(1) is significantly higher than the risk for A(1). Also, from about $E = 1.05$ the risk for B(E) increases with E.

For X, investment in area A is possible for E greater than 1.33 and so X will choose A(1.5). For Y, investment in area B is not possible for any value of E, and hence Y will also choose A(1.5). For Z, the risk for B(2) is 0.05491, which is greater than R_0^Z . It is then found that the maximum value of E for which the risk on B(E) is not greater than this threshold is 1.87. Accordingly, Z will choose B(1.87).

39. These results can be summarised as follows :

	Normal risk aversion	High risk aversion
X	A (1.5)	A (1.5)
Y	B (1.98)	A (1.5)
Z	B (2)	B (1.87)

Clearly these results are consistent with general reasoning. Since Y is investing less than X, any particular investment is "less risky" to him in view of the lower probability of any given absolute level of shortfall. Also, with his different threshold of absolute return,

Z will perceive a particular investment as "less risky" than perceived by Y. Finally, for each investor, investments in area B are perceived as being much riskier on the higher degree of aversion to risk on account of the much higher weighting attached to high values of shortfall which cannot occur with investments in area A.

4. COMPARISON WITH THE MARKOWITZ MODEL

4.1. By far the most important development in the theory of portfolio selection was the paper by Markowitz (2) in which he suggested that the process should be tackled in three stages :

- (i) estimating the future return from securities ;
- (ii) determining from these estimates an "efficient" set of portfolios ; and
- (iii) selecting from that set the portfolio best suited to a particular investor's preferences.

4.2. The key axiom introduced by Markowitz is that for portfolios with the same expected return E an investor will prefer the portfolio with the lowest variance of return V , and such a portfolio is defined as being "efficient". The situation is illustrated in Figure 13. The attainable portfolios are represented as a convex region in the E - V diagram and the efficient set consists of the portfolios lying along the frontier ABC .

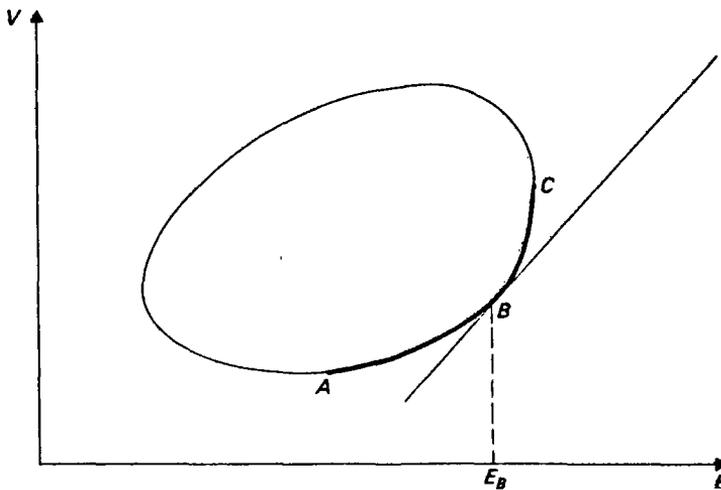


FIGURE 13

If the tangent at B has gradient λ , the portfolio represented by point B is found by maximising the function $U = \lambda E - V$ over all portfolios which have expected value E_B .

Markowitz showed how the required maximum could be found using his "critical line" method ; all points on the efficient frontier could then be calculated by repeating the process for different values of λ from zero to infinity.

4.3. Markowitz assumed that reasonable estimates of the expected returns and covariances of returns could be obtained for all the available securities, but he emphasised that his paper did not attempt to cover this first stage in the portfolio selection process. Also, he did not elaborate on the final selection of a portfolio from the efficient frontier, but instead merely commented that "The investor, being aware of what (E, V) combinations were available, could state which he preferred".

4.4. This pioneering work of Markowitz was virtually ignored for several years until other researchers, notably Tobin [6] and Sharpe [7] [8], took up the challenge and modified and extended the original theory. The theory underlying the use of models for portfolio selection became known as portfolio theory, and a very useful description of the principal methods is given by Moore [9].

4.5. As regards the third stage of the Markowitz approach, namely selecting from the efficient set the portfolio best suited to a particular investor's requirements, the general statement that "investors choose between portfolios solely on the basis of expected returns and variance of returns" gradually evolved into the much more precise statement that "investors choose between portfolios by maximising the utility function $U = \lambda E - V$, where the selection of λ by an investor is equivalent to his specifying his attitude towards risk". This value of λ corresponds to the gradient of the tangent in Figure 13.

4.6. Portfolio theory, which is essentially the Markowitz approach to portfolio selection as modified to the extent described in section 4.5, is similar in many ways to the general theory developed in Part 2, except that the concept of a maximum level of risk does not appear. To highlight these similarities, we can express the Markowitz formulation in terms of axioms which are remarkably similar to those set out in section 2.31 :

Axiom 1M Investment risk is a function of the uncertainty of the return.

Axiom 2M Investment risk can be measured by the dispersion of the return.

Axiom 3M For investments with the same expected return, an investor will prefer the investment with the lowest risk (Identical to Axiom 3).

Axiom 4M No equivalent.

Axiom 5M An investor will choose between investments on the "efficient frontier" identified by Axiom 3M by maximising a utility function U of expected return E and risk.

Axiom 6M Different investors may differ in their degree of aversion to risk by using different utility functions U .

Axiom 7M For most practical and theoretical work there is very little loss of generality in assuming that dispersion can be measured by the variance of return V .

Axiom 8M For most practical and theoretical work it can be assumed that the utility function U is : $U = \lambda E - V$.

47. We **now** examine what results are obtained by applying portfolio theory to the detailed example in Part 3. The relevant $E \cdot V$ diagram is shown in Figure 14.

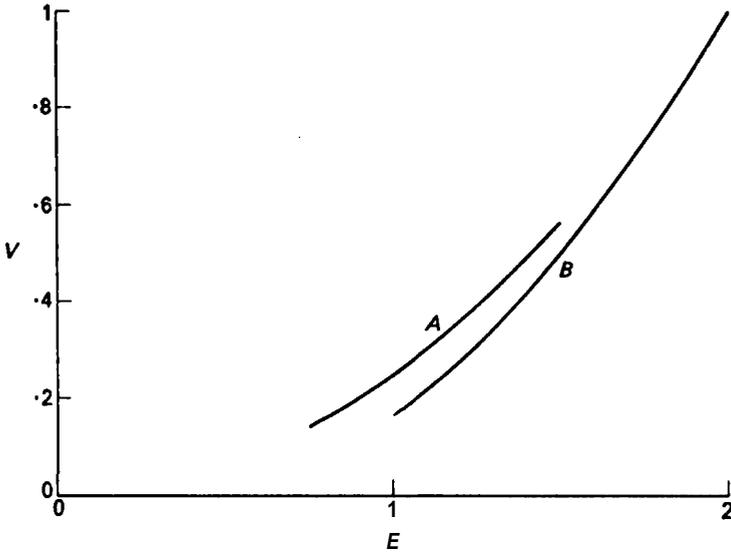


FIGURE 14

For values of E from 1 to 1.5, the variance of return on investment B is always lower than that on investment A, and the efficient frontier therefore **consists of** all investments B together with investments A for values of E from 0.75 to 1. There are then four **cases** for the final investment chosen, depending on the values of λ , the gradient of the tangent to the efficient frontier :

A	Investment
$0 \leq \lambda < .0775$	A (.75)
$.0775 \leq \lambda < .48$	B (1)
$.48 < \lambda < 1.2$	B (E) where $1 < E < 2$
$\lambda \geq 1.2$	B (2)

Although the portfolio theory approach does not specify what numerical values of λ are appropriate, it is reasonable to make the following deductions :

- (i) For a particular degree of aversion to risk, the circumstances of the three investors X, Y and Z are such that Y can choose a "riskier" investment than X and that Z can choose a "riskier" investment than Y, and hence $\lambda_x < \lambda_y < \lambda_z$

- (ii) Each investor will use a lower value of λ in the case of the higher degree of aversion to risk compared with the value he would use on the basis of the normal degree of aversion to risk Possible values for λ might then be :

	Normal aversion to risk	High aversion to risk
X	.5	.25
Y	1	.5
Z	2	1

4.8. On the basis of these values of λ the solutions obtained using portfolio theory, together with the solutions obtained earlier, are :

	Portfolio theory		General theory	
	Normal aversion to risk	High aversion to risk	Normal aversion to risk	High aversion to risk
X	B (1.03)	B (1)	A (1.5)	A (1.5)
Y	B (1.72)	B (1.03)	B (1.98)	A (1.5)
Z	B (2)	B (1.72)	B (2)	B (1.87)

4.9. This example highlights three key differences between the two approaches :

Using portfolio theory, the information about the **circumstances** and preferences of the different investors cannot be used to specify precise values of A. **Accordingly**, the values of λ derived in section 4.7 are arbitrary in the extreme.

Using portfolio theory, there is **the** possibility of selecting A(0.75) as the best investment, whereas (for the reasons set out in section 1.2) A(1.5) is for all practical purposes a "lower **risk**" investment and also has twice the expected return.

Using portfolio theory, A(1.5) can never be chosen as the best investment **since** it is not on the efficient frontier, regardless of the degree of aversion to risk. In **terms** of the general theory, however, A(1.5) has a significantly lower risk than B(1.5) for high degrees of aversion to risk.

4.10. It might be argued that, **&spite** the serious shortcomings exhibited in the case of this detailed example, the portfolio theory approach could be a sufficiently accurate special case of the general theory in certain circumstances, such as when **families** of **normal** distributions are being compared.

4.11. Suppose that we take as the "risk standard" an investment where the density function of end-period value is normally distributed with unit expected value and unit variance. For a particular value a_0 of the risk aversion parameter and any given value of the expected return E, there will be a unique value of variance $V_{a_0}(E)$ at which the risk (**based** on $L = 1$) is the same as that on the "risk standard. Plotting this value against E in the E • V diagram gives an equal risk contour with positive gradient. Using values of 1.5, 2, 2.5 and 3 for the risk aversion parameter we obtain the pattern shown in Figure 15.

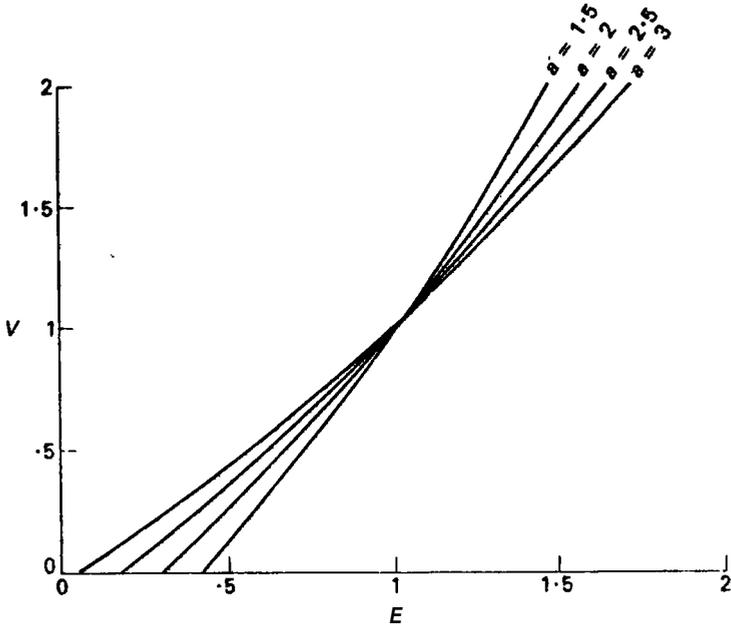


FIGURE 15

All these equal risk contours are concave upwards but over short ranges of E they are not far from being rectilinear, with the gradient decreasing as the risk aversion parameter increases.

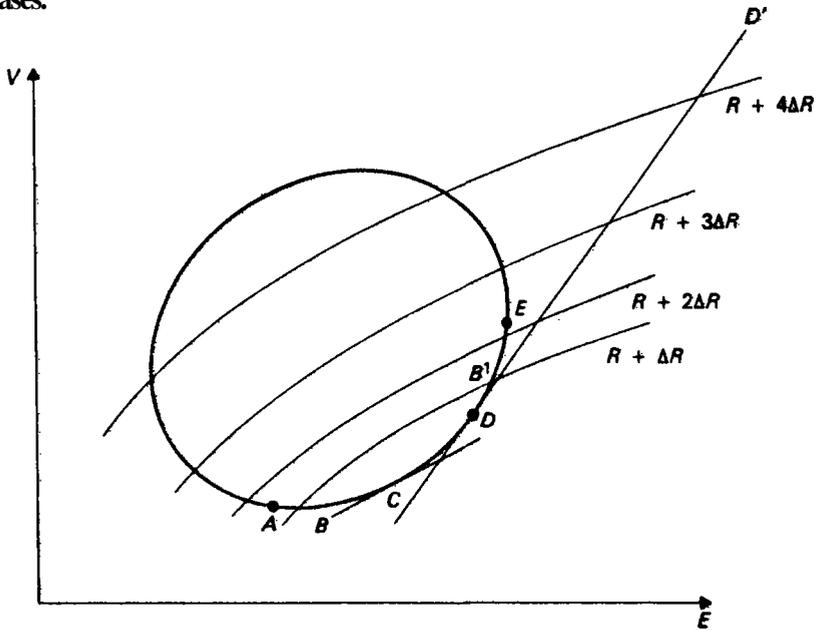


FIGURE 16

4.12. It might be thought that these contours correspond to the $E \cdot V$ indifference curves used in portfolio theory, since in that case also the gradient decreases as the degree of aversion to risk increases. However, as can be seen from Figure 16, this is not the case.

If the equal risk contour joining all portfolios with risk R touches the region of attainable portfolios at C , this point represents the portfolio with the lowest value of risk and will not be the portfolio selected using the general theory unless (by coincidence) R is equal to the risk threshold R_0 . If a tangent with a smaller gradient is used and touches the efficient frontier AE at B , this point cannot be the portfolio selected using the general theory since there is another portfolio at B^1 with the same value of risk but a higher value of expected return. The gradient of the appropriate indifference line must therefore be greater than the gradient of the equal risk contour at C . In this case, the portfolio theory approach will select the same portfolio as the general theory only if the equal risk contour R_0 intersects the efficient frontier at a point D between C and E and the tangent touches the frontier at D . In general it is highly unlikely that both these conditions will be satisfied.

4.13. There is a further serious objection to using the tangent at point D as an indifference line, namely that no upper bound is placed on the risk of the selected portfolio. For a different set of attainable portfolios, the point of contact with the efficient frontier could be at D' which had a very high value of risk compared with point D . The analogy with the dangerous sports example is that every individual would be prepared to take part in hang-gliding if the "enjoyment value" were sufficiently high. This is clearly unrealistic, and again with investment portfolios it is unrealistic to assume that there is no upper bound on the risk to which an investor will expose himself.

4.14. The above arguments suggest that there are serious practical and theoretical objections to the use of a family of indifference curves of the form $U = \lambda E \cdot V$, even when all the distributions involved are of similar shape. In the more general case, even the concept of the efficient frontier is of doubtful validity.

In the example in Part 3 it was assumed for simplicity that each investor could invest entirely in area A or entirely in area B and could not split his investment between the two areas. Assume now that this restriction is removed, and assume further that the covariance between investments in area A and investments in area B is always zero. (If the investments in area B consist entirely of futures and options contracts, this is not an unreasonable assumption since these contracts can be equally profitable in both rising and falling markets). If a fraction t of the total is invested in A (E) and the balance of $1 - t$ of the total is invested in B (E) for any value of E in the range 1 to 1.5, the values of risk and variance for the composite investments (which also have expected value E) vary between the respective values for A (E) and B (E). For any value of expected return in this range, investment A has lower risk and higher variance than investment B . Hence, as t increases from 0 to 1, risk increases but variance decreases. This is a realistic example of the type of situation shown in Figure 17, with the equal risk contours within the region of attainable portfolios decreasing in value as variance increases. In this case the efficient frontier is CB on the top half of the boundary rather than the bottom right hand quadrant AB .

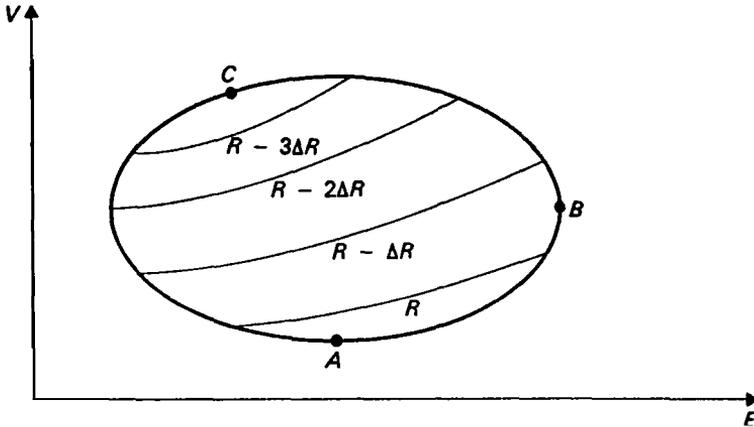


FIGURE 17

4.15. It is generally believed that portfolio theory and the Markowitz approach to **portfolio** selection are essentially one and the same thing. For example, Sharpe [10] states **the position** as follows :

"Markowitz' contribution **was** so monumental that it must be noted explicitly. Others have extended, **modified** and tested his original theory, but the core remains unchanged. In fact, many prefer the term *Markowitz theory* to *portfolio theory*. The **terms** are, for all practical purposes, synonymous".

Given the very **serious** inconsistencies between the general **theory developed** in Part 2 and portfolio theory, it might appear that **the** general theory must also be inconsistent with the Markowitz approach. However, I believe that **the true** situation is quite different, namely that my general theory represents a **straightforward** extension of the Markowitz approach whereas portfolio theory excludes two key elements of the Markowitz formulation and accordingly has very little theoretical or practical validity.

4.16. Consider **first of** all investments with the same expected return. For families of distributions with the same general shape, it can readily be seen that the Markowitz approach will give the same result as the general **theory**, since **the investment** with the lowest variance also has the lowest risk. Take, for example, normal distributions with unit expected value and differing values of variance. The values of risk on **the** basis of widely differing values of **the** risk threshold L and the risk aversion **parameter** α are set out below.

a= Variance	L=.5		L=1		L=1.5	
	1.5	3	1.5	3	1.5	3
.1	.005	.001	.076	.025	.412	.276
.2	.018	.006	.129	.071	.470	.432
.3	.037	.019	.174	.131	.523	.594
.4	.058	.038	.216	.202	.573	.763
.5	.080	.065	.256	.282	.619	.940
.6	.103	.098	.293	.371	.664	1.123
.7	.126	.138	.329	.467	.706	1.312
.8	.149	.183	.364	.571	.747	1.508
.9	.172	.234	.397	.681	.786	1.709
1.0	.195	.291	.430	.798	.824	1.916

The efficient frontiers for both approaches will be identical, since minimum risk on every basis is equivalent to minimum variance.

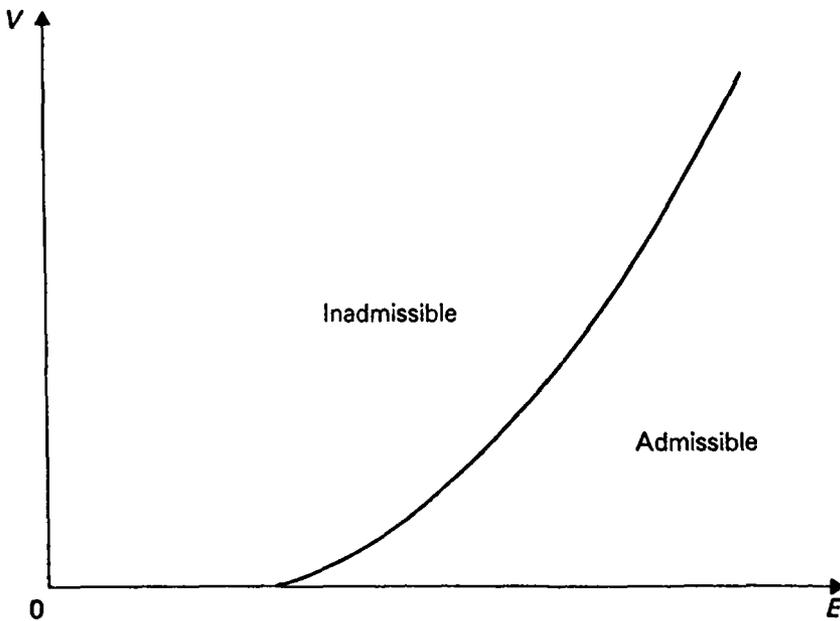


FIGURE 18

4.17. Consider **now the** way in which **an** investor should **tackle** the third stage of **the** Markowitz **approach**, namely selecting from the efficient frontier the portfolio that is best suited **to** his requirements. For **any** given expected value, portfolios decline in **attractiveness as** variance increases, and there will in general be **some** value **of** variance above which the investor regards a portfolio as unsuitable. Accordingly, we can begin the third stage by **asking** the investor **the** question "For each value of expected return, what is **the maximum** value of variance that you are prepared to accept?" From his response we can draw a line in the E - V diagram as show in Figure 18 such that portfolios lying below this frontier are "admissible" and portfolios lying above this frontier are "inadmissible".

4.18. We now examine the general properties of this admissibility frontier. **Different** individuals with the same financial circumstances as regards existing assets and liabilities may differ in their attitude to risk, and hence if investor X is **more** risk-averse than investor Y his admissibility frontier **will** be lower than that of Y as shown in Figure 19.

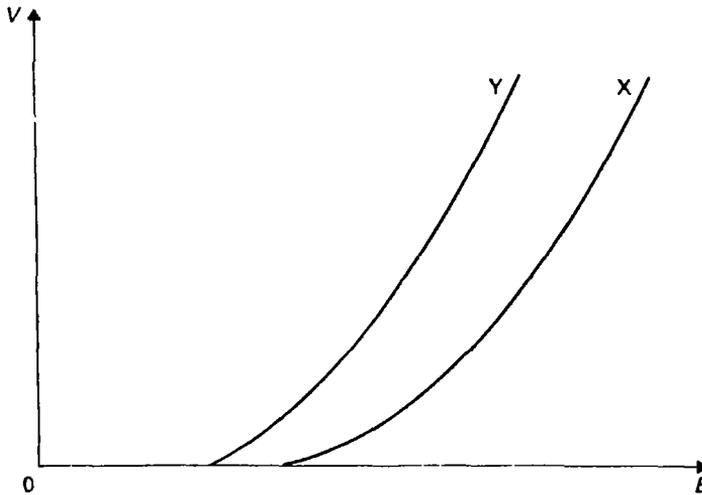


FIGURE 19

4.19. Consider now two investors X and Y who would be equally averse to risk if **their financial** circumstances were identical and assume that Y is **now** considerably wealthier than X. The uncertainty or variance of return **on** a particular investment is of less **consequence** to Y than to X, **and** accordingly Y's **admissibility** frontier will be higher than that of X. Again the situation is portrayed in Figure 19.

4.20. Although **Markowitz** used the variance of **return** as the measure of risk, it was

always clear that he believed that the **semi-variance** was a better measure. For example, **Sharpe [8]** comments as follows :

"Under certain conditions the mean-variance approach can be shown to lead to unsatisfactory **predictions** of behaviour. **Markowitz** suggests that a **model** based on the semi-variance (the average of the squared deviations below the mean) would be preferable ; in the light of the formidable computational problems, however, he bases his analysis on the variance and standard deviation".

Where **distributions** of similar shape are being considered, the ratio of semi-variance to variance **will** be roughly constant, and the variance is a **convenient** and **sufficiently** accurate proxy for the semi-variance in these circumstances. Where, however, the **skewness** varies to any marked extent, a **significant** distortion could arise if the variance alone were used in the analysis. To allow for this, the admissibility frontier **should** be higher the greater the degree of skewness to the right. For example, the lognormal distributions for investment in area **A** in Part 3 have a **semi-variance** of about 32 % of the variance as against 50 % for a normal distribution or any other **distribution** that is symmetric about its mean. Hence each investor will use differing admissibility frontiers for skewed and symmetric distributions as shown in Figure 20.

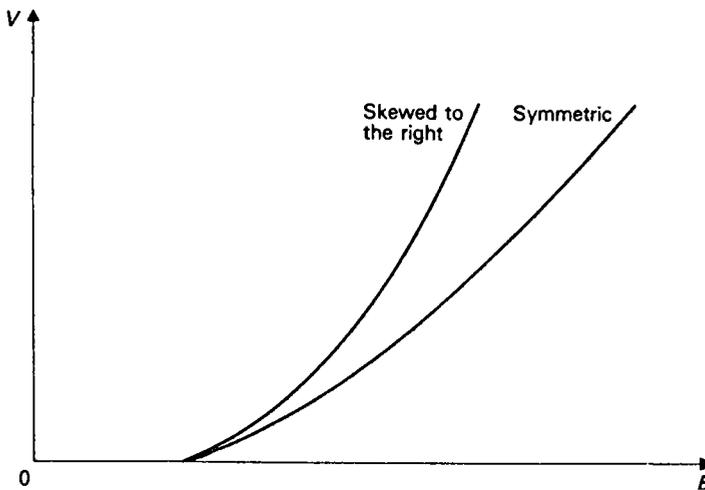


FIGURE 20

421. We now combine the concepts of the admissibility **frontier** and the efficient **frontier** to obtain a decision rule by which the investor will choose the portfolio best suited to his preferences. There are three cases, as shown in Figure 21, depending on whether the admissibility frontier lies above, intersects, or lies below, the **efficient** frontier.

In the first case, consider the portfolio at point C, which has the **maximum** expected return of any portfolio on the efficient frontier. If the investor were to select a portfolio on expected return alone it would be this one. As regards **risk**, this portfolio has a lower risk than the portfolio at point C' on the admissibility frontier which has the same expected return. In other words, the investor is not averse to holding the portfolio at point C on considerations of risk. Accordingly, the selection of this portfolio is fully consistent with the **Markowitz** concept of assessing both **expected** return and risk and then selecting the "best" portfolio in the light of the investor's E - V preferences. Similarly, in the **second** case, the obvious portfolio to select is the **one** at point B, the intersection of the two **frontiers**, since it **maximises** the expected return subject to the risk constraint expressed in terms of the admissibility frontier. In the third case, no portfolio is suitable.

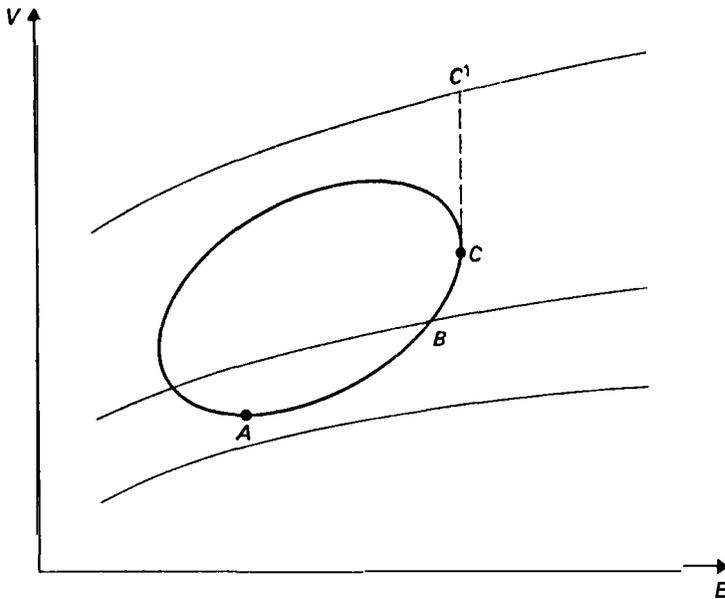


FIGURE 21

4.22. The decision rule employed above can be expressed as follows :

"Within the admissible region (which excludes portfolios with an unacceptable level of risk for the particular value of expected return) select the portfolio (if any) on the efficient frontier which **maximises** the expected return".

4.23. The very close equivalence **between** the above interpretation of the Markowitz approach and the general theory can now be stated very simply : the admissibility frontiers used in the E - V diagram correspond exactly to lines of equal risk as calculated in terms of the general theory.

4.24. This equivalence can be demonstrated by applying the above interpretation of the

Markowitz approach to the numerical example in Part 3. Figure 22 shows the admissibility frontiers in the case of the "normal" degree of aversion to risk and Figure 23 represents the corresponding situation in the case of the higher degree of aversion to risk.

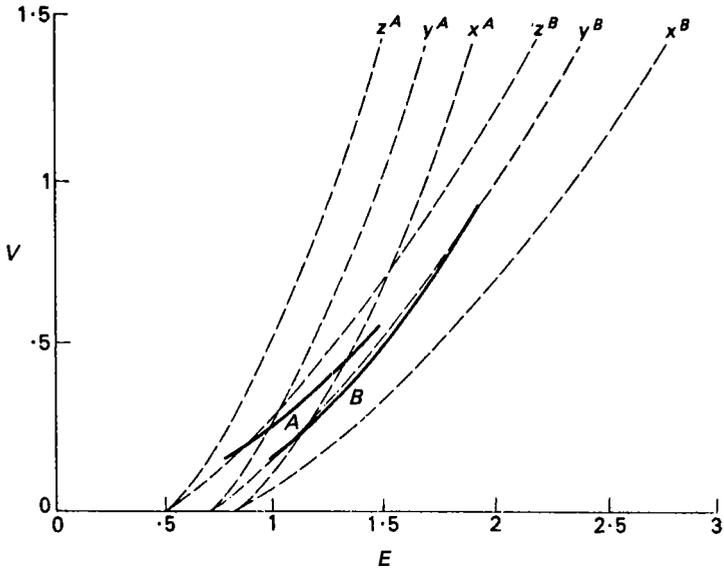


FIGURE 22

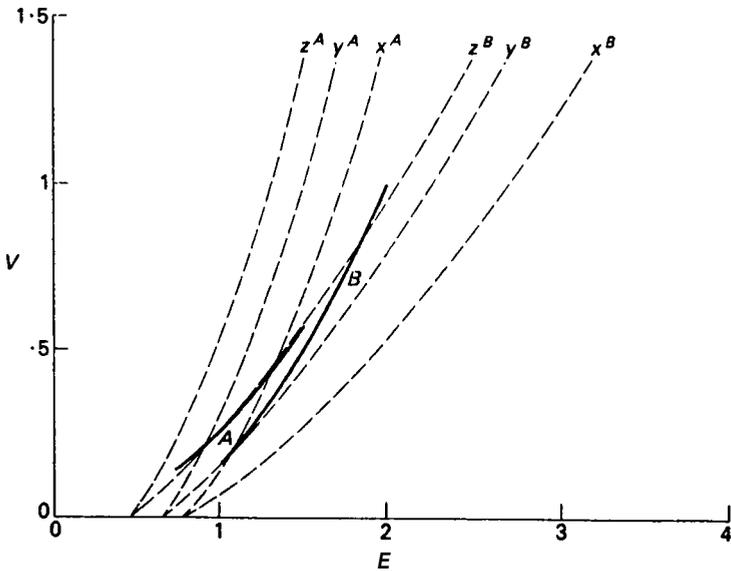


FIGURE 23

The various admissibility frontiers exhibit all three types of variation described in sections 4.18, 4.19 and 4.20 above. Since risk as calculated in terms of the general theory is a monotonic increasing function of variance for any given expected value, these E - V diagrams are effectively identical to Figures 11 and 12, and hence the application of the decision rule set out in section 4.22 above leads to the selection of the same investments as is the case when the general theory is used.

4.25. The integral expression for semi-variance, which is the underlying measure of risk in the Markowitz formulation, is, in an obvious notation :

$$SV = \int_{-\infty}^E (E-r)^2 p(r) dr.$$

Given the striking similarity of this expression to the measure of risk derived from a first principles approach in Part 2, it is quite probable that Markowitz followed almost identical lines of reasoning to many of those set out earlier in this paper. The integral measure of risk defined by Axioms 2 and 7 can therefore be regarded as a more detailed statement of the intuitive concepts underlying the Markowitz formulation. Also the introduction of the concept of the admissibility frontier as described above is fully consistent with Markowitz' original formulation and corresponds to Axiom 4 of the general theory. Finally, the simple decision rule described in section 4.22 is equivalent to Axiom 5 of the general theory.

The general theory can therefore be regarded as a straightforward extension of the original Markowitz formulation.

4.26. The relationship between the general theory and portfolio theory is, however, quite different. It is clear that Markowitz always regarded semi-variance as the fundamental measure of risk; variance was then introduced as a proxy for semi-variance to simplify the computational aspects of the portfolio selection problem. It would have been both logical and highly desirable for later researchers to have examined in detail the nature of the approximations (if any) involved in using variance rather than semi-variance as the measure of risk. However, the reverse occurred; later researchers made little or no reference to semi-variance, and even in textbooks on portfolio theory which claim to follow a rigorous approach, the possibility that any significant error could arise from the use of variance (or its equivalent, the standard deviation σ) as the measure of risk is ignored. Sharpe^[10], for instance, comments as follows :

"The desirability of a portfolio is expressed by the values of E_p and σ_p . Two portfolios with quite different probability distributions might nonetheless have the same E_p and the same σ_p . The theory assumes that any investor would consider such portfolios equivalent - he would just as soon have one as the other. This may not be strictly true in every instance. As always, abstraction may lead to error. But the chance of error may be small; and the error, if made, may not be serious".

As shown in the example in Part 3, the use of the variance or semi-variance as the underlying measure of risk is equivalent to using a particular fixed value (namely 2) for the risk aversion parameter. This represents a very serious loss of generality, unless (as is

the case in the Markowitz approach) variance or **semi-variance** is being used only for the purpose of identifying the **minimum** risk investment for a particular value of expected return. The **distortions** which can arise from these two sources (i.e. different shape of distribution and differing degree of aversion to risk) are discussed in item (iii) of section 4.9 above.

4.27. The other heroic simplifying assumption of portfolio theory relates to the implementation of the third stage of the **Markowitz** approach, namely selecting from the efficient set the portfolio best suited to a particular investor's preferences. This assumption is that an investor will select the portfolio on the **efficient** frontier which maximises the utility function $U = \lambda E - V$, where λ is a constant which **characterises** this attitude towards risk. This is equivalent to stating that the investor's **preferences** can be represented by straight line indifference curves in the $E - V$ diagram as shown in Figure 24.

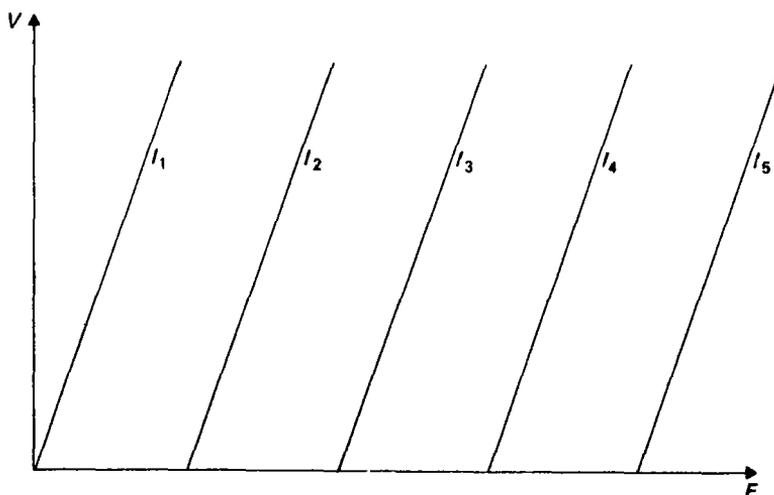


FIGURE 24

Thus points on l_2 are preferred to those on l_1 , and points on l_3 are preferred to points on l_2 , etc.

4.28. The indifference curves corresponding to Axioms 4 and 5 of the general theory are vertical straight lines below the risk threshold R_0 as shown in Figure 25.

The underlying axioms of investment behaviour are

- (i) an investor will not select an investment with a value of risk greater than his risk threshold R_0 , and
- (ii) for investments with a value of risk lower than his risk threshold the investor is **indifferent** to risk.

The first of these axioms needs no further justification. The second is consistent with human behaviour in areas other than investment. For example, if an individual is prepared to accept the risks involved in flying, he will normally base his decision on whether to travel by **aeroplane or train** for a particular business or holiday trip on the basis of other considerations (primarily convenience and possibly also cost) and will virtually ignore the higher risk associated with flying.

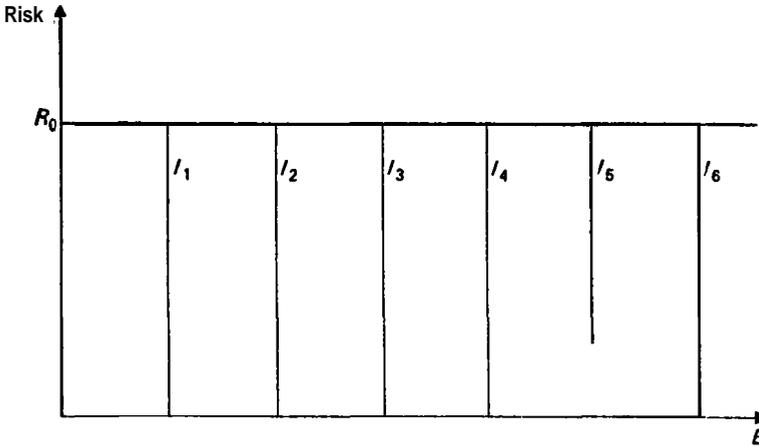


FIGURE 25

4.29. There is, however, an element of simplification in the second axiom. If one investment has a value of risk which is just below the threshold R_0 and there is another investment with very slightly lower expected value but minimal risk, it is possible that this latter investment will be preferred. A more accurate portrayal of the situation is given in Figure 26, where the indifference curves are asymptotic to vertical straight lines for small values of risk, decrease in gradient as risk increases, and then are asymptotic to the risk threshold R_0 .

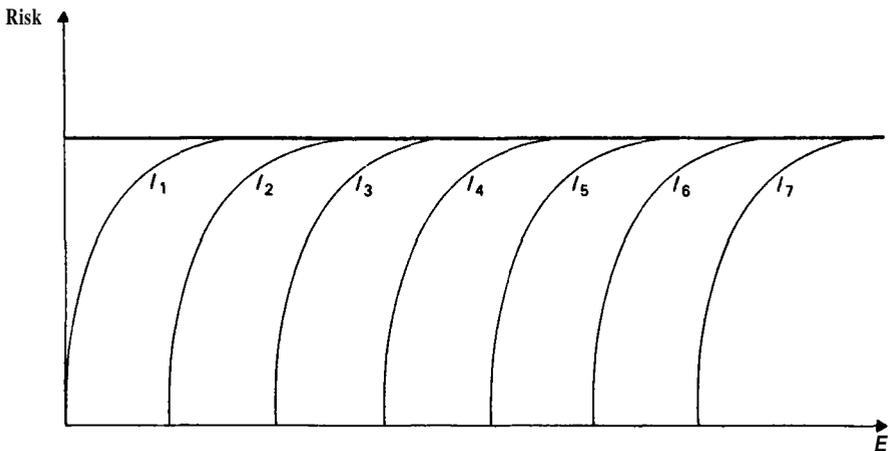


FIGURE 26

Given the estimation errors involved in quantifying investment returns and **investors' preferences**, the **approximations** involved in using the simplified **situation** as portrayed in Figure 25 are unlikely to **be** of any practical consequence. It is, however, important to **realise** that Axiom 5 is a convenient **practical** approximation to the **more** general axiom :

Axiom 5G An investor will select between investments which have a lower value of risk than his risk threshold R_0 on the basis of $E - R$ indifference curves which exhibit the following properties :

- (i) for small values of risk the curves are asymptotic to vertical straight lines,
- (ii) the curves decrease in gradient as risk increases, and
- (iii) the curves are asymptotic to the horizontal straight line which represents his risk threshold R_0 .

430. When the pattern of investor preferences portrayed in Figure 26 is translated into the $E - V$ diagram the situation is as shown in Figure 27, with the **indifference** curves being asymptotic to the admissibility frontier.

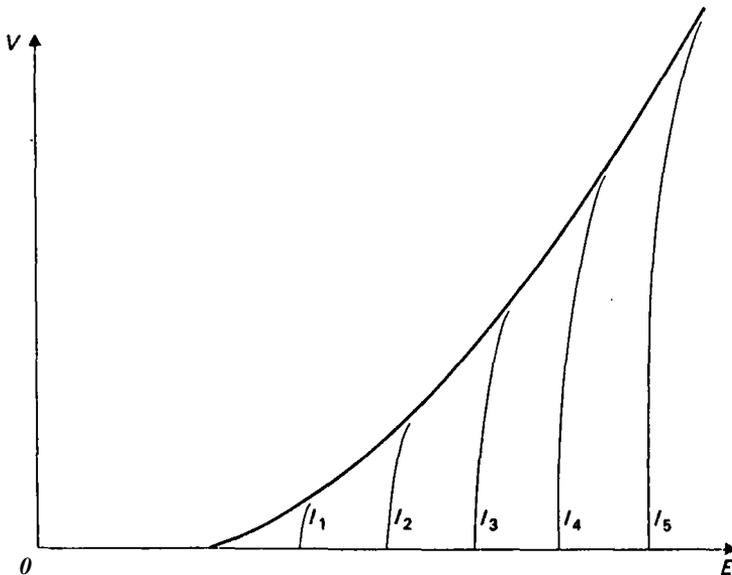


FIGURE 27

This pattern of investor preferences bears virtually no resemblance to the pattern portrayed in Figure 24. Accordingly, if it is accepted that the pattern of investor preferences portrayed in Figure 26 is consistent with investor **behaviour** in the real world, then the **portfolio** theory approach to the third stage of the **Markowitz** portfolio selection problem has no **theoretical** or practical validity.

431. In section 15 I suggested that there was a strong parallel between the more general theory of risk developed in this paper and my **gilt-edged** model which provided a more general theoretical and practical framework than previous models in that area. The two essential features of any satisfactory mathematical model are that it encapsulates the key elements of real life behaviour in a particular area and that it assists in decision-making within that area. In the case of the **gilt-edged** model, I began with the very general hypothesis that the underlying price structure was such that no "blatant anomalies" existed and that the prices of individual stocks oscillated about the "true" prices implied by that price structure, and I then derived the most general expression for the price structure that was consistent with this general hypothesis. The next (and most difficult) stage was to identify, for particular practical applications, a specific expression which was sufficiently flexible to reflect the salient features of the price structure but involved as few variable parameters as possible to simplify the computational aspects and ensure that the model was statistically stable. The final stage was to describe Mean Absolute Deviation (i.e. control theory) decision rules for the identification of cheap and dear stocks and to specify suitable parameters for the implementation of these decision rules. Care was taken to ensure that the specific expression for the price structure and the detailed decision rules were consistent with the intuitive concepts underlying the general hypothesis from which the model was derived. The various stages and interrelationships are summarised in Figure 28.

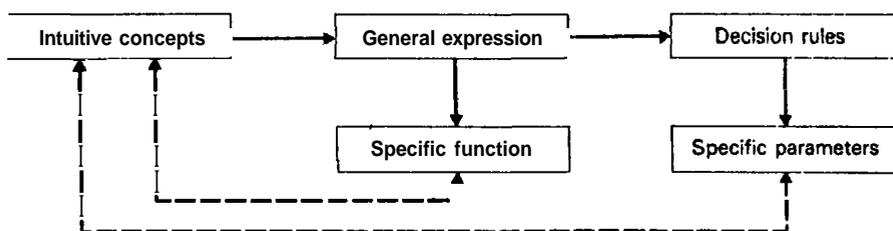


FIGURE 28

432. In the Markowitz model, the "general expression" of risk is the semi-variance of return for portfolios with the same value of expected return, and the "specific function" used to simplify the computational aspects is the variance of return. Also, the model contains two decision rules, the first of which is very specific and the second of which is stated only in general terms :

- (i) for portfolios with the same value of expected return, the portfolio with the minimum variance of return is preferred, and
- (ii) the investor will choose between portfolios on the efficient frontier in accordance with his own circumstances and preferences as regards risk and expected return

433. In the general theory, the "intuitive concepts" are set out explicitly in Axioms 1 and 6, the "general expression" is the much more detailed integral measure of risk defined in terms of Axiom 2, and the "specific function" is the integral measure of risk defined in

terms of Axiom 7. There are also three very specific decision rules, namely **Axioms** 3, 4 and 5, and **Axiom** 8 specifies how the parameter R_0 can be **quantified**. Finally, the "special **function**" is sufficiently general to be **consistent** with the underlying concepts, and the **method** of calculating R_0 relates directly to the general hypothesis set out in **Axiom** 1, namely that investment risk relates to the severity of the consequences of **certain** outcomes and also to the probabilities of these outcomes.

434. In the case of portfolio theory, the "specific **function**" is variance of **return**, and the decision rules **comprise** the key Markowitz axiom that **defines the efficient frontier** and also **the axiom** that each investor will select investments on the efficient frontier by maximising the utility function $U = \lambda E - V$. There is no explicit statement of the underlying concepts or of the "general **expression**".

435. The overall conclusion must be that the general theory developed in Part 2 **offers** a framework within which the full potential of the Markowitz approach can be realised, whereas portfolio theory is a somewhat narrow interpretation of the Markowitz approach and hence is of limited theoretical and practical validity.

5. FURTHER COMMENTS ON THE GENERAL THEORY

5.1. Part 4 essentially completes the objectives of this paper as set out in the introduction, namely to expand the concept of investment risk as described in Section 7 of Clarkson & Plymen into a general theory and to compare and contrast the resulting theoretical framework with the Markowitz approach. It may, however, be useful to refer very briefly to various other aspects **and** implications of the general theory.

5.2. The theoretical approach pursued in this paper supports the main practical conclusions set out in Clarkson & Plymen, namely that investment risk **as** measured by the variance of return is essentially irrelevant in the practical management of investment portfolios and that attention should **be** focused instead on using advanced analytic techniques to improve **the** expected **return**.

5.3. If it is accepted that investor behaviour in the real world is **inconsistent** with the decision rules of portfolio theory, then the general body of work known as Capital Market Theory or **Modern Portfolio** Theory has no theoretical validity. An alternative description of **the** price formation process in capital markets is developed in **Section** 5 of Clarkson & Plymen ; the key features are that prices **oscillate** about a "**correct**" price which is related to fundamental attributes and that the magnitude of the "**random noise**" represented by these oscillations is a reflection of the scale of the estimation errors involved in quantifying the **fundamental** attributes. **If** this alternative description is a more accurate representation of real life behaviour, then methods such as those developed by Weaver & Hall ^[11] (**relating** equity prices to fundamental **attributes**) and Plymen & Prevett ^[12] (**analysing** price oscillations using control theory techniques) would **appear** to offer **the** best prospect of achieving **consistent** superior performance. It is interesting to note that similar but much less rigorous work was carried out in **the** United States by Whitbeck & Kisor ^[13] (**analysing** equity prices to earnings per share and earnings **growth** rates) and Cootner ^[14] (**analysing** prices deviations **from their**

"current worth" value). However, the work of these researchers receives no mention **whatsoever** in later Capital Market Theory textbooks such as Sharpe [10].

5.4. Although the discussion in Part 2 relates to individual investors, the **concepts** are clearly also relevant in the life office or pension fund context. In particular, **considerations** of solvency in these cases lead directly to the existence of an upper limit on the risk value that is deemed to be acceptable, and the general theory (unlike the portfolio theory approach) allows such an upper limit to be used explicitly in **determining** optimum investment distributions.

5.5. When the paper on portfolio theory by Moore [9] was discussed at the Institute of Actuaries, **various** speakers were very critical of the general approach, particularly **as** regards **the** use of the variance of return as the measure of risk. Grimes suggested that a probability of ruin should be included explicitly, and he also gave an example of two investments (one with an expected return of 10 % and standard deviation of 1 % and the other with an expected return of 100 % and standard deviation of 10 %) where the latter was obviously the "less risky" investment regardless of its variance being **one** hundred times that of the other. Melnikoff suggested that the concept of variability was not necessarily **synonymous** with risk and that it would be most appropriate to **distinguish** between uncertainty, which was perhaps more directly related to volatility and variability, and risk, which might be better defined as the chance **of** missing a target and by how much. Also, in a written contribution Joseph suggested that **attention** should be focused instead on **maximising** the expected return.

The general theory of investment risk as developed in Part 2 is consistent with all these observations. It incorporates an explicit upper bound of risk analogous to a probability of ruin, Grimes' counterexample is very similar to that described in section 1.2 above, and Axioms 1 and 2 formalise the approach suggested by Melnikoff. Finally, as shown in the numerical example in the special case of **lognormal** distributions, **for** most "realistic" distributions risk decreases as variance and expected return increase, so that - as suggested by Joseph - the variance of return is irrelevant.

5.6. An important recent contribution to matching theory was the paper by Wise [15] in which he showed **how** a unique "positive match" is an excellent yardstick for actuarial valuation and for measuring the degree of risk inherent in any other portfolios. Wilkie [16] then showed how these results were in many ways an extension of conventional **portfolio theory**. He also pointed **out** that the "positive match" portfolio was **inefficient** in portfolio theory **terms**, and suggested **various** ways in which a **better** optimum portfolio could be selected. In particular, he suggested the concept of a " $k\sigma$ - solvency region" which involves precisely the type of upper bound of risk that is **defined** by my Axiom 4. Wise [17] subsequently generalised his approach to encompass many of Wilkie's observations, and in particular **introduced** a definition of the "degree of risk" (namely $v = dV/dP$) while still referring to the variance V of ultimate surplus as

the "risk" of the portfolio. This is equivalent to using the portfolio theory approach, which as explained in Part 4 can lead to conclusions that are **inconsistent** with real life behaviour.

5.7. **Certain** aspects of **Markowitz'** original **formulation** (e.g. the use of variance rather than **semi-variance**) were introduced to simplify the computational work in practical applications. Although for a particular family of density functions which vary in respect of both **mean** and variance the general theory developed in **Part 2** involves no fewer than five parameters (the **risk aversion factor** a , **the** scale factor k , the risk **threshold** L , the **expected** value E and the standard deviation σ), the results of **sections 2.27** and **2.28** show that, for a particular value of the risk aversion factor a , the risk function is in effect a function of only one variable. If we regard the risk aversion factor a as **corresponding** to the actuarial interest rate i , there is a remarkable similarity between this single variable risk **function** and actuarial commutation functions such as N_x . **Accordingly**, the **axioms** described in section 2.31 above define a mathematical system in which both theoretical development and practical problem-solving are greatly facilitated by these functional relationships. Furthermore, if **the** density function of return is known, **the** risk values can be calculated by approximate integration

5.8. I wish to thank Scott **Jamieson** for **writing** the approximate integration computer programmes used to calculate risk values. My main debt of gratitude is to Jack **Plymen**, without whose assistance and encouragement this paper would never have been written. In particular, most of the ideas developed in this paper, including the use of examples **from** other areas of **human behaviour** to **demonstrate** general principles, arose from our discussions during the preparation of our earlier joint paper.

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APPENDIX

1.1. If the expected returns are E_A and E_B , then :

$$\begin{aligned} E_A &= \int_0^{\infty} r p_A(r) dr \\ &= \int_0^{\infty} \frac{1}{2} r \cdot p_B\left(\frac{r}{2}\right) \cdot 2 \cdot d\left(\frac{r}{2}\right) \\ &= 2 \int_0^{\infty} s p_B(s) ds \quad \text{putting } s = \frac{r}{2} \\ &= 2 E_B. \end{aligned}$$

If the variances are V_A and V_B , then :

$$\begin{aligned} V_A &= \int_0^{\infty} (r - E_A)^2 p_A(r) dr \\ &= \int_0^{\infty} \left(\frac{2r}{2} - 2E_B\right)^2 \cdot \frac{1}{2} \cdot p_B\left(\frac{r}{2}\right) \cdot 2 \cdot d\left(\frac{r}{2}\right) \\ &= 4 \int_0^{\infty} \left(\frac{r}{2} - E_B\right)^2 p_B\left(\frac{r}{2}\right) d\left(\frac{r}{2}\right) \\ &= 4 \int_0^{\infty} (s - E_B)^2 p_B(s) ds \quad \text{putting } s = \frac{r}{2} \\ &= 4 V_B. \end{aligned}$$

1.2. $\int_0^x p_A(r) dr < \int_0^x p_B(r) dr + \int_x^{2x} p_B(r) dr$ assuming $p_A(r)$ is continuous and $p_A(r) > 0$ for some $r \in [x, 2x]$

$$\begin{aligned} &= \int_0^{2x} p_A(r) dr \\ &= \int_0^{2x} \frac{1}{2} \cdot p_B\left(\frac{r}{2}\right) \cdot 2 \cdot d\left(\frac{r}{2}\right) \\ &= \int_0^x p_B(r) dr. \end{aligned}$$

2.1. Let the values of risk on the original and revised distributions be R_A and R_B respectively.

$$\begin{aligned}
 \text{Then } R_B &= \int_{-\infty}^L W(L-r)p(r)dr \\
 &= \int_{-\infty}^L W(L-X-r+X)p_X(r+X)dr \\
 &= \int_{-\infty}^{L-X} W(L-X-s)p_X(s)ds \quad \text{putting } s = r + X \\
 &< \int_{-\infty}^{L-X} W(L-s)p_X(s)ds \quad \text{assuming } p_X(s) \text{ is continuous and} \\
 &\quad p_X(s) > 0 \text{ for some } s < L-X. \\
 &< \int_{-\infty}^{L-X} W(L-s)p_X(s)ds + \int_{L-X}^L W(L-s)p_X(s)ds \quad \text{assuming } p_X(s) \\
 &\quad \text{is continuous and} \\
 &\quad p_X(s) > 0 \\
 &\quad \text{for some} \\
 &\quad s \in [L-X, L]. \\
 &= \int_{-a}^L W(L-s)p_X(s)ds \\
 &= R_A
 \end{aligned}$$

2.3. Let the values of risk be R_A and R_B and let the density functions intersect at r_1 .

$$\begin{aligned}
 \text{Then } R_A &= \int_{-\infty}^{r_1} W(L-r)p_A(r)dr + \int_{r_1}^L W(L-r)p_A(r)dr, \\
 R_B &= \int_{-\infty}^{r_1} W(L-r)p_B(r)dr + \int_{r_1}^L W(L-r)p_B(r)dr, \\
 \text{Area 1} &= \int_{r_1}^L (p_A(r) - p_B(r))dr, \\
 \text{Area 2} &= \int_{-\infty}^{r_1} (p_B(r) - p_A(r))dr,
 \end{aligned}$$

$$\begin{aligned}
 \text{Hence } R_B - R, &= \int_{-\infty}^{r_1} W(L-r)(p_A(r) - p_B(r))dr \\
 &\quad - \int_{r_1}^L W(L-r)(p_A(r) - p_B(r))dr \\
 &= W(L-r_2) \int_{-\infty}^{r_1} (p_B(r) - p_A(r))dr \\
 &\quad - W(L-r_3) \int_{r_1}^L (p_A(r) - p_B(r))dr
 \end{aligned}$$

where $r_2 < r_1$ and $r_1 < r_3 < L$

$$\begin{aligned}
 &= W(L-r_2) \times \text{Area 2} - W(L-r_3) \times \text{Area 1} \\
 &\geq \text{Area 1} \times [W(L-r_2) - W(L-r_3)] \\
 &> 0 \quad \text{since} \quad W(L-r_2) > W(L-r_3),
 \end{aligned}$$

i.e. $R_1 < R_2$.

23. The increase in risk is AR where

$$\begin{aligned}
 AR &= \Delta r \Delta p W(L-(r-r_1)) - 2\Delta r \Delta p W(L-r) + \Delta r \Delta p W(L-(r+r_1)) \\
 &= \Delta r \Delta p [W((L-r) + r_1) - 2W(L-r) + W((L-r) - r_1)] \\
 &= \Delta r \Delta p [W(L-r) + r_1 W^i(L-r) + \frac{r_1^2}{2} W^{ii}(L-r) + \frac{r_1^3}{6} W^{iii}(L-r) \dots \\
 &\quad - 2W(L-r) \\
 &\quad + W(L-r) - r_1 W^i(L-r) + \frac{r_1^2}{2} W^{ii}(L-r) - \frac{r_1^3}{6} W^{iii}(L-r) \dots] \\
 &= \Delta r \Delta p r_1^2 [W^{ii}(L-r) + \frac{r_1^2}{12} W^{iv}(L-r) \dots] \\
 &> 0
 \end{aligned}$$

Letting $r_1 \rightarrow 0$ gives $W^{ii}(L-r) > 0$ and hence $\frac{d^2W}{ds^2} > 0$.

2.4.

$$\begin{aligned}
 R_L^a(\mu, \sigma) &= \int_{-\infty}^L \left(\frac{L-r}{k}\right)^a p(r, \mu, \sigma) dr \\
 &= \int_{-\infty}^L \left(\frac{L-r}{k}\right)^a \cdot \frac{1}{\sigma} \cdot p\left(\frac{r-\mu}{\sigma}\right) dr \\
 &= \int_{-\infty}^{(L-\mu)/\sigma} \left(\frac{L-\mu-s\sigma}{k}\right)^a p(s) ds \quad \text{putting } s = \frac{r-\mu}{\sigma} \\
 &= \sigma^a \int_{-\infty}^{(L-\mu)/\sigma} \left(\frac{\frac{L-\mu}{\sigma}-s}{k}\right)^a p(s) ds \\
 &= \sigma^a R_{(L-\mu)/\sigma}^a(0, 1).
 \end{aligned}$$

2.5.

$$\begin{aligned}
 R_L^a(\lambda) &= \int_{-\infty}^L \left(\frac{L-r}{k}\right)^a \cdot \frac{1}{\lambda} \cdot p\left(\frac{r}{\lambda}\right) dr \\
 &= \int_{-\infty}^{L/\lambda} \left(\frac{L-\lambda s}{k}\right)^a p(s) ds \quad \text{putting } s = \frac{r}{\lambda} \\
 &= \lambda^a \int_{-\infty}^{L/\lambda} \left(\frac{\frac{L}{\lambda}-s}{k}\right)^a p(s) ds \\
 &= \lambda^a R_{L/\lambda}^a(1).
 \end{aligned}$$

3.1. Risk Values

Log normal distribution : $\mu = 1, \sigma = 0.5$ (Based on $y = 2.1169 \log_c x$)

a =	1.5	2	2.5	3
L				
0.0	0.00000	0.00000	0.00000	0.00000
0.1	0.00000	0.00000	0.00000	0.00000
0.2	0.00000	0.00000	0.00000	0.00000
0.3	0.00011	0.00003	0.00001	0.00000
0.4	0.00091	0.00031	0.00011	0.00004
0.5	0.00383	0.00153	0.00064	0.00028
0.6	0.01091	0.00497	0.00237	0.00118
0.7	0.02431	0.01241	0.00662	0.00366
0.8	0.04591	0.02589	0.01521	0.00922
0.9	0.07711	0.04750	0.03038	0.02001
1.0	0.11880	0.07919	0.05464	0.03873
1.1	0.17145	0.12273	0.09069	0.06871
1.2	0.23517	0.17964	0.14129	0.11371
1.3	0.30984	0.25118	0.20919	0.17795
1.4	0.39520	0.33843	0.29710	0.26599
1.5	0.40987	0.44223	0.40765	0.38267
1.6	0.59643	0.56329	0.54336	0.53306
1.7	0.71146	0.70215	0.70665	0.72242
1.8	0.83551	0.85927	0.89984	0.95617
1.9	0.96819	1.03499	1.12513	1.23984
2.0	1.10908	1.22960	1.38462	1.57905

Normal distribution : $\mu = 0, \sigma = 1$

a :	1.5	2	2.5	3
L				
-1.0	0.07567	0.07534	0.08056	0.09129
-0.9	0.09270	0.09367	0.10155	0.11656
-0.8	0.11279	0.11569	0.12717	0.14786
-0.7	0.13631	0.14195	0.15823	0.18639
-0.6	0.16365	0.17305	0.19564	0.23352
-0.5	0.19520	0.20964	0.24040	0.29077
-0.4	0.23138	0.25240	0.29362	0.35992
-0.3	0.27257	0.30206	0.35651	0.44290
-0.2	0.31916	0.35936	0.43036	0.54192
-0.1	0.37153	0.42508	0.51657	0.65936
0.0	0.43002	0.50000	0.61663	0.79788
0.1	0.49496	0.58492	0.73212	0.96036
0.2	0.56664	0.68064	0.86467	1.14992
0.3	0.64531	0.78794	1.01602	1.36990
0.4	0.73119	0.90760	1.18793	1.62392
0.5	0.82445	1.04036	1.38223	1.91577
0.6	0.92522	1.18695	1.60078	2.24952
0.7	1.03360	1.34805	1.84547	2.62939
0.8	1.14962	1.52431	2.11822	3.05986
0.9	1.27330	1.71633	2.42092	3.53556
1.0	1.40460	1.92466	2.75550	4.09129