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PAR / BY

Wolfgang BUHLER

RFA / Germany

EVALUATION DE BONS DE SOUSCRIPTION D'OBLIGATIONS

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RESUME

Cet article présente un modèle temporel continu d'évaluation des bons de souscription d'obligations, dans lequel le cours des bons est une variable exogène et le taux d'interêt à court terme est parfaitement lit5 à ce cours. La dynamique du cours des bons de souscription est définie en sorte que ce cours soit borné par ces conditions, que sa volatilité décroisse dans le temps et que sa valeur à maturité soit égale à la valeur de remboursement avec une probabilité Cgale à un. Des données chronologiques et transversales du march6 obligataire allemand sont utilisées pour estimer les paramètres dévolution du cours. Les valeurs limites des options d'achat et de vente européennes et américaines sont dérivées de conditions sans arbitrage, sans distribution. Enfin, des résultats comparatifs statiques sont obtenus par une résolution numérique du modèle d'évaluation.

VALUATION OF BOND WARRANTS

WOLFGANG BÜHLER*

*University of Dortmund
Lehrstuhl Investition und Finanzierung
Postbox 50 05 00
4600 Dortmund 50 West Germany

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ABSTRACT

In this paper a time continuous model to value bond warrants is developed. The model is based on the bond price as an exogeneous variable and a short-term interest rate which is linked perfectly to the bond price. The price dynamics of the bond are defined such that the bond price is bounded from above, its volatility decreases with,time, and at maturity its value is equal to the redemption value with probability one. Time-series and cross-sectional data from the German bond market are used to estimate the parameters of the price process. The boundary values for European and American puts and calls are &rived from distribution-free no arbitrage conditions. Finally comparative static results are obtained by solving the valuation model numerically.

The primary markets for bonds have seen a variety of innovative structures during the last years. Of great importance, and difficult to value, are options and bond warrants. In principle, options can be attached to all bond features. The most important are coupon, life, redemption and issue price. Bond warrants offer a particularly large variety of features. In the Euromarket call and put warrants of the European and American type are issued Moreover, the warrants can entitle the holder to the purchase of the host band, of another outstanding bond, or to the purchase of a not yet issued bond. There are redeemable and interest bearing warrants as well as warrants which enable the holder to choose between bonds and stocks of the issuing corporation[1].

For a number of reasons the well-known valuation models for stock options are not immediately applicable to the valuation of bond options:

- Excluding perpetual bonds and credit risk, the bond price must equal the redemption value with probability one at maturity.

- Bond prices are bounded from above. If negative interest rates are not feasible, the maximum bond price is the sum of the redemption value and the future interest accruing during the residual lifetime.

- The probability distribution of bond price returns and with it the volatility of bond prices are not stationary.

- The bond price and the price of a zero bond, due at the expiration date of the option, must be compatible with one another so that no negative forward rate for the period from the expiration date until the maturity of the bond is possible.

- At the purchase and the sale of bonds interest accrues, depending on the time of transaction, the coupon dates and the coupon.

These five differences can lead to substantial deviations between the values of bond and stock options. A European bond option, for example, cannot increase monotonously with the time to expiration. This follows from the observation that if the time to expiration equals the maturity of the bond, and the exercise price is equal to the redemption value, then this option must be worthless.

Two approaches have been suggested to value bond options. The first, as discussed by Courtadon (5), Brennan and Schwartz (3), Dietrich-Campbell and Schwartz (6), Cox, Ross and Ingersoll (4), and Heath, Jarrow and Morton (9) rely upon one or two interest rates as exogeneous variables. The stochastic behavior of these variables determine in equilibrium the current term structure of interest rates, the current bond price and option value. Two problems are associated with this approach. Firstly, the current interest rates must be adapted in such a way that the equilibrium bond price equals the quoted bond price. Secondly, with the exception of Heath, Jarrow and Morton, the option value depends on the risk preferences of the investors. The second line of reasoning, as presented in the papers by Ball and Torous (1), Schöbel (14), Ho and Lee (9), and Schaefer and Schwartz (13) use bond prices as exogeneous variables. As in the Black-Scholes-Merton model (2, 12) the resulting valuation equation does not depend on the risk preference of the agents. However, the bond price dynamics result in negative yields.

1 See Mason (10) for a detailed description of different bond options and bond warrants.

The model presented in this paper follows the second approach. The closest affinity exists with the model by Schaefer and Schwartz. It differs from this model in that the short-term interest rate is linked to the bond price and that bond prices are bounded from above. It can be used and is applied to value calls and puts of the European and American type with fixed or time-dependent exercise prices.

The paper is organized as follows. First distribution-free bounds for European and American options are derived. In Section II the dynamics of the bond price are specified and the valuation model is developed. Empirical results from estimating the parameters of the stochastic bond prices are presented in Section III. In Section IV the findings of comparative statics are discussed.

I. DISTRIBUTION-FREE BOUNDS

In deriving bounds for bond options the following simple framework is presumed :[2]

Table I

Investment Opportunities ($t < t_1 < T$)

time	curent date t	coupon date t_1	expiration date T
Bond	$B(t) + c_t$	c	$B(t) + c_T$
Zeron Bond I	D_1	1	
Zero Bond II	D_T		1

Besides the options considered, and the possibility to keep funds in cash, three securities exist : The coupon bond, on which calls and puts are written, and two zero bonds maturing at the first coupon date t_1 and at the expiration date T of the option, respectively. The current prices of these bonds are denoted by B (t), D_1 and D_T. At t_1 the first coupon c after the current date t falls due. Accrued interest at t and T amounts to $c_t = c \cdot (t-t_0)$ and $c_T = c \cdot (T-t_1)$, where t_0 denotes the last coupon date before t. Normally the bond price $\tilde{B}$ (T) at the expiration date of the option is uncertain

When exercising an option, the exercise price plus accrued interest must be paid in exchange for the bond whose value depends on its price and accrued interest Hence, for the intrinsic value of a bond option the amount of accrued interest is of no importance.

The following additional assumptions are made :

(i) Transaction costs and taxes are zero.
(ii) Short selling is possible without penalties.
(iii) The options and bonds are infinitely divisible.
(iv) Interest rates may vary between zero and infinity.
(v) The exercise price E is smaller than the maximum bond price B_{max} (T) on the expiration date T of the option.

2 Without incrasing the complexity of the problem, more than one coupon date between t ant T could be considered. The generalisation would, however, not permit new insights, but rather increase the notational requirements.

A violation of the last assumption would mean that a European call is worthless and the value of a European put has a certain component which amounts to $(E-B_{max}(T))D_T$.

A. European Call Options

To derive upper and lower bounds for calls three portfolios are considered

Portfolio I : Purchase of the bond.

Portfolio II : Purchase of one European call, c units of the zero bond with maturity date t_1 and $E+c_T$ units of the second zero bond.

Portfolio III : Purchase of $B_{max}(T)/(B_{max}(T)-E)$ units of the call, c units of the zero bond with maturity date t_1 and c_T units of the second zero bond[3]

The cash flows of these portfolios are summarized in Table II

Table II

Cash flows of portfolios I, II and III

Portfolio	t	t_1	T: $\tilde{B}(T)<E$	T: $\tilde{B}(T)\geq E$
I	$-B(t)-c_t$	c	$\tilde{B}(T)+c_T$	$\tilde{B}(T)+c_T$
II	$-C_e-(E+c_T)D_T-cD_1$	c	$E+c_T$	$\tilde{B}(T)+c_T$
III	$-\frac{B_{max}(T)}{B_{max}(T)-E}\cdot C_e-c_TD_T-cD_1$	c	c_T	$c_T+B_{max}(T)\frac{\tilde{B}(T)-E}{B_{max}(T)-E}$

At the expiration date T portfolio II dominates portfolio I which for its part dominates portfolio III. This implies by the usual argumenation[4] :

$$\frac{B_{max}(T)}{B_{max}(T)-E}\cdot C_e + c_TD_T + cD_1 \leq B(t)+c_t \leq C_e + (E+c_T)D_T + cD_1. \qquad (1)$$

(1) together with the inequality[5]

$$B(t) + c_t - cD_1 - c_TD_T \leq B_{max}(T)\cdot D_T \qquad (2)$$

3 This portfolio was suggested by J. Aase Nielson, University of Aarhus.
4 See for example Smith (15) pp. 9
5 In words : The current value of the bond cannot be larger than the discounted maximum value of the bond at T plus the discounted coupon payment at point of time t_1.

implies that the European call is bounded from above by

$$C_e \le (B_{max}(T)-E)\cdot\frac{B(t)+c_t-c_T D_T-cD_1}{B_{max}(T)} \le (B_{max}(T)-E)\cdot D_T, \tag{3}$$

i.e. the value of a European call is not greater than the discounted maximum exercise value.

Inequality (1) and the limited liability of an option implies the lower bound

$$C_e \ge \max\{0;B(t)+c_t-(E+c_T)D_T-cD_1\}, \tag{4}$$

i.e. the value of a European call is non-negative and not smaller than the current bond value minus the discounted payments when the call is exercised and minus the discounted coupon which the option holder does not receive.

These bounds can now be used to derive the call values at the boundaries B_{max} (t) and B_{min} (t) of the feasible bond prices. The maximum bond price corresponds with minimum interest rates which are assumed to be zero. Zero interest rates result in the maximum discount structure $D_{\tau-t}=1$ $(\tau>t)$ and in the maximum bond price

$$B_{max}(t) = c\cdot(T_B-t) + RV, \tag{5}$$

where T_B denotes the maturity date of the bond and RV its redemption value.

The minimum bond price corresponds with infinitely high interest rates which lead to the discount structure $D_{\tau-t}=0$ $(\tau>t)$ and the minimum bond value[6]

$$B_{min}(t) + c_t = 0. \tag{6}$$

If B (t) equals B_{max} (t), then (3), and (4) together with $D_1 = D_T = 1$ imply

$$C_e(B_{max}(t),t) = B_{max}(T)-E, \tag{7}$$

i.e. the value of a European call option for the current maximum bond price is equal to the maximum bond price at the expiration date minus the exercise price.

If several exercise dates T_1 ...., T_n with corresponding exercise prices E_1 ....., E_n exist, as is typical for a redemption right, then (7) must be replaced by

$$C_e(B_{max}(t),t) = \max\{B_{max}(T_i) - E_i \mid i=1,\ldots,n\}. \tag{8}$$

For the smallest possible bond price B_{min} from (3) and (4) follows

$$C_e(B_{min}(t),t) = 0. \tag{9}$$

6 It should be noted that the institutionally practised calculation of accrued interest, in the case of infinitely high interest rates, results in a negative current bond price B_{min} (t) = $-c_t$. Theoretically every higher pice offers arbitrage opportunities.

B. American Call Options

An American call with a time-independent exercise price and the possibility of being exercised immediately has for the maximum bond price the value

$$C_a(B_{max}(t),t) = B_{max}(t) - E, \tag{10}$$

i.e. it is optimal to exercise the call. A larger value of the call would result in a riskless profit opportunity for the option writer for the following reason : If the call is never exercised, the profit is equal to C_a ; if it is exercised at time τ, $t<\tau\leq T$, the writer realizes at least the profit

$$C_a - (B_{max}(\tau)-E) > B_{max}(t) - B_{max}(\tau) = c\cdot(\tau-t) > 0.$$

Since B_{max} (t) decreases monotonously in t, it follows from (10) that for sufficiently high bond prices it is always optimal to exercise an American call on a bond. This is also true for zero bonds. Merton's well-known result that a premature exercise of American calls on non-dividend paying stocks is not optimal, does not apply to non-coupon paying bonds? The reason for this difference is the mutual dependence of B (t), D_1 and D_T on one another.

If the American call can be exercised for the first time at time $\bar{t}>t$, than the right hand side of (10) must be replaced by $B_{max}(\bar{t})$-E. If the exercise price is timedependent such that in the time intervals $[T_1, T_2)$, $[T_2, T_3)$, ..., $[T_n, T_{n+1}]$ the exercise prim E_1, E2, ..., E_n hold, then

$$C_a(B_{max}(t),t) = \max\{B_{max}(T_i)-E_i \mid i=1,\ldots,n;\ T_1 \geq t\} \tag{11}$$

holds true. Obviously for time-dependent exercise price. an immediate exercise of an American call for B (t) = B_{max} (t) is not necessarily optimal.

As in the European case the American call is worthless

$$C_a(B_{min}(t),t) = 0 \tag{12}$$

for the smallest possible bond price.

C. European Put Options

Again three portfolios are considered in order to derive upper and lower bounds for the put value. The corresponding cash flows are shown in Table III.

7 Note that in the zero bond case it is not claimed that - if (10) holds - t is the only point of time in which it makes sense to exercise the American call.

Table III

Cash flows of three portfolios in order to derive bounds for put values

Portfolio	t	t_1	T	
			$\tilde{B}(T)<E$	$\tilde{B}(T)\geq E$
I	$-(E+c_T)D_T-cD_1$	c	$E+c_T$	$E+c_T$
II	$-P_e-B(t)-c_t$	c	$E+c_T$	$\tilde{B}(T)+c_T$
III	$-\frac{B_{max}(T)-E}{B_{max}(T)}[B(t)+c_t-c_TD_T-cD_1]-(E+c_T)D_T-cD_1$	c	$\frac{B_{max}(T)-E}{B_{max}(T)}\tilde{B}(T)+E+c_T$	$\frac{B_{max}(T)-E}{B_{max}(T)}\tilde{B}(T)+E+c_T$

Portfolio I is composed of two zero bonds due at t_1 and T, portfolio II is made up of the put and the bond, whereas portfolio III consists of the bond and the two zero bonds of portfolio I.[8] At the expiration date portfolio III dominates portfolio II which for its part dominates portfolio I. Therefore, at the current date the put value is bounded from above by

$$P_e \leq E\cdot D_T - \frac{E}{B_{max}(T)}[B(t)+c_t - cD_1 - c_TD_T] \leq E\cdot D_T \qquad (13)$$

and from below by

$$P_e \geq \max\{0;\ (E+c_T)D_T + cD_1 - B(t) - c_t\} \qquad (14)$$

Again, these bounds can be used to derive the put values for the extreme bond prices $B_{max}(t)$ and $B_{min}(t)$:[9]

$$P_e(B_{max}(t),t) = 0 \qquad (15)$$

$$P_e(B_{min}(t),t) = 0 \qquad (16)$$

(15) in conjunction with (16) leads to the - at the first glance - surprising observation that the value of a European put cannot be a monotonic decreasing function of the bond price. Again, the reason for this result is the mutual dependence of the bond price and the zero bond prices D_1 and D_T.

8 The third portfolio has been suggested by J. Aase Nielsen, University of Aarhus

9 Note that $B(t) = B_{max}(t)$ implies $D_1 = D_T = 1$ and $B(t) = B_{min}(t)$ implies $D_1 = D_T = 0$

The equations (15) and (16) are independent of the exercise price and the expiration date. As long as the current date does not coincide with an exercise date, they remain valid if the option can be exercised at several dates $T_1, ..., T_n$ at exercise prices $E_1, ..., E_n$.

D. American Put Options

If an American put can be exercised at time t at exercise price E its value for the minimum bond price is equal to the exercise price

$$P_a(B_{min}(t),t) = E; \tag{17}$$

if it can be first exercised at a later date $\bar{t}$ > t, the put is worthless. Both results are hue, regardless of whether the exercise price is fixed or varies with time.

In order to derive the value of an American put at B (t) in the Appendix will be shown that the upper bound

$$P_a(B(t),t) \leq B_{max}(t) - B(t) \tag{18}$$

holds. From this inequality follows immediately

$$P_a(B_{max}(t),t) = 0. \tag{19}$$

II. A VALUATION MODEL FOR BOND WARRANTS

In addition to the assumptions (i) to (v) formulated at the beginning of Section I, the following five conditions are stipulated

(vi) There exists an instantaneously risk-free investment opportunity, perfectly linked with the bond price.[10]

(vii) The markets operate continuously.

(viii) The minimum bond price is assumed to be zero.[11]

(ix) The bond price follows a diffusion process in the interval (0, T).

(x) The c o u p is positive.[12]

A. Price Dynamics

The objective of the subsequent considerations is to construct a diffusion process $\tilde{B}(t)$ of bond prices which behaves in the following way

- B (t) and B (t) = 0 ($t<T_B$) are natural boundaries.
- At the maturity date T_B of the bond, $\tilde{B}(T_B)$ = RV with probability one.
- The instantaneous volatility of the bond return c.p. decreases monotonously with time.

10 This assumption is made for computational convenience. It results in a model with one state variable only.
11 The assumption B_{min} (t) = 0 avoids a number of numerical problems with the sawtooth pattern of the accrued interest function.
12 Sane of the following arguments do not hold for zero bonds. To avoid the lengthy discussion of different cases, zero bonds are excluded.

Based on assumption (ix) the local behavior of the bond **price** $\tilde{B}$ (t) can be described by an **Itô** stochastic differential equation

$$d\tilde{B}(t) = \mu dt - \sigma d\tilde{W}(t) \qquad (0 \leq t < T_B) \tag{20}$$

with respect to the standard Wiener process $\tilde{W}$ (t). The instantaneous volatility of **the** bond's **price** change will be defined by

$$\sigma = k \cdot B(t) \cdot \frac{B_{max}(t) - B(t)}{B_{max}(t) - RV} \text{ Duration } (B(t), RV, c, T_B). \qquad (0 \leq t \leq T_B). \tag{21}$$

In (21) k denotes a constant which is independent of the terms of the bond. The duration of the bond is defined by means of the bond's yield to maturity under continuous **compounding.**[13] B_{max} (t) - B (t) is divided by B_{max} (t) - RV, since without this standardization the **factor** k would **strongly** depend on the coupon.

o has the following **desirable** propertie

• σ is equal to zero at B_{max} (t) and B (t) = 0, **i.e.** o satisfies the necessary condition far B_{max} **(t)** and B **(t)** = 0 **being** natural boundaries.

• The instantaneous volatility σ I B (t) of the bond's return $d\tilde{B}$ (t) I B (t) is a monotonously decreasing function of t.

The drift is not defied explicitely, since its explicit form does not affect the option values. In the Appendix an intuitive justification will be given that there exist a **drift**[14] which together with definition (21) result in the desired behavior of the bond price.

The instantaneous risk-free interest rate **r** is linked to the yield to maturity of the bond and, therefore, to the bond price by the relation

$$r(\tilde{B}(t)) = s(t) * [\text{yield to maturity } (\tilde{B}(t))] \tag{22}$$

The factor **s** (t) characterizes the yield spread between a short-term investment of funds and the purchase of the bond at time t. To avoid **arbitrage** possibilities, s (t) must converge to one as the time to maturity approaches zero. In the Appendix it will be shown that the behavior of the bond price and of the short-term interest rate is internally consistent in the sense that for every bond price B **(t)** and every short-term interest rate **r** (B (t)) there exists a monotonously decreasing discount-structure $D_{\tau-t}$ ($\tau \geq t$) such that

$$B(t) + c_t = \sum_{j=1}^{n} c D_{\tau_j - t} + FV \cdot D_{\tau_n - t} \tag{23}$$

13 The yield to maturity is used, as in the framework of the model no term-structures of interest rates are available. The differences between the durations calculated by means of the current term-structure of interest rates and those on the basis of the yield to maturity are negligible. A study of the German bond market f a the period between 1970 and 1985 shows a maximum difference of 1.3%. A similar result for the U.S. bond market has been reported by Ingersoll (10). pp. 166.

14 This drift does not satisfy the growth condition of the existence and uniqueness theorem of stochastic differential equations. This may result in problems if the expiration date of the option and the maturity date of h e bond coincide.

$$\text{and} \lim_{\tau \to t+} [- \ln D_{\tau-t}/(\tau-t)] = r(B(t)) \tag{24}$$

hold. In (24) τ_j denote the future coupon dates of the bond ($\tau_n = T_B$).

B. Valuation Equation and Boundary Conditions

Applying the now "classical" procedure to duplicate the cash flows of an option by a continuously rebalanced portfolio composed of the bond and the instantaneously risk-free investment, the following valuation equation for the unknown value U of an option is obtained

$$1/2\sigma^2 \frac{\partial^2 U}{\partial B^2} + [r(B)(B+c\cdot(t-t_0(t)) - c] \frac{\partial U}{\partial B} + \frac{\partial U}{\partial t} - r(B)U = 0 \tag{25}$$

$$0 < B < B_{max}(t); \quad 0 \leq t < T < T_B.$$

(25) defines a linear parabolic differential equation of the second order with state variable B and time variable t. U stands for the unknown value C_e, C_a, P_e or P_a of a call or put of the European or American type. Compared with the well-known valuation equations for stock options, noteworthy is only that the coefficient of the derivative a $\frac{\partial U}{\partial B}$ is not continuous. The accrued interest function $c \cdot (t-t_0(t))$, where $t_0(t)$ is the last coupon date before t, exhibits a sawtooth pattern with jumps at the coupon dares. This discontinuous term enters into the differential equation as the bond's instantaneous return at time t

$$[d\tilde{B}(t) + c\cdot dt]/[B(t) + c\cdot(t-t_0(t))]$$

depends on accrued interest.

The terminal and boundary conditions can be taken from Section I. They are summarized in Table IV for the special case that the exercise price is fixed and an American option can be exercised immediately.

Table IV

Terminal and Boundary Conditions

	European Call	European Put
$B(t)=B_{max}(t)$ } $t<T$	$B_{max}(T)-E$	0
$B(t)=0$ } $t<T$	0	0
$t=T$	$\max\{0;B(T)-E\}$	$\max\{0;E-B(T)\}$
	American Call	**American Put**
$B(t)=B_{max}(t)$ } $t<T$	$B_{max}(T)-E$	0
$B(t)=0$ } $t<T$	0	E
$t=T$	$\max\{0;B(T)-E\}$	$\max\{0;E-B(T)\}$
$0<B(t)<B_{max}(t)$	$C_a(B(t),t)\geq\max\{0;B(t)-E\}$	$P_a(B(t),t)\geq\max\{0,E-B(t)\}$

The boundary conditions for timedependent exercise prices or delayed exercise periods are given in Section I. They are not to be repeated here.

Problem (25) together with the terminal and boundary conditions is well defined if the expiration date T of the option is smaller than the bond's maturity T_B. If T and T_B coincide, the value U (T_B) is defined as the limit of U (T) as T converges to T_B.

C. Numerical Considerations

The solution of the partial differential equation (26) in conjunction with the terminal and boundary conditions has to be done numerically. For reasons of stability the implicit Crank-Nicholson method was implemented. For this method as with all numerical methods, a grid size has to be chosen which takes into account conflicting goals like stability, accumulation of rounding errors, and computer time.

Based on a number of test runs, an equidistant step size of $\Delta B = B(0)/100$ was chosen When fixing the time step Δt it must be noted that for options with timedependent exercise price, the option should be numerically evaluated at these specific dates in order to increase accuracy. Therefore, equidistant time steps cannot always be observed. For typical options and warrants a subdivision of the interval (0, T) into 20 subintervals is sufficient.

Apart from the grid size the computer time also strongly depends on the maturity of the bond, as in every grid point the yield to maturity and the duration of the bond must be calculated. The computer time for an option on a 10-year bond with a 10% coupon using 20 time steps is about one minute on an IBM-AT 03 with mathematical coprocessor.

III. PARAMETER ESTIMATION

The bond price dynamics (20) depend on the factor k. To estimate k statistically (20) is approximated by the following time-discrete model with heterosketastic error term

$$\tilde{Y} = -kB(t)\cdot\frac{B_{max}(t)-B(t)}{B_{max}(t)-FV}\ \text{Duration}\cdot\tilde{\varepsilon}(t), \tag{26}$$

where $\tilde{Y}(t)$ is defined as

$$\tilde{Y}(t) = \tilde{B}(t+\Delta t) - B(t) - \mu\cdot\Delta t.$$

With respect to the error variable it is assumed, in accordance with equation (20), that $\tilde{\varepsilon}$ is normally distributed with mean 0 and standard deviation $\sqrt{\Delta t}$. Furthermore, it is assumed that $\tilde{\varepsilon}(t)$ and $\tilde{\varepsilon}(\tau)$ for $t \neq \tau$ are uncorrelated.

A. Longitudional Analysis

First k is estimated by means of the maximum likelihood principle.[15] Accordingly time-series of bond prices were collected from the Frankfurt security exchange. The bonds are non-callable straight bonds, issued by the German Government, the German National Railway and the German National Post Office.

The first sample consists of 105 Friday prices per bond collected from January 1, 1981 to December, 31, 1982. This period is characterized by high interest rates and partly inverse term-structures of interest rates. The second sample is composed of 104 Friday prices per bond for the period from January 1, 1984 to December, 31, 1985. In this second period interest rates generally fell and the term-structures of interest rates behaved "normally".

Both samples have been compiled in such a manner that the time to maturity of the bonds fall into three well seperated groups. This design has been chosen in order to analyse the stability of the estimated values $\hat{k}_i$ of k with respect to the bond's time to maturity. The details of the sample are given in Table V below

15 Support for the statistical work by J. Käsler is greatfully acknowledged.

Table V

Structure of the Samples

	Sample 1 1981/82	Sample 2 1984/85
Number of bonds	28	30
Time to maturity at January, 1		
Group 1 } number of bonds ; time to maturity	13; 1.00 - 1.92	3; 1.16 - 1.92
Group 2	3; 4.17 - 4.92	12; 4.00 - 4.84
Group 3	12; 7.00 - 7.83	15; 7.01 - 7.84
Coupon	5.25 % - 10.25 %	5.25 % - 10.25 %
Yield index	7.75 % - 11.42 %	6.22 % - 8.30 %
DG-Price index minimum	86.57	100.29
DG-Price index maximum	103.14	111.49

For each bond k was estimated on the basis of 105 and 104 prices respectively. In addition, for the second period k is re-estimated using 24 monthly prices. The results are summarized in Table VI.

Table VI

Estimation of the Factor k by Analysis of Time-Series-Data

	Sample 1 1981/82	Sample 2 1984/85	
		Weekly Prices	Monthly Prices
Sample mean $\hat{k}$	0.0126	0.0072	0.0076
Coefficient of variation	0.0135	0.0126	0.025
Sample minimum	0.0079	0.0056	0.0061
Sample maximum	0.0198	0.0105	0.0104
Subsample means			
Group 1	0.0167	0.0098	0.0096
Group 2	0.0087	0.0077	0.0081
Group 3	0.0090	0.0062	0.0068

The maximum likelihood estimation of k permits the following interpretation

• For a 10-year, 10 %-bond quoted at par the instantaneous volatility of the bond's return was on average about

$0.0126 \cdot 6 \cdot 759 \cdot 100 = 8.5$ % p.a. in the first period and

$0.0072 \cdot 6 \cdot 759 \cdot 100 = 4.9$ % p.a. in the sond period.

As was expected, k is not stationary and in the first time span is about 75 % higher than in the second.

• A comparison of the estimation for weekly and monthly data shows that the time interval used to collect price changes has on average no strong impact on the estimation. The coefficient of variation is two times larger for monthly data than for weekly prices. This reduction in precision can be attributed completely to the smaller sample size.

• The minima of the estimated values k_i (i = 1, ..., 28 and 30 respectively) occur for long-term bonds, the maxima for short-term bonds. This result is confirmed by the decrease of the subsample means with time to maturity. Furthermore, in both periods the minimal $\hat{k}_i$ in group 1 is larger than the maximal k_i in the other two groups. Since in the first sample the majority of bonds quote below par and in the second sample above par, it seems that the decrease of the subsample means with time to maturity does not primarily depend on the bond price or on the factor $(B_{max} - B(t)) / (B_{max} - RV)$ in the definition of the volatility.

B. Cross-Sectional Analysis

The estimation procedure under A. is supplemented by a cross-sectional analysis for two reasons. First, it is interesting in itself to know whether two basically different statistical designs result in different estimations for k. Second, the systematic influence of the time to maturity on the subsample means leads to the assumption that the impact of the duration on the instantaneous volatility is too large. This influence can be reduced if the exponent of the duration term is diminished from one to $\gamma<1$.

The cross-section study was done using a linear regression.[16] For this purpose (26) was divided by the second and third term of the right hand side, and the new left hand side is denoted by Z (t). If further Z (t) and e (t) are replaced by their standard deviations and, logarithms are taken, then the following linear equation results

$$\sigma(\tilde{Z}(t)) = \ln k + \gamma \cdot \ln \text{Duration}(t) + \ln \sqrt{\Delta t} \tag{27}$$

The estimation of k and γ from (27) poses at least two problems. First, in every point of time t only one observation os $\tilde{Z}(t)$ is available which means that an estimation of $\sigma(\tilde{Z}(t))$, using this information only, is not possible. Second, if a sequence of observations from different points of time is used, an error in variable problem with the independent variable Duration (t) occurs.

16 For a comparable study see Schaefer and Schwartz (13), pp. 1118

Despite these reservations k and y were estimated for six different periods of 12 weeks each by a cross-sectional linear regression. Each of the six samples consists of all non-callable bonds by the same issuers as described in part A. For each bond based on 12 successive Friday prices the arithmetic average of the corresponding duration values is defined as one observation of the exogeneous variable "Duration (t)" and the standard deviation of the 12 $\tilde{2}$-values as one observation of the endogeneous variable.

Table VII shows the first Friday of the estimation periods, the sample size, the estimated values of k and γ, and the coefficient of determination

Table VII

Results of the Cross-Section Analysis

Estimation Period	2.1.81	2.1.82	7.1.83	6.1.84	4.1.85	3.1.86
Sample Size	92	91	90	93	93	102
$\hat{k}$[17]	0.021	0.0076	0.012	0.0086	0.011	0.0068
$\hat{\gamma}$	0.74	0.86	0.63	0.74	0.90	1.02
R^2	91.18 %	90.15 %	82.49 %	87.92 %	'87.76%	87.02 %

A comparison of the values $\hat{k}$ with those obtained from the analysis of time series data shows that they are of a similar size. The arithmetic mean of the three estimates for the periods 1981 through 1983, for example, gives a value of 0.0135 for $\hat{k}$ compared with 0.0126 for the whole period 1981/82. The analogous comparison for the last three samples results in an average value for $\hat{k}$ of 0.0088 compared with 0.0072 from the time series analysis.

The majority of estimations of γ are less than one, as conjectured. The arithmetic mean of the six sample values is 0.82. The estimates $\hat{\gamma}$ show no total stability over time. Besides sample variation the negative correlation between $\hat{k}$ and $\hat{\gamma}$ can explain this instability. An increase in $\hat{k}$ can be partly offset by a reduction in $\hat{\gamma}$. This effect can possibly explain the value $\hat{k} = 0.012$ in the third sample which is high compared with the observed price fluctuations in this period. The same offsetting effect possibly explains the large value of $\hat{\gamma}$ in the last sample.

17, $\hat{k}$,$\hat{\gamma}$ and ($\hat{k}$, $\hat{\gamma}$) are significant on the 0.1 % level. Due to the above formulated reservations these values are not very meaningful.

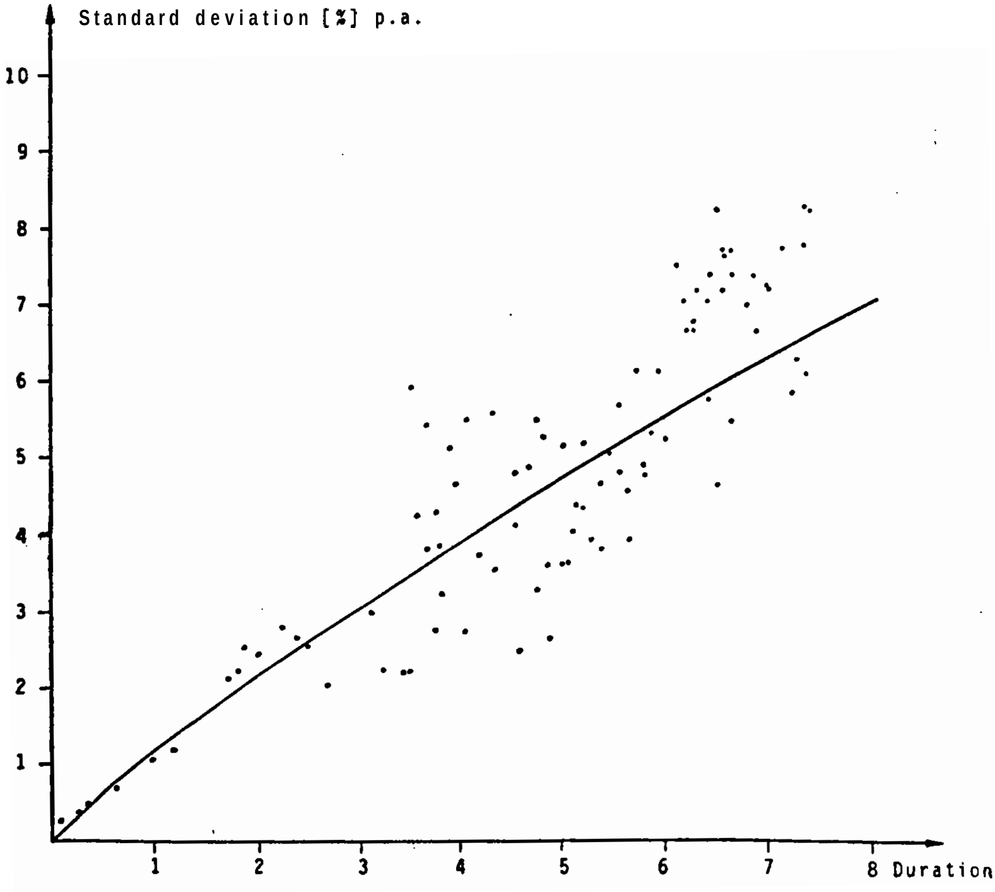

Figure 1 : Observations and regression function for the fifth sample.

IV. COMPARATIVE STATICS ANALYSIS

The subsequent Figures show the call and put values as a function of the most important determinants. All valuations are based on the following assumptions : The exercise price is equal to the redemption value which is normed to 100 %. The factor k is fixed such that a 10-year, 10 %-bond quoted at par has an instantaneous volatility of 10 % p.a. initially. From (21) k has the value 0.1/6.759 = 0.015. The parameter y is fixed to one. The values given for the short-term interest rate refer to the beginning of the option life. Later this rate like the bond price develop stochastically.

Figure 2 shows American call values for a ten and a four year bond as a function of the current bond price. The premiums for the at-the-money option are 3.26 % and 1.47 %. The substantially higher values for the call on the ten-year bond are due to its higher instantaneous price volatility.

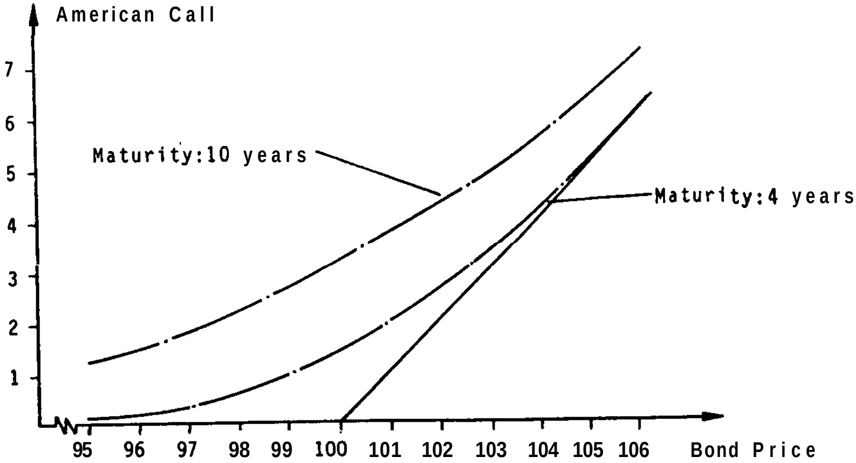

Figure 2 : American call values as a function of the bond price. Coupon 8 % ; short-term interest rate 8 % ; expiration date 0.75 years.

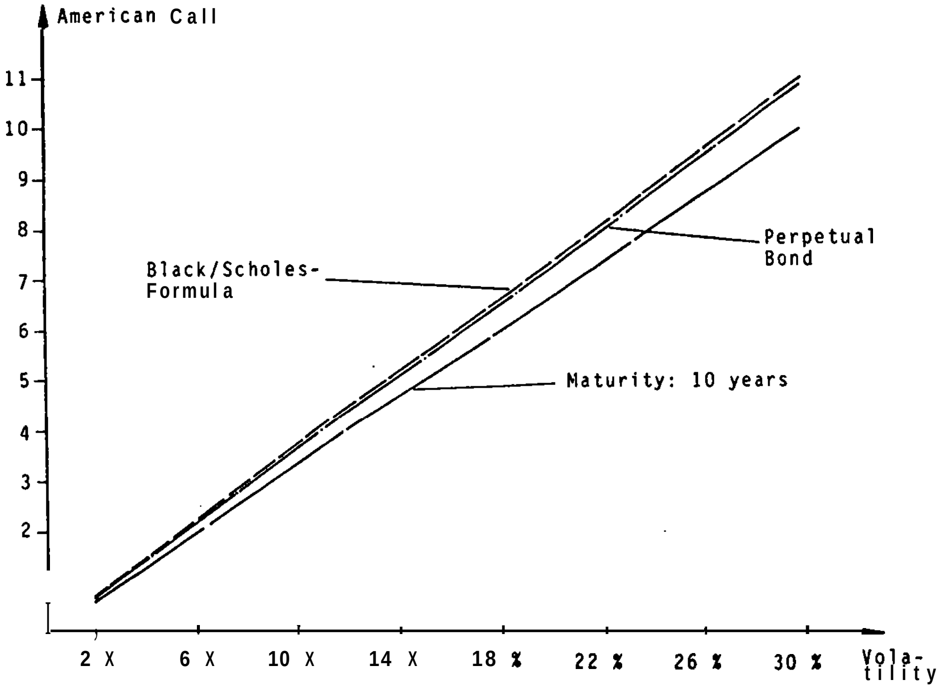

Figure 3 : American call values as a function of the volatility. Coupon 10 % ; short-term interest rate 10 % ; expiration date 1 year.

Figure 3 shows the value of an American call as a function of the initial instantaneous volatility. In the volatility range considered this function is approximately linear. The values of a call on a 10-year bond are supplemented by the values of the same call on a perpetual bond and by Black-Scholes values.[18] For a volatility of 6 % p.a. the difference between the values of the call on the perpetual bond and the 10-year bond is 8.96 % in terms of the call value of the ten year bond If must, however, be borne in mind that the comparison is based on the assumption that the instantaneous volatilities of the two bonds are initially identical. The difference is, therefore, caused by the smaller profit potential and the decreasing instantaneous volatility of the 10-year bond during the time to expiration. This effect can be observed even more clearly for long exercise periods.

Figure 4 shows American call values for these two bonds as a function of the time to expiration.

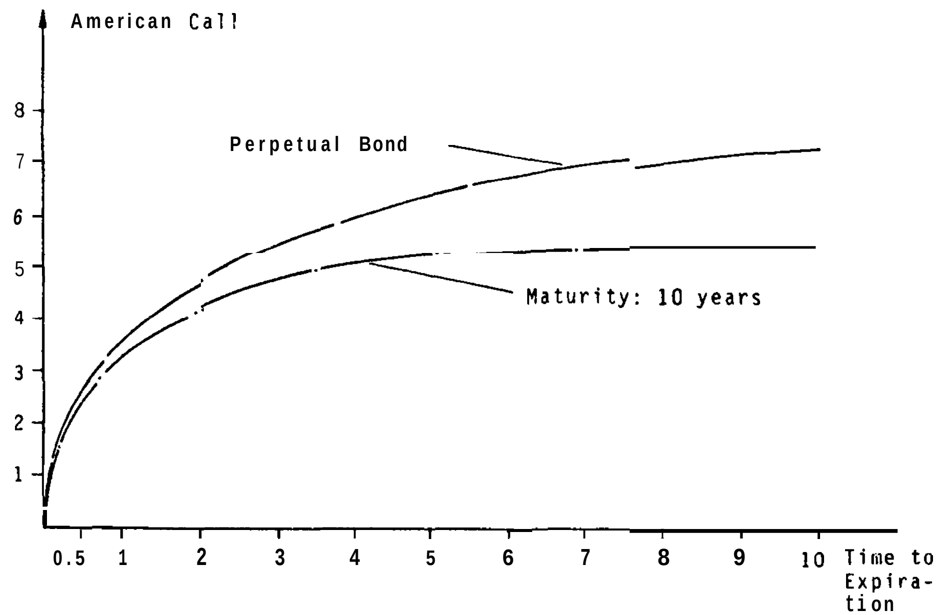

Figure 4 : American call values as a function of the time to expiration. Bond price 100 % ; coupon 10 % ; short-term interest rate 10 %.

18 Here h e Black-Scholes formula for a European call on a stock with a constant dividend rate is used. The dividend rate is substituted by h e current yield to maturity of the bond.

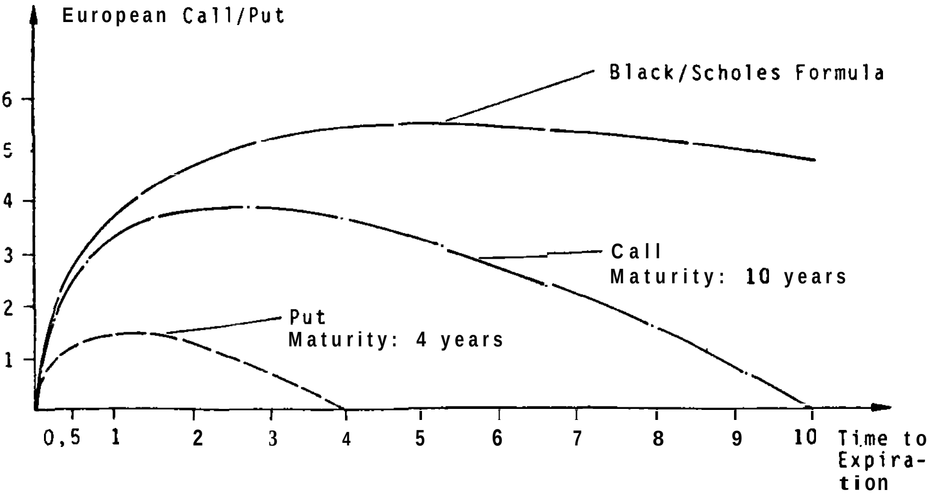

Figure 5 : European call and put values as a function of the time to expiration. Bond price 100 % ; coupon 10 % ; short-term interest rate 10 %.

It is si cant that for times to expiration of more than 5 years the values of the call on the 10-year bond run almost parallel to the abscissa, while the call on the perpetual bond increases strictly monotonously as in the case of stock options. The waning profit potential, the decreasing instantaneous volatility and a total volatility of the bond price which rises and then falls over time prevent any further increase in the call values for times of expiration of more than five years. In the case of European options this effect can be demonstrated even more clearly.

Figure 5 shows European call values on a 10-year bond and European put values on a 4-year bond as a function of the time to expiration. Both values - unlike the values of stock options - begin to decline for times to expiration in excess of about 30 % of the maturity of the bonds and are worthless as soon as the time to expiration and residual life coincide. For comparative purposes, Figure 5 also displays the call values produced by using the Black-Scholes formula.

In Figures 2 to 5 coupon and short-term interest rate were assumed to be identical. This choice of parameters was designed to eliminate possible influences from differences in the bond yield and the rate of return of the alternative investment. Figure 6 shows the values of American and European options for a coupon of 6.5 % and a short-term interest rate of initially 5 %.

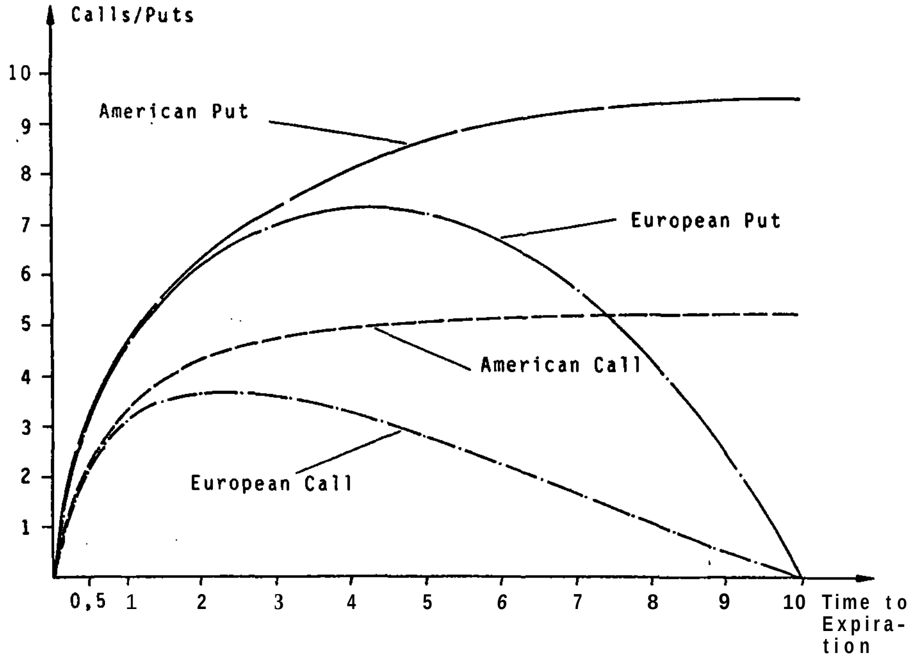

Figure 6 : Call and put values as a function of the time to expiration. Bond price 100 % ; coupon 6.5 % ; short-term interest rate 5 %.

The greater the difference between bond yield and short-term interest rate, the smaller the call values and the greater the put values. This effect can be explained best from the point of view of the option writer. The writer of a put option posts an annual opportunity loss depending on the difference between the bond yield and the short-term interest rate. His compensation for this loss, which gets larger as the difference increases, is partially reflected in the option price.

A comparison of Figure 6 with Figures 4 and 5 shows that the European call is reduced mare in value than its American counterpart. The reason is that in the American case the interest gain for the writer is partly offset due to the fact that the exercise probability increase. with decreasing short-term interest rate.

The interest rate disadvantage suffered by the writer of a put also results in the fact that in Figure 6 the value of the American put - in contrast to the value of the American call - continues to increase for times to expiration of more than 5 years. In principle, this argument can also be applied to the European put. Since, however, the total volatility of the bond price decreases beyond a critical point of time which is not exactly known for the price model defined in Section II, the average loss for the writer, if the option is exercised, also decreases. This effect offsets the interest rate disadvantage of the writer and results in a decreasing put value for times to expiration of more than 4 years.

Figure 7 hows the values of American calls and puts as a function of the short-term interest rate

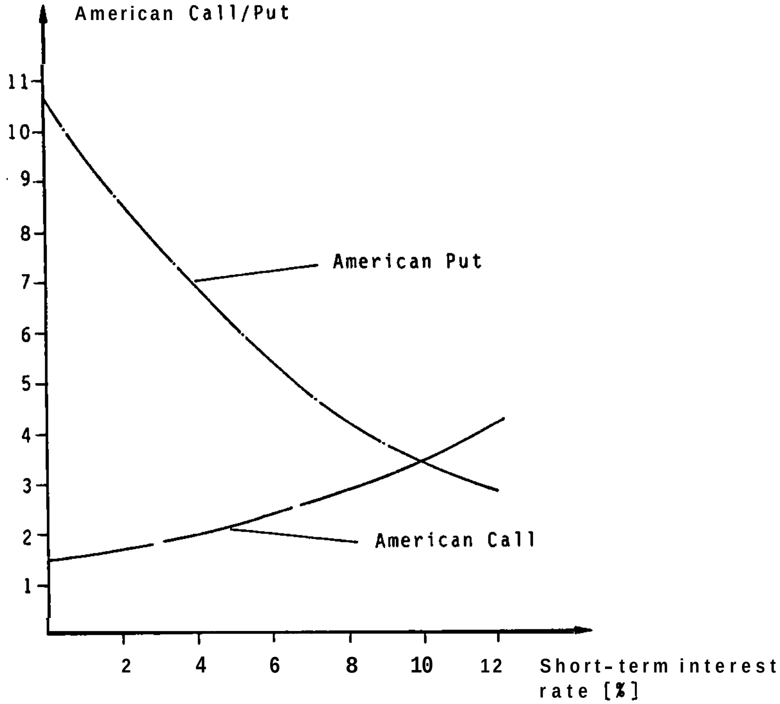

Figure 7 : Call and put values as a function of the short-term interest rate. Bond price 100 % ; coupon 10 % ; time to expiration 1 year ; time to maturity 10 years.

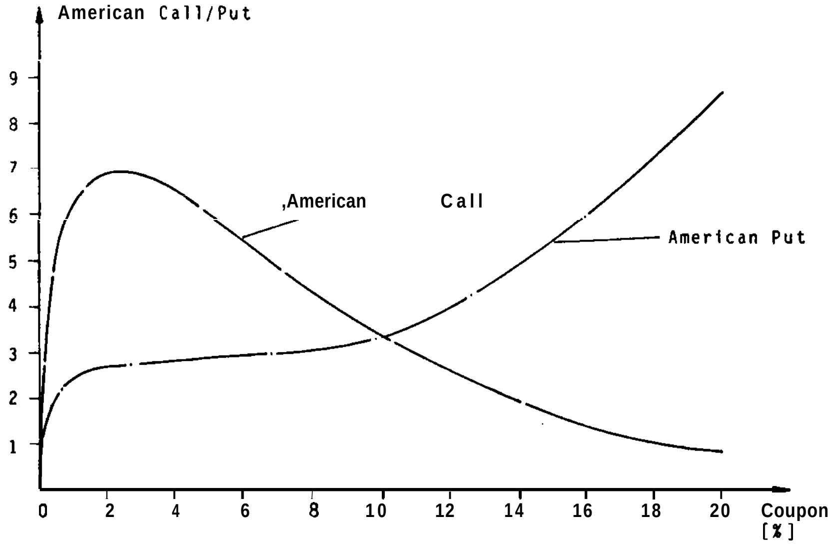

Figure 8 : Call and put values as a function of the coupon. Bond price 100 % ; short-term interest rate 10 % ; time to expiration 1 year ; time to maturity 10 years.

As discussed above, the put increases and the call decreases if the short-term interest rate decreases. To get some further insights, the option values for the short-term interest rates of 0 % and 10 % are compiled in the following table.

Table VIII

Option Values for Two Short-Term Interest Rates

	Calls		Puts	
	American	European	American	European
r = 10 %	3.35	3.20	3.42	3.20
r = 0 %	1.44	0.67	10.67	10.67

Table VIII shows that for the short-term interest rate of 0 % the right of premature exercise is worthless for the American put, whereas for the American call it amounts to 53 % of its value. If the short-term interest rate is low, the owner of a call has a strong incentive to obtain the bond by exercising his option.[19] The owner of the put an the other hand will exercise his option only if the bond price decreases ,considerably.A more detailed analysis shows that within the accuracy of the numerical method used, for the put the right of premature exercise is worthless if the short-term interest rate is 6 % or less.

For the low interest rate scenario the difference between the European put and call values is - as put - call parity demands - equal to the coupon. The interest disadvantage for the put writer in this example amounts to 93.7 % of the put value whereas the average profit of exercising the put contributes only 6.3 %.

Figure 8 shows the values of American options as a function of the coupon.

It follows from the boundary conditions discussed in Section I that the call and the put are worthless for a zero coupon.[20] As the coupon increases $B_{max}(t) = c \cdot (T_B - t) + RV$ becomes steeper and steeper, and the profit potential of the call increases. This together with an increasing total variance of B (t) $(0 \leq t \leq 1)$ as a function of c justifies the very strong growth of the call values for small coupons, peaking at a coupon of about 2 %, with a maximum value of 6.94 %. Two effects might be responsible for the non-monotonicity of the call value function

- First, it can be shown that the instantaneous standard deviation is monotonously decreasing with the coupon and is bounded on the interval $c \geq \varepsilon > 0$. Assuming for a moment that the drift of the bond's price return is constant, than it follows that the total volatility of B (t) $(0 \leq t \leq 1)$ is a bounded function of the coupon. Therefore, an increase of the coupon beyond a critical value does not increase essentially the mean exercise value of the call.

- Second, the increase of the coupon reduces the interest disadvantage for the writer of the call and, therefore, this part of the call price. This second effect leads to a decrease of the call value function

19 Note that in this comparative static analysis the initial yield to maturity of the bond does not vary with the short-term interest rate.

20 If the coupon is zero for a fixed bond price of 100 % the yield to maturity and, therefore, the short-term interest rate are zero.

The put value function also increases steeply to begin with because the total volatility of the bond price increases initially. The subsequent phase of the put value's lower sensitivity can be explained by the fact that put values do not contain a premium for the writer's interest disadvantage and that - as argued above - the total volatility of B (t) is bounded in the coupon. A compensation for the interest disadvantage is paid for higher coupons and then results in a steeper increase of the put value function.

V. SUMMARY

In this paper a model was developed permitting the valuation of puts and calls on bonds, of the European or American style, with constant and varying exercise prices as well as for exercise periods which can be characterized by a single time interval and a series of isolated time intervals or points of time. In order to come up with reasonable option values it turned out to be essential that in defining the bond price dynamics and in solving the resulting partial differential equation, the bounded variation of the bond price and its convergence towards the redemption value had to be considered.

The most important results of the comparative statics for at-the-money options are:

- Call and put values are very sensitive to changes in bond maturity,

- European call and put values as a function of the time to expiration increase initially and then decrease.

- If coupon and short-term interest rate are identical, then American calls and puts do not increase beyond a critical time of expiration of roughly 40 % of the bond's maturity.

- If short-term interest rate is lower than the coupon of a par bond, this result remains for American calls but not for American puts.

- Call values increase, put values decrease, as the short-term interest rate rises.

- An increase in the coupon results in monotonously increasing American put values and initially strongly increasing and than decreasing American call values.

The next step will be to replace the short-term interest rate by a zero bond with maturity equal to the time to expiration.

APPENDIX

A. Proof of the inequality $P_a(B(t), t) \leq B_{max}(t) - B(t)$

To prove this inequality assume that the put value is larger than the difference between the maximum bond price and the current bond price. The portfolio composed of

- one unit of the bond sold short until the expiration date T of the option,
- one written put and
- cash to the amount of $P_a + B(t) + c_t > B_{max}(t) + c_t$

will then result in a riskless future profit. Three cases must be distinguished in order to demonstrate this assertion.

Case 1 : The put is not exercised At point of time T the available cash amounts to

$$P_a + B(t) + c_t - c > B_{max}(t) + c_t - c$$

$$= c \cdot (T_B - t) + RV + c \cdot (t - t_o) - c \cdot (t_1 - t_o)$$

$$= c \cdot (T_B - T) + RV + c_T = B_{max}(T) + c_T$$

Therefore, even in the worst case the bond can be bought back without spending the entire available cash.

Case 2 : The put is exercised at time $\tau \leq t_1$.

After having paid the exercise price E_τ and accrued interest c_τ, the riskless profit amounts to

$$P_a + B(t) + c_t - E_\tau - c_\tau > B_{max}(t) + c_t - E_\tau - c_\tau$$

$$= c \cdot (T_B - t) + RV + c \cdot (t - t_o) - E_\tau - c \cdot (\tau - to)$$

$$= c \cdot (T_B - \tau) + RV - E_\tau = B_{max}(\tau) - E_\tau \geq 0$$

and the short sale commitments can be met by the bond purchased at time τ.

Case 3 : the put is exercised at time $\tau > t_1$.

The cash outflows consist of the coupon payment $c \cdot (t_1 - t_o)$ at time t_1, the exercise price E_τ and accrued interest $c_\tau = c \cdot (\tau - t_1)$ at time τ. Again a positive riskless, profit remains

$$P_a + B(t) + c_t - E_\tau - c \cdot (t_1 - t_o) - c \cdot (\tau - t_1) >$$

$$B_{max}(t) - E_\tau + c \cdot (t - t_o) - c \cdot (t_1 - t_o) - c \cdot (\tau - t_1)$$

$$= B_{max}(\tau) - E_\tau \geq 0.$$

Summing up, the upper bound (18) for the American put has been proven.

B. Bond Price Dynamics

To the knowledge of the author there are no general characterizations available of the boundary behavior of a diffusion process if the drift and the instantaneous volatility depend on time. The following presentation is an intuitive justification that $B_{max}(t)$ and $B(t) = 0$ are natural boundaries if the instantaneous standard deviation is defined as in (21) and a suitable drift is chosen

Consider the stochastic process defined by

$$\tilde{F}(t) = B_{max}(t)/[1+(T_B-t)e^{b\tilde{W}(t)}], \qquad (28)$$

where b is a positive parameter. $-\infty$ and $+\infty$ are natural boundaries of the Wiener process. Therefore, F (t) has $B_{max}(t)$ and $B(t) = 0$ as natural boundaries. In addition, $F(T_B) = 1$ with probability one. The stochastic differential equation of F (t) reads

$$d\tilde{F}(t) = \mu_F dt + \sigma_F d\tilde{W}(t)$$

$$= \frac{F(t)}{B_{max}(t)}\cdot\left[\frac{B_{max}(t)-F(t)}{T_B-t} - c + b^2\cdot(B_{max}(t)-F(t))\cdot\left(\frac{1}{2}-\frac{F(t)}{B_{max}(t)}\right)\right]dt$$

$$- b\cdot\frac{(B_{max}(t)-F(t))\cdot F(t)}{B_{max}(t)}\, d\tilde{W}(t) \qquad (0\le t<T_B) \qquad (29)$$

The drift of $\tilde{B}(t)$ will be defined as

$$\mu_B = -\frac{B(t)}{T_B-t}\cdot\ln[B(t)/RV] \qquad (0\le t\le T_B-\varepsilon) \qquad (30)$$

The sign of the drift is identical to the sign of RV - B (t), i.e. when the bond price deviates from the redemption value it always moves back toward RV, as long as no random disturbance occurs. For the minimum bond price the drift is zero.

At $B_{max}(t)$ it can be shown that μ_B is smaller than the slope $-c$ of $B_{max}(t) = c\cdot(T_B-t) + RV$. This means that the drift directs towards the interior of the region of feasible bond prices at the upper boundary, whereas the drift of F(t) is equal to $-c$ at $B_{max}(t)$. This means that the force driving the bond price back is in a neighbourhood of $B_{max}(t)$, larger für B (t) than f a F (t).

In a neighbourhood of B (t) = 0 the drift of B (t) is larger than the drift of F (t). Therefore, at the second boundary B (t) is also more strongly driven back than F (t).

Finally, b can be chosen such that

$$b \cdot \frac{B(t)\cdot(B_{max}(t)-B(t))}{B_{max}} > k \cdot \frac{B(t)\cdot(B_{max}(t)-B(t))}{c}$$

$$\geq k \cdot \frac{B(t)\cdot(B_{max}(t)-B(t))}{B_{max}(t)-RV} \cdot \text{Duration} = \sigma$$

holds. Hence, on the interval $0 \leq t < T_B$ the process B (t) compared with F (t) has a smaller instantaneous volatility and absolutely greater drift terms in appropriately chosen neighbourhoods of the boundaries. Therefore, it is conjectured that $B_{max}(t)$ and $B(t) = 0$ are also natural boundaries for B (t).

It is not clear whether σ as defined in (21) and μ_B guarantee that $B(T_B) = RV$ with probability one. Therefore, on the interval $\varepsilon / 2 \leq T_B - t \leq \varepsilon$ the drift of B (t) is continuously transformed to the drift of F (t), and on the interval $0 < T_B - t \leq \varepsilon / 2$ the drift μ_B is substituted by μ_F.

C. Consistency of B (t) and r (B (t))

Let $t_1 > t$ be the next coupon date and $y^* = y(B(t), t)$ the continuously compounded yield to maturity of the bond. Define

$$D_{\tau-t} = e^{-y^*(\tau-t)} \qquad \text{for } t_1 \leq \tau$$

$$D_{\tau-t} = e^{-r^*(\tau-t)} \qquad \text{for } t \leq \tau \leq t+\varepsilon < t_1, \text{ where}$$

$r = s(t) \cdot y(B(t), t)$ and $\varepsilon > 0$ is chosen such that $D_\varepsilon > D_{t_1-t}$

Finally, on the interval $t + \varepsilon \leq \tau \leq t_1$ the discount function is defined as a linear function in τ with $D_\varepsilon = e^{-r^*\varepsilon}$ and $D_{t_1-t} = e^{-y^*(t_1-t)}$.
Hence, $D_{\tau-t}$ is strictly monotonously decreasing and $\lim_{\tau \to t} [-\ln D_{\tau-t}/\tau-t] = r^* = r(B(t)_1\, t)$

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