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STOCHASTIC EQUILIBRIUM AND PREMIUMS IN INSURANCE

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EQUILIBRE STOCHASTIQUE ET PRIME D'ASSURANCE

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RESUME

Cet article présente une étude de la structure des primes de contrats d'assurance en équilibre économique. On considère une économie d'échange comportant des agents adverses de risque, et où les risques sont représentés par des processus stochastiques.

On indique d'abord les conditions suffisantes, en matière de préférences, d'existence d'un équilibre général, dans le cas d'un processus de prix fictif, qui dépend uniquement des spécificités des risques.

On montre ensuite comment ce résultat peut - Bee utilisé pow calculer des primes d'assurance, par une transformation en un modèle à risques neutres, dans lequel les portefeuilles sont des martingales.

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ABSTRACT

This paper investigates the structure of premiums of insurance contracts in economic equilibrium. We consider an exchange economy with risk averse agents and where the risks are represented by stochastic processes.

First, we give sufficient conditions on preferences for a general equilibrium to exist, with a shadow price process that only depends on specifics of the risks.

Second, we show how this result can be used to compute premiums in insurance by the transformation to risk neutrality where portfolios are martingales.

1 .INTRODUCTION

1.1 In the **first** part of this paper, we introduce a reinsurance setting, where we discuss **the** general **problem** of the pricing of **contracts** and the **reallocation** of risks under uncertainty. We consider continuous time, where risks are represented by **stochastic** processes. The discrete time **case** is effectively subsumed in our setting.

1.2. We wak with processes rather than with one period **pay-offs,** and thus we use functional analysis to derive the market's marginal utility process, ***or,*** the shadow price process consistent with an economic equilibrium. It turns **out** that, under our **assumptions,** as soon as a "risk-space" has been established, the shadow price process **only** depends on properties of this space rather than on specifics of preferences. This means, among *other* things, that our **assumptions** can be tested by statistical inference (f a stochastic processes). This is not the case when **the** shadow **price** process **depends *on*** specifics of **preferences.** To be more concrete, **suppose** preferences can be represented by von Neuman-Morgenstem expected utility functionals. Then the earlier pioneering results by **Borch** (1960-85) typically give the market's marginal utility process as a function of the underlying Bemoulli utility functions. In order to obtain explicit results, Borch then assumed **that** all the participants in the exchange economy possess preferences that can be represented by the same class of Bemoulli utility functions, like quadratic, logarithmic or exponential. Such an assumption is of **course** very hard to test, let alone the specification (estimation) of the **parameters** of the individual utility functions, once a class has been determined. Such results are thus basically of theoretical interest only.

1.3. Considering the progress that is presently being made in the field of statistical inference for stochastic processes, on the other hand, it should be possible also in insurance to utilize' the theory presented in this paper. Similar results are constantly being used in the pricing of financial options and in the pricing of stocks.

The second part of the paper indicates how these results can be used in the computation of premiums for risks typically occurring in the insurance industry.

2. THE REINSURANCE ECONOMY

In this section we describe the primitives for a stochastic insurance economy; a model for uncertainty and revelation of infamation over time, a space of **stochastic** processes representing the risks in the market, endowments and initial portfolios, and preferences. Also the definition of a general **(ADB)-equilibrium** in this model is given.

2.1. The probabilistic setting and the revelation of information over time

In this subsection we start with the **description** of a general model of uncertainty and the dynamics by which infamation is revealed over time.

We consider an insurance economy of I companies indexed by $\mathbf{i} \in \ = \{1, 2, ..., I\}$, a **riskless** asset $\mathbf{x_0(t)}$ and a **time** interval $\mathbf{T} = [0, \mathbf{T}]$ $t \in T$. Let $\boldsymbol{\Omega}$ be the set of all **possible** states of the world which could occur. A state of the world $\omega \in \Omega$ is an exogenous sequence of circumstances occurring from time 0 to time T which determined the realization of every exogenous random variable relevant to the economy. Furthermore,

$(\Omega, \mathcal{F}, F=\{\mathcal{F}_t, 0 \le t \le T\}, P)$ is a probability space with the σ-algebra $\mathcal{F}_t$ *representing* the information available at time t, $\mathcal{F}_t \supseteq \mathcal{F}_s$, $t > s$, $\mathcal{F} = \mathcal{F}_T$. The filtration F we assume to be continuous and $\mathcal{F}_0$ is almost trivial (meaning that Ω is the only event of non-zero probability in $\mathcal{F}_0$). We assume that the participants in the economy possess subjective probability measures P_i on $(R, \mathcal{F}, F)$ that are uniformly absolutely continuous with respect to one another. This means that there are bounds on the heterogeneity of probability assessments. In other words, there exist constants c_1 and c_2 such that for any $B \in \mathcal{F}$ and any company i.

$$(2.1) \qquad c_1 P_i(B) \le P_j(B) \le c_2 P_j(B), j \in I$$

In this paper we shall discuss a class of finite variance variables, which will play a central role in the analysis. This class of variables is preserved under a change of measure of the above type. Furthermore, all the topological properties of the risk space described in the next section are invariant under changes of probabilities of this sort, so there is no loss in generality to assume that the participants all have common probability assessments given by the probability measure P, which is uniformly absolutely continuous with respect to the probability measures P_i, $i \in I$.

2.2. The risk space X

The vector of initial portfolios $(x_1(t,\omega), x_2(t,\omega), ..., x_I(t,\omega))$ and the corresponding vector of final portfolios after the reinsurance treaties $(y_1(t,\omega), y_2(t,\omega), ... y_I(t,\omega))$ we assume to be stochastic (vector) processes adapted to the filtration F. This means that the values of $x_i(t)$ and $y_i(t), i \in I$, depend only on the information revealed by $\mathcal{F}_t$ and available to all the participants at time t. From now on we restrict attention to a class of *optional* stochastic processes called *semimartingales.* Each such process $x = \{x(t), 0 \le t \le T\}$ has a decomposition of the form

$$(2.2) \qquad x(t) = m(t) + a(t), t \in T$$

where m is a (P, F_t)-square integrable martingale; $m(0) = 0$, $E\{(m(T))^2\} < \infty$ and $E(m(t) \mid \mathcal{F}_s) = m(s)$ for $t \ge s$, where E is the expectation corresponding to P. The process $a = \{a(t), t \in T\}$ is a predictable, finite variation process satisfying $\sup_{t \in T} E\{(a)^2\} < \infty$.

Thus each $a(t)$ can be written as a difference of two square-integrable increasing processes. Note that by construction a semimartingale can always be taken to be right-continuous with left-limits (RCLL). A process is optional if it is measurable with respect to the optional σ-algebra O generated by the RCLL-process. Predictable means measurable with respect to the σ-algebra ρ on $\Omega \times [0,T]$ generated by the left-continuous F-adapted processes. A semimartingale where the bounded variation process is predictable is sometimes called a special semimartingale. Intuitively, a(t) is a predictable process if the value of a(t) can be determined from information available up to, bur *not* including, time t, for each $t \in T$.

Each element x defined above can be considered as a member of $L^2(\Omega \times [0,T], \rho, P \times l)$ where l denotes the Lebesque measure. Notice that it follows from our assumptions that

$$E\left(\int_0^T x^2(t)dt\right) < \infty. \tag{2.3}$$

This square-integrability assumption is normally not restrictive, but it excludes certain stable processes of some interest to financial economics (see the literature on processes with stable increments, e.g. McCulloch (1978) with further references). In insurance we do not consider (2.3.) to be restrictive.

We denote the set of semimartingales defined above by X. Our definition is slightly more restrictive than is sometimes used in the literature (see Jacod (1977) for instance).

The market value of an endogeneously determined risk $y \in X$, its premium π, is naturally a functional on X. At this stage we mention that the premium must be a positive linear functional in economic equilibrium. In the risk space we consider, (below we shall equip X with a norm so that it becomes a Hilbert lattice) such functionals are also continuous, or equivalently, bounded. Economic considerations lead to positive, continuous linear functionals as equilibrium premiums: If, for example, linearity of π is not met, there would exist an *arbitrage opportunity*. As an illustration, consider the following situation $\pi(y_1+y_2) > \pi(y_1) + \pi(y_2)$. Then an industrious company agent could perform the following strategy: Sell the insurance (y_1+y_2), reinsure separately y_1 and y_2. The cash flow at time zero is $(\pi(y_1+y_2) - \pi(y_1) - \pi(y_2)) > 0$, and the cash flow at time T is $-(y_1(T) + y_2(T)) + y_1(T) + y_2(T) = 0$. This is a money pump. Each agent would jump at the opportunity to hold this strategic portfolio, and each one would do so in unlimited quantities.

In the actuarial science many "premium principles" that are in use do not satisfy this property. In actuarial terms, the premium is a property of the risk (and nothing else). Examples may be (one period problems).

$$\pi_1^a(y) = E(y) + \alpha \operatorname{var}(y)$$

$$\pi_2^a(y) = E(y) + \beta \operatorname{SD}(y).$$

The intended interpretation is that the premium of a risk equals the net *premium* E(y) (the "actuarially fair" premium) plus a term compensating for the risk bearing, here proportional to the variance (or the standard deviation) of the risk y. First of all neither π_1^a nor π_2^a are linear functionals and can accordingly not be consistent with an economic equilibrium. Second, actuarial premium systems do not depend on other economic quantities (risks) in the market (see e.g. the survey by Goovaerts et al. (1983)). We argue that the risk premium must naturally be a part of the total premium π, but it must be determined by the market, and thus by all the relevant economic parameters, equilibrium mechanisms, risks and attitudes towards risk that constitute this market, and not solely by exogeneous properties of the risk itself. We return to concrete examples later.

23. The Hilbert space structure of X

In order to exploit the properties of the risk space X, consider the optional quadratic variation process $[x,x]_t$, the predictable quadratic variation process $\langle x,x\rangle_t$ and their polarized extensions.

$$[x,y] = \frac{1}{2}([x+y, x+y] - [x,x] - [y,y])$$

$$\langle x,y\rangle = \frac{1}{2}(\langle x+y, x+y\rangle - \langle x,x\rangle - \langle y,y\rangle).$$

We equip the space X with the norm

$$\|x\|^2 = E([x,x]_T) = E(\langle x,x\rangle_T). \tag{2.4}$$

This norm is equivalent to the product L^2-norm $\| x \|_2^2 = E\{ \int_0^T x^2(t)dt\}$ in that they generate the same topology. Furthermore, the definition

$$\langle\langle x,y\rangle\rangle = E(\langle x,y\rangle_T) \tag{2.5}$$

satisfies the properties of an inner product. Since X is complete (Jacod (1979)) it is a Hilbert space. Let X_+ denote the positive cone of X, i.e. $x \geq 0$ means that $x \in X_+$, $x > 0$ means $x \in X_+$, $x \neq 0$. X is also a Hilbert lattice. This follows from the equivalence of the product L^2-norm $\| . \|_2$ and the norm $\| . \|$, and from the fact that an L^p-space is a Banach lattice. (Jacod (1979), Schaefer (1971). Meyer (1975-76), Doob (1984), Le Cam (1986) among others). The lattice property enables us to use general equilibrium theory and can not be dispensed with (Mas Colell (1986)).

For an element $x \in X$ we define the set

$$L^2(x) = \{\text{predictable processes } h\text{: } \|h\|^2_{L^2(x)} = E(\int_{0-}^T h^2(t)d\langle x,x\rangle_t) < \infty\}. \tag{2.6}$$

The linear map $h \to \int_{0-}^{t} h(s)dx(s)$ (from $L^2(x)$ into X defines a stochastic integral with respect to the semimartingale x.

Let $\mathcal{M}^2$ denote the set of square integrable $(P, \mathcal{F}_t)$—martingales and consider $\eta(t) = \int_{0-}^{t} h(s)dm(s)$ for $m \in \mathcal{M}^2$ and $h \in L^2(m)$. The following results are known in stochastic analysis:

$$(2.7) \quad \langle \eta, y \rangle_t = \int_{0-}^{t} h(s)\, d\langle m,y \rangle_s$$

and

$$(2.8) \quad E(\eta(T)y(T)) = E\left(\int_{0-}^{T} h(t)\, d\langle m,y \rangle_t\right)$$

for any $y \in X$.

This we make use of to characterize premiums in equilibrium. As we have pointed out, the premium of a risk must be a linear functional π on X By the Riez representation theorem there exists a unique $\xi \in X$ such that

$$(2.9) \quad \pi(y) = \langle \xi, y \rangle = E([\xi,y]_T) = E(\langle \xi,y \rangle_T)$$

for all $y \in X$. Such a linear functional can be represented by a martingale measure P* under which all $x \in X$ are $\mathcal{F}_t$ –martingales. In order to construct this martingale measure, consider a process ξ_t given as

$$(2.10) \quad \xi_t = \xi_0 + \int_{0-}^{t} \xi(s-)\, dm^x(s), \quad E(\xi_0) = 1,$$

where m^x is a $(P, \mathcal{F}_t)$—martingale. Here

$$(2.11) \quad x(t) = \sum_{i=1}^{I} x_i(t), \quad t \in T$$

and the connection between $x \in X$ and m^X we return to later. The equation (2.10) has a unique solution given by

(2.12) $$\xi_t = \xi_0 \exp(\varphi_t)$$

where

(2.13) $$\varphi_t = m_t^x - \frac{1}{2}\langle m^{xc}, m^{xc}\rangle_t + \sum_{s\le t}\{\ln(1+\Delta m_s^x) - \Delta m_s^x\}$$

(see e.g. Jacod (1977), Meyer (1975), Memin (1980) among others).

Here m^{xc} is the continuous part of the process m^x and $\Delta m_s^x = m_s^x - m_{s-}^x$. We assume that $\Delta m_s^x > -1$ for all $s \in [0,T]$, in which case $\xi > 0$.

Now we define a new probability measure P* on (Ω, F) by

(2.14) $$\xi_T = \frac{dP^*}{dP}.$$

Under the assumption that $E(\xi_T) = 1$, it can be shown that the RCLL version of $E(\xi_T|\mathcal{F}_t)$ and xt are equal almost everywhere, so ξ_t is a positive martingale (which is uniformly integrable if $T < \infty$). Since only relative prices are of importance, the normalization $E(\xi_T) = 1$ is no real restriction (sufficient conditions will be provided in the applications). The measures P and P* are equivalent, that is $P^* << P$ and $P^* >> P$. Furthermore, using the properties of stochastic integrals and the representation (2.10).

$\langle \xi, y\rangle_t = \int_{0-}^t \xi(s-)d\langle m^x, y\rangle_s$ so that

(2.15) $$\langle\xi, y\rangle = E\left(\int_{0-}^{T} \xi(t-)\, d\langle m^x, y\rangle_t\right) = E(\xi_T y_T)$$

where we have used (2.7) and (2.8).

Notice that because ξ_T is the Radon-Nikodym derivative of P* with respect to P by (2.14). it follows that

(2.16) $$\langle\xi, y\rangle = E(\xi_T y_T) = E^*(y_T)$$

where E* is the expectation operator with respect to the probability measure P*. Again, the RCLL version of the conditional expectation $E^*(y_T|\mathcal{F}_t)$ is indistinguishable from y_t, so that y_t is really a $(P^*, \mathcal{F}_t)$—martingale for any $y \in X$, in particular for $y = x = \sum_{i=1}^{I} x_i$, so that the accumulated initial portfolio x is a $(P^*, \mathcal{F}_t)$—martingale. Thus we have constructed the equivalent martingale measure as promised. The semimartingale property is preserved under an absolutely continuous change of probability measure. The particular choice represented by the equations (2.10) — (2.14) gives us a space of $(P^*, \mathcal{F}_t)$—martingales under new probability assessments P*. In the examples we show the power of this result in practice.

In the next section one of our basic assumptions is that $x > 0$. Since we have assumed that $\mathcal{F}_0$ is almost trivial, this means that

$$(2.17)\quad E^*(x_T) = E^*(x_0) = x_0 > 0.$$

The economic interpretation of the process ξ is that ξ (t,ω) is the *shadow* price at time t if the state of the world is ω, i.e. ξ is the accumulated (or the market's) marginal *utility* process. If we can establish the existence of an economic equilibrium with the shadow price process ξ given in (2.10), we have at the same time an equilibrium where the market's marginal utility process is derived from stochastics, i.e. from distributional assumptions, rather than from assumptions about specific utility functions representing preferences. However, now it is time to turn to preferences.

2A. Risk aversion

Each insurance company may be characterized by an initial portfolio $x_i = \{x_i(t), 0 \leq t \leq T\}$, $x_i \in X$, and a complete transitive preference order $\geq_i$ on X_+($\succ_i$ denotes the strict preference relation induced by $\geq_i$). A preference relation $\geq$ on X_+ is *uniformly* proper *if* there exists some scalar $\varepsilon > 0$ and a portfolio $x \in X_+$ such that for all $w \in X_+$ the relation $w - \alpha x + z \geq w$ for $z \in X$ and $a \in R_+$, implies that $\| z \| \geq \alpha\varepsilon$. That is, the portfolio x is so desirable that z can only compensate f a some loss of x if z is sufficiently large in norm (*see* Mas-Colell (1986), Richard (1985)). The same portfolio x in this definition is said to be extremely desirable for $\geq$, (Yannelis and Zame (1986)). Alternatively, properness means that there exists an open cone $\Gamma \subset X$ containing a positive vector such that $(-\Gamma) \cap \{z \cdot w \in X_+ : z \geq w\} = \emptyset$ for all $w \in X$. We now make the following assumptions for each company $i \in I$:

<u>Assumption 2.1</u>.

(i) $x \in X_+$ and $w > 0$ imply $x + w \succ_i x$.

(ii) The graph of $\succ_i$ is relatively open.

(iii) $x = \sum_{j \in I} x_j$ is extremely desirable for $\geq_i$.

(iv) For all $x \in X_+$, $\{y \in X_+ : y \succsim_i w\}$ is convex.

(v) $x_i > 0$.

These assumptions may be interpreted as : (i) strictly monotonic preferences, (ii) continuous preferences, (iii) the aggregate initial portfolio of the insurance companies is extremely desirable, (iv) convex preferences, and (v) positive initial portfolios.

Assumption (iii) can be considered as a smoothness condition on preferences, and a strengthening of monotonicity. It holds automatically if there is a continuous, positive linear functional π such that $\pi(z) \geq \pi(w)$ whenever $z \geq w$. Conversely, if $\geq$ is convex, then uniform propemess implies the existence of such a functional (Mas-Colell (1986)). Thus, under risk aversion uniform properness is equivalent to the linear pricing rule that we want to establish, or phrased differently, individual uniform propemess is equivalent to social supportability of x. So, even if the individual companies have different probability assessments P_i and different preference relations $\geq_i$, there may exist an aggregate marginal utility ξ_T, or a state space shadow price process $\xi_t(\omega)$, $t \in T$.

Assumption (v) implies that the insurance companies have strictly positive initial portfolios in [0,T]. This is not an assumption about solvency, since the reinsurance agreements leading to y (t) are not restricted.

These assumptions are not necessarily the weakest that can be found. If the preference relations are represented by von Neuman-Morgenstem type utility functionals of the form $E\{\int_0^T u_i(y(t))dt\}$, then sufficient conditions for assumptions (i) — (iv) are that u_i be concave, strictly increasing with a right derivative at zero, and that $(\sum x_j$ be bounded away from zero. These conditions are far more restrictive than (i) — (iv) above. In most applications to insurance even more restrictive assumptions are common, such that u_i be twice continuously differentiable. Assumptions (i) - (vi) can be weakened, in particular the completeness and transitivity assumptions on preferences can be eliminated (Zame (1987), Yannelis and Zame (1986)).

2.5. The existence of a competitive market equilibrium

In this section we demonstrate the existence of a competitive equilibrium with a Pareto Optimal allocation. We do this by demonstrating an Arrow-Debreu-Borch (ADB) equilibrium for the economy $\mathcal{E} = (X_+, x_i, \geq_i ; i \in \mathcal{I})$. For such an economy every time-state "Arrow-Debreu security" is assumed available for reinsurance treaties at time zero, leaving no incentive for markets to remain open after time zero. The introduction of an ADB-economy in a dynamical setting becomes purely a matter of one's imagination, since the number of states in $\Omega \times [0,T]$-space is uncountable (if time is continuous, for example). In fact, the dynamic feature of the economy is not transparent and cannot be exploited in this framework. Nevertheless, it turns out to be a useful construction in the development of a dynamic description of an equilibrium (Aase (1988b)).

An **ADB-equilibrium** for ε is **defined as** a nonzero premium functional π on X, initial **portfolios** $x_i \in X_+$ and reinsurance **treaties** $y_i \in X_+$ **satisfying** for all i:

$$\pi(y_i) = \pi(x_i), \tag{2.18}$$

$$v \succ_i y_i \Rightarrow \pi(v) > \pi(y_i) \text{ for all } v \in X_+ \tag{2.19}$$

and

$$\sum_{i \in \mathcal{I}} y_i = \sum_{i \in \mathcal{I}} x_i = x. \tag{2.20}$$

In a **reinsurance** setting, this definition of an **equilibrium** was first formulated by **Borch** (1962) in a **one-period** model. Condition (2.18) corresponds to the budget constraint in conventional **microeconomic** analysis. The insurance **compagny** may improve its **position** from a **risk-sharing** perspective in accordance with its preference relation $\geq_i$, but the market value of its portfolio will not change (increase). Condition (2.19) states that each company's final portfolio, after reinsurance treaties (at time zero), is optimal **according** to the company's preferences. **Condition** (2.20) follows since the I **companies are assumed** to exchange parts of their portfolios only among themselves.

Proposition 2.1.

Under Assumption 2.1, $\mathcal{E}$ **has** an **(ADB)-equilibrium** with a **Pareto** optimal **allocation,** where the premium functional is

$$\pi(y) = E(\xi(T)\, y(T)) = E^*(y(T)).$$

Here $\xi(T)$ is given in **(2.10)**, P^* in (2.14) and E^* is the expectation with respect to P^*.

Proof

First we **demonstrate** that the **economy** has a ***quasiequilibrium***, defined by the existence **of** a $\pi \in X_+^*$ (the dual space of X_+), $\pi \neq 0$, satisfying $\pi(x_i) = \pi(y_i)$ and $\pi(v) \geq \pi(y_i)$ whenever $v \geq_i y_i$, $i \in \mathcal{I}$. A quasiequilibrium is an equilibrium if $v \geq_i y_i$ implies $\pi(v) > \pi(y_i)$ for all i. The latter property holds at a **quasiequilibrium if** $\pi(x_i) > 0$ for all i.

To this end **define neighbourhoods** V_i, $i \in \mathcal{I}$, of zero as in one of the equivalent **definitions** of **uniform** properness. Put $V = \bigcap_{i \in \mathcal{I}} V_i$ and let $\Gamma \subset X$ be the open, convex cone spanned by $(x) + V$.

For any given weakly **optimal** allocation y_i, $i \in \mathcal{I}$ define $Z = \{\sum_{i=1}^{I}(v - y_i) : v \geq_i y_i\} \subset X_+$.

Then **because** of convexity of preferences the open set $Z + \Gamma$ is **convex.** Also because of uniform properness, $0 \notin Z + \Gamma$, because $Z \cap (-\Gamma) = \emptyset$. Since X is a topological vector lattice by Mas-Colell (1986), Proposition 3, there exists a $\pi \in X^*$ that **supports** $(y_1, y_2, \ldots y_I)$, **i.e.** there exists a **quasiequilibrium.** Since X is a **Hilbert** space, $X = X^*$ and π can be represented as $\pi(v) = \langle\langle \xi, v \rangle\rangle$, for some $\xi \in X$. **Thus** $\pi(v) = E(\langle \xi, v \rangle_T)$. Now by

Huang (1985) there exists an equivalent martingale measure P* **representing** π, and **this** construction is as carried out in (2.10) - (2.14). We now show that the **resulting** price functional **supports** a weakly optimal y_i, $i \in \mathcal{I}$. By construction $E^*(x(T)) = E^*(x(0)) = x(0) > 0$ from Assumption 2.1 and (2.17). Also, $[x,y]_t$ is ***invariant*** under a ***substitution*** of an equivalent probability measure, as is the **semimartingale** property, so that **uniform properness** does not change **under** P*. It follows that $E^*(z(T)) > 0$ for $z \in Z + \Gamma$. Since Γ, is a cone and $0 \in Z$, this implies that $E^*(z(T)) \geq 0$ for all $z \in \Gamma$. Thus if $z \geq_i y_i$, $i \in \mathcal{I}$, then $z - y_i \in Z$ and so $E^*(z(T) - y_i(T)) \geq 0$ or $E^*(z(T)) \geq E^*(y_i(T))$. We conclude that π as given in (2.16) supports y_i and the **economy** $\mathcal{E}$ has a **quasiequilibrium.**

Second, we **demonstrate** that it is also an equilibrium. Since we have shown that $E^*(\sum_{i=1}^{I} x_i(T)) > 0$, then $E^*(x_j(T)) > 0$ for some company **j**. **Suppose,** f a some **nonzero** $v \in X_+$, that $E^*(v(T)) = 0$. **By** strict **monotonicity of preferences,** $z = v + x_j >_j x_j$ and $E^*(z(T)) = E^*(x_j(T))$. Then, for some $\varepsilon \in (0,1)$, by continuity of preferences, we would have $\varepsilon z \geq_j x_j$ and $E^*(\varepsilon z(T)) = \varepsilon E^*(z(T)) < E^*(x_j(T))$. **This** contradicts the definition of a **quasiequilibrium,** so $E^*(x_i(T)) > 0$ for **all i,** and the **quasiequilibrium** is in fact **also an equilibrium. Finally,** the usual **convexity** and **continuity conditions** ensuring Pareto **optimality** for a **Walrasian** allocation have been assumed (see, for example Duffie **(1986a)).**

Remarks

-**The** risk premium is seen to equal $E(\xi_T y_T) - E(y_T) = E((\xi_T - 1)y_T)$. In accordance with usual ***actuarial*** terminology we may say that the premium of the risk $y \in X$ **equals**

$$E(y(T)) + E((\xi_T - 1)y_T), \tag{2.21}$$

i.e. the **actuarially** fair premium plus a term compensating for risk beariig.

-**If** a company has an infinite number of identically distributed, stochastically independent risks, then by the strong law of large numbers the risk premium can be set equal to zero, (as for example in life insurance). The other interesting case where the risk premium is zero is under risk neutrality. The **P*-economy** can be thought of as a world in which the participants are risk neutral, since here the risk premium equals $E^*((1-1)y_T) = 0$. **Thus** we *can* compute premiums in the original economy, by the transformation to an economy with risk neutral agents, but with ***altered probability assessments*** **P*** instead of **P**. Since the change of measure given in (2.14) can be carried out explicitly for a wide range of models, we can use Proposition 2.1 to compute premiums for a large class of risks.

So far we have not utilized the increasing information flow $\mathcal{F}_t$, nor have we discussed the construction of the dynamic strategies producing the optimal portfolios $y_i(t)$, $i \in \mathcal{I}$, $t \in T$. In **order** for the theory to be useful, there must exist some strategic reinsurance treaty for each company, such that the companies can adjust their portfolios in accordance with preferences as uncertainty is resolved bit by bit as time progresses. In Aase

(1988b) it is shown how the "static", infinite dimensional problem can be reduced to a finite dimensional one, but with an explicit dynamic description.

3. THE COMPUTATION OF PREMIUMS IN INSURANCE

3.1. At this point we want to illustrate some of the ideas contained in the preceding section. We start by using the principles to compute premiums of typical insurance risks. It is then natural to consider an economy where the risks are marked point processes. Here we want to illustrate what conditions should be met for a risk to be insurable, and we want to find explicitly the shadow price process ξ.

3.3. Consider an economy where a portfolio $y(t)$ satisfies the following dynamic equation

$$(3.1)\qquad y(t) = y(0) + \int_0^t \int_R y\, \nu(ds,dy),$$

where $\nu = \nu(\omega,.,.)$ is a *random measure*. The intended interpretation is that at random *times* $\tau_1(\omega), \tau_2(\omega), \ldots$ "events" happen, and a sequence of *marks* $y(\tau_1), y(\tau_2), \ldots$ are r e a l i i We assume that there exists a $(P,\mathcal{F}_t)$-predictable *intensity kernel*

$$(3.2)\qquad \lambda^y(t;dy) = \lambda_t^y\, F_t^y(dy)$$

where $\lambda_t^y = \lambda_t^y(\omega)$ is a nonnegative $\mathcal{F}_t$-predictable process and $F_t^y(dy)$ is, for each t, a probability distribution in **R** (or more precisely, a probability transition kernel from $(\Omega \times [0,T], \mathcal{F} @+)$ into $(R,\mathcal{B})$, where $\mathcal{B}_+$ and $\mathcal{B}$ are the Borel σ-fields on $[0,T]$ and **R** respectively). The following two relations hold

$$(3.3)\qquad E\{\int_0^T \int_R y\, \nu(dt,dy)\} = E\{\int_0^T \int_R y\, \lambda^y(t)F_t^y(dy)dt\}$$

and

$$(3.4)\qquad \int_0^t \int_R y\, \nu(ds,dy) = \sum_{n=1}^{\infty} y(\tau_n)1(\tau_n \le t) = \sum_{n=1}^{N(t)} y(\tau_n)$$

where $1(A)$ is the indicator of the set $A \subset R$, and the counting process $N(t)$ = number of "events" by time $t \in T$.

Define $\tilde{\nu}(dt,dy) = \nu(dt,dy) - \lambda^y(t)F_t^y(dy)$.

Then, if

$$(3.5) \quad E\{\int_0^T \int_R y^2 \lambda_t^y F_t^y(dy)dt\} < \infty,$$

we have that

$$(3.6) \quad m(t) = \int_0^t \int_R y\, \tilde{\nu}(ds,dy)$$

is a $(P,\mathcal{F}_t)$-square integrable martingale.

3.3 Dynamic completeness: Suppose the history $\mathcal{F}_t$ is the one generated by the process y(t) itself, i.e.

$$(3.7) \quad \mathcal{F}_t = \mathcal{F}_0 \vee \mathcal{F}_t^y \quad , \quad t \in T.$$

In this case it can be shown that any square integrable martingale $m(t) \in \mathcal{M}^2$ has a representation.

$$(3.8) \quad m(t) = m(0) + \int_0^t \int_R \theta(s,y)\, \tilde{\nu}(ds,dy)$$

for some unique R-indexed $\mathcal{F}_t$-predictable process $\theta(t,y)$ such that

$$(3.9) \quad E[\int_0^T \int_R |\theta(t,y)|^2 \lambda^y(t) F_t^y(dy)] < \infty.$$

(see e.g. Bremaud (1981)). When this property holds, the model is said to be dynamically complete.

3.4. Suppose the portfolio y(t) of an insurance company can be written as

$$(3.10) \quad y(t) = x(t) - z(t)$$

where x(t) represents the assets and z(t) the liabilities of the company at time t. Suppose that the liabilities satisfy a dynamic equation of the type (3.1) a (3.4), so that z(t) is a random, marked point process. Alternatively, the liabilities at the **expiration** time could be a function $\psi(z(T))$ of $z(T)$, where ψ **determines** an insurance contract. **The** interpretation is that if a claim occurs against the company at **time** τ_n, then the size of the claim **equals** $z(\tau_n) = z_n$.

The process **x(t)** mainly represents income from premiums. Thus x may be taken to be a predictable process with the following representation.

$$(3.11) \qquad x(t) = x(0) + \int_0^t \int_R x \lambda^x F_s^x(dx) ds .$$

In order for z to be insurable in our simple economic **model,** y must be a **(P^*,$\mathcal{F}_t$)-martingale.** The **conditions** for this to be **the** case are:

There must exist a non-negative $\mathcal{F}_t$-predictable **process** $\mu(t)$ and an $\mathcal{F}_t$**-predictable,** R-indexed non-negative process **f(t,y)** such that

$$(3.12) \qquad F_t^x(dx) = f(t,x)\, F_t^z(dx)$$

i.e. such that

$$(3.13) \qquad \int_R f(t,x)\, F_t^z(dx) = 1 \qquad P \text{ a.s. } t \in T$$

and

$$(3.14) \qquad \lambda^x(t,\omega) = \mu(t,\omega)\, \lambda^z(t,\omega)$$

such that

$$(3.15) \qquad \int_0^T \mu(s)\, \lambda^z(s) ds < \infty \qquad P \text{ a.s.} .$$

By **defining** the new probability measure P^* as $dP^*/dP = \xi_T$, where

$$(3.16) \qquad \xi_T = \prod_{n \geq 1} \mu(\tau_n) f(\tau_n, z_n) 1(\tau_n \leq T) \exp\{\int_0^T \int_R (1-\mu_t f(t,z)) \lambda^z(t) F_t^z(dz) dt\}$$

$y(t)$ now becomes a $(P^*,\mathcal{F}_t)$-martingale, and the premium of the contract $\psi(z)$ equals

$$(3.17) \qquad \pi(\psi(z)) = E(\xi_T\, \psi(z(T))) = E^*(\psi(z(T))).$$

35. As an example, let $z(t)$ be a time-homogeneous compound Poisson process. In the insurance literature this is known as the classical Lundberg risk model. In this case $\lambda^z(t) = \lambda^z$, $F_t^z(dz)$ } $F^z(dz)$ for all $t \in T$, where $\lambda^z > 0$ is a constant and where $F^z(z)$ is a cumulative probability distribution function. In this case (3.12)-(3.15) mean that there exist a constant $\mu > 0$ and a function $f(z)$ such that $\lambda^x = \mu\lambda^z$ and $f(z) = F^x(dz)/F^z(dz)$, where x refers to the corresponding characteristics for the assets process. Here the shadow price ξ takes on the form.

$$\xi_T = \prod_{n\geq 1} \mu f(z_n) 1(\tau_n \leq T) \exp\{\int_0^T \int_R (1-\mu f(z))\, \lambda F^z(dz)dt\},$$

which can be reduced to

$$(3.18) \qquad \xi_T = \mu^{N(T)} \prod_{n=1}^{N(t)} f(z_n)\, e^{\lambda^z(1-\mu)T}$$

Here we have used the independence of the jump sizes $z_1, z_2, \ldots$, the exponential distribution of $\tau_1, \tau_2, \ldots$ and the fact that $N(t)$ is here a Poisson process. Suppose we want to compute the premium of a stop loss contract $\psi(z(T)) = (z(T)-d)^+ = \max\{(z(T)-d),0)$ where d is the deductible. Using (3.18) we get

$$\pi = E^*((z(T)-d)^+) = E^*\{E^*\{(z(T)-d)^+ | N(T)\}\}, \text{ or}$$

$$(3.19) \qquad \pi(\psi(z)) = \sum_{n=0}^{\infty} \frac{e^{-\mu\lambda^z T}(\mu\lambda^z T)^n}{n!} \int_d^{\infty} (z-d)(fF^z)^{n*}(dz)$$

where $(fF^z)^{n*}$ is the n-th convolution of $(fF^z)(dz)$ with itself.

Alternatively we can compute this premium using
and the expression for ξ_T given in (3.18): Taking double expectations under the measure P we get

$$\pi(\psi(z)) = E\{E\{\xi_T(z(T)-d)^+ | N(T)\}\} =$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda^z T)^n e^{-\lambda^z T}}{n!} \mu^n E\Big[\prod_{k=1}^{n} f(z_k)\Big(\sum_{k=1}^{n} z_k - d\Big)^+ \Big| N(T) = n\Big] \cdot e^{\lambda^z(1-\mu)T}$$

$$= \sum_{n=0}^{\infty} \frac{e^{-\mu\lambda^z T}(\mu\lambda^z T)^n}{n!} \int_R \cdots \int_R \prod_{k=1}^{n} f(z_k)\Big(\sum_{k=1}^{n} z_k - d\Big)^+ \prod_{k=1}^{n} F^z(dz_k)$$

$$= \sum_{n=0}^{\infty} \frac{e^{-\mu\lambda^z T}(\mu\lambda^z T)^n}{n!} \int_d^{\infty} (z-d)(fF^z)^{n*}(dz)$$

which equals (3.19): From this example we notice that the premium of a risk typically depends on:

(i) The stochastic properties of the risk itself, here represented by the local characteristics

(ii) The stochastic relationship between the particular risk and *mother* economic quantity, here represented by the assets process or the total portfolio of the company. In more complicated applications this quantity would be a market index, such as in Section 2. In the above example this stochastic relationship is represented by the term μ and the function $f(z)$.

(iii) The market's attitude towards risk represented by the market's marginal utility process, or the shadow price process $\xi(t, \omega)$, again determined by μ and $f(z)$.

3.6. Notice that only in the case where $\mu(t, \omega) \equiv 1$ and $f(t, z, \omega) \equiv 1$, the premium π equals the "actuarially fair" value, since in this case $y(t)$ is a $(P,\mathcal{F}_t)$-martingale and $P=P^*$. In this simple economy this means that the insurance company's assets process $x(t)$ can probabilistically "match" the liabilities process $z(t)$. On the other hand, if there does not exist processes μ and f satisfying (3.12)-(3.15) the risk $\psi(z)$ is uninsurable, since it cannot be given a premium in market equilibrium. The insurance company cannot establish an income process $x(t)$, such that the claims process $z(t)$ *can* be matched to $x(t)$ after an absolutely continuous change of probability measure. The "economic difference" between z and x may here be thought of as measured by ξ, and if ξ does not exist, z cannot be matched to the P*--economy, so there does not exist a market price for any risk $\psi(z)$.

3.7. Actually, the original line of attack in comparing two probability measures P and P* has been through a metric as follows : The ***Kakutani-Hellinger distance*** $\rho(P,P^*)$ between P and P* is the nonnegative number whose square is

$$\rho^2(P,P^*) = \frac{1}{2}\int_\Omega \Big(\sqrt{\frac{dP}{dQ}} - \sqrt{\frac{dP^*}{dQ}}\Big)^2 dQ,$$

where $Q = \frac{1}{2}(P+P^*)$. It can be shown that ρ is a metric on the set of all probability measures on $(\Omega,\mathcal{F})$, which does not depend on the measure Q. This motivates the following notation

$$\rho^2(P,P^*) = \frac{1}{2}\int_{\Omega}(\sqrt{dP} - \sqrt{dP^*})^2.$$

If the risk premium is large, it seems reasonable to conjecture that the Kakutani-Hellinger distance between P and P^* is also large. Now, "predictable criteria for absolute continuity" can be expressed through the Hellinger metric. The results for the present stochastic economy have been presented in (3.12)-(3.16) above. The "Kakutani alternative" says that either $P \sim P^*$ or $P \perp P^*$ (P and P^* are mutually singular). In the latter case $\rho^2 = 1$, its maximal value. In this case the risk is uninsurable in our terminology. We do not elaborate further the connection between risk premiums and the Kakutani-Hellinger distance at this point.

3.8. It may finally be noticed that a similar approach as utilized above can also be employed in the pricing of options, contingent claims and other financial assets. For details, see Aase (1988), with further references.

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