Single-Factor Immunizing Duration of an Interest Rate Swap

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Summary

The present paper derives general results for the (immunizing) duration of (default-free) generic interest rate swaps within the framework of single-factor duration models (SFDM). Results on the duration of floating-rate notes (FRNs) are obtained as a special case.

The problem of determining the duration of an interest rate swap has been analyzed by GOODMAN (1991) for the special case of a flat term structure of interest rates, however, these results are flawed.

Durée d'immunisation à un seul facteur pour un swap de taux d'intérêt

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Résumé

Le présent exposé dérive des résultats d'ordre général pour la durée (d'immunisation) de swaps de taux d'intérêt génériques (sans défaut) dans le cadre de modèles de durée à un seul facteur (SFDM). Des résultats sur la durée d'obligations à taux variable (FRN) sont obtenus sous forme de cas particulier.

Le problème de détermination de la durée d'un swap de taux d'intérêt a été analysé par GOODMAN (1991) pour un cas particulier d'une structure inchangée des taux d'intérêt, mais ces résultats présentaient des erreurs.

1. Introduction

The concept of duration plays a central role in the measurement of interest rate risk, whereas interest rate swaps are an important tool in interest rate risk management. Therefore, the calculation of the duration of an interest rate swap seems to be of some interest, first results are contained in GOODMAN (1991) for the case of a flat yield curve.

In the present paper we derive the (immunizing) duration of an interest rate swap within the framework of traditional single-factor duration models (SFDM), cf. BIERWAG (1987, pp. 312-314). In SFDMs, the change of deterministic forms of the term structure of interest rates depends on the change of a one-dimensional parameter.

Generic interest rate swaps consist of an exchange of fixed for floating interest payments and therefore their value can be looked at as the difference between the values of a fixed-rate bond and a floating-rate bond. Thus, the problem of determining the duration of a floating-rate note (FRN) is closely related to the problem considered in this paper.

Our results correct results of GOODMAN (1987) in the case of interest rate swaps and generalize results of CHANCE (1983) and MORGAN (1986) in the case of FRNs. The generalization of our derivations to multiple factor duration models (MFDM), where the change of deterministic forms of the term structure depends on the change of a multi-dimensional parameter, is straightforward. However, an analysis within the framework of stochastic models of the term structure of interest rates, as done for FRNs by RAMASWAMY/SUNDARESAN (1986) and SUNDARESAN (1991), is beyond the scope of this paper. The interest rate swaps considered are always assumed to be default-free.

2. Modelling the Term Structure of Interest Rates

For the ease of exposition, we are working with a time continuous model of interest rates. Let i(s,u) denote the *instantaneous interest rate* at time u as observed at time s, which implies that one unit of money at time s will increase in value up to time $t \geq s$ to the amount

$$q(s,t) = \exp\left(\int_s^t i(s,u)\, du \right), \tag{1}$$

which determines the accumulation factors within the given interest rate model. The value, b(s,t), at time s of a pure discount bond maturing at time $t \geq s$ with unit maturity value is given by

$$b(s,t) = \exp\left(-\int_s^t i(s,u)\, du \right), \tag{2}$$

which determines the *discounting factors* within the given interest rate model. The *term structure of interest rates*, r(s,t), at time s is given by the internal rate of return of this discount bond, that is,

$$r(s,t) = -\frac{1}{t-s}\ln b(s,t) = \frac{1}{t-s}\int_s^t i(s,u)\,du \quad . \tag{3}$$

Comparing (1) with (2) and (1) with (3) we obtain (1') and (2') respectively:

$$q(s,t) = \exp[(t-s)r(s,t)] \tag{1'}$$

$$b(s,t) = \exp[-(t-s)r(s,t)] \quad . \tag{2'}$$

Finally, let f(s,t,v) denote ($v > t \geq s$) the *implied forward accumulation factors* at time s with respect to the given term stucture, i.e. the accumulation of one unit of money given at time t up to the time v under the interest rate process as given at time s. We have

$$\begin{aligned} f(s,t,v) &= \exp\left(\int_t^v i(s,u)\,du\right) \\ &= \exp\left(\int_s^v i(s,u)\,du - \int_s^t i(s,u)\,du\right) \\ &= \exp\left[(v-s)r(s,v) - (t-s)r(s,t)\right] \quad . \end{aligned} \tag{4}$$

Sometimes one wants to transform results derived within this continuous time model to results expressed within a discrete time interest rate model or vice versa. This can be done as follows. Let r*(s,t) denote the current interest rate at time s of a discrete time interest rate model, then the corresponding accumulation factor is given by

$$q(s,t) = [1+r^*(s,t)]^{t-s} \quad , \tag{5}$$

which implies

$$r^*(s,t) = \exp r(s,t) - 1 \tag{6}$$

as well as

$$r(s,t) = \ln[1+r^*(s,t)] \quad . \tag{6'}$$

Example 1 : Flat term structure

A flat term structure at time s is defined within the discrete time interest rate model by

$$r^*(s,t) \equiv r_s \quad \text{for all } t \geq s \quad , \tag{7}$$

which implies for the continuous time model

$$r(s,t) \equiv \ln(1+r_s) \quad \blacksquare \tag{7'}$$

3. The General Structure of an Interest Rate Swap

A (generic) interest rate swap consists of the exchange of floating-rate interest payments for fixed-rate payments. The underlying debt instrument, however, is *not* being exchanged. We assume in the following that the two underlying debt instuments have an identical notional amount (principal), N, and that the exchange payments take place at times t_0 (:=0) < t_1 < ... < t_n. The notional amount N is not exchanged. However, in order to obtain duration measures being interpretable for the fixed and the floating rate part of the swap, we assume a fictitious exchange of N at time t_n, leaving the net position unchanged. The participant in the swap *paying* fixed interest rates, let us say of amount $Z_F(t_j)$, and receiving floating interest rates, let us say of amount $Z_V(t_i)$, will be called *fixed-rate payer*, the counter-party, *floating-rate payer*. Now we are able to illustrate the general structure of a generic interest rate swap graphically as it is done in Figure 1.

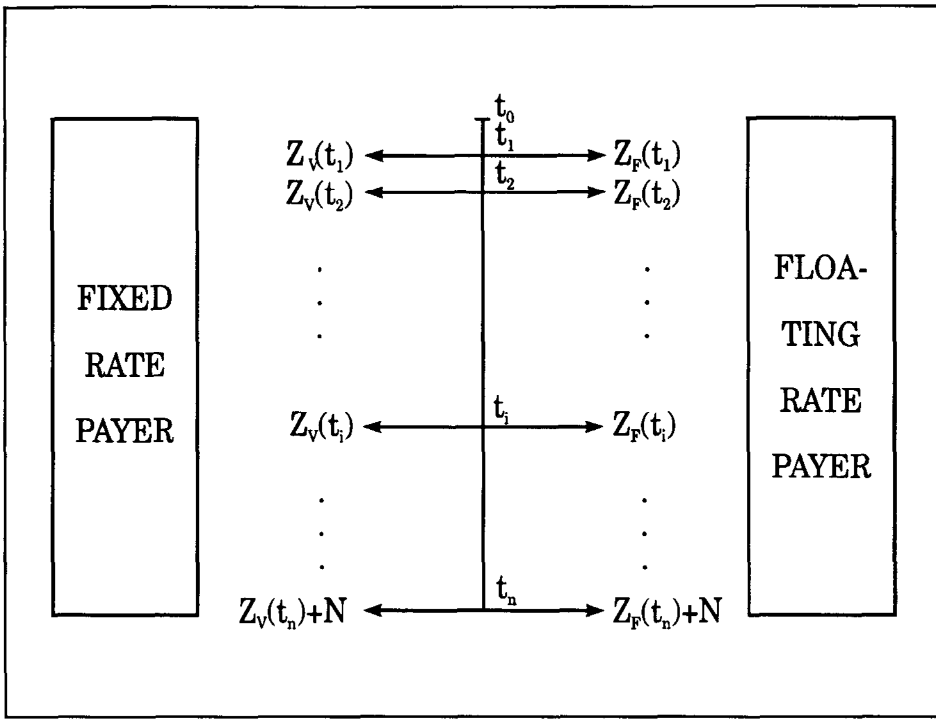

Figure 1: The general structure of an interest rate swap

In the following, we always assume N = 1 as well as equidistant times of payment, i.e.

$$t_i = t_{i-1} + h = ih \quad i = 1, \ldots, n \quad . \tag{8}$$

We will allow $Z_F(t_i) \equiv 0$ and $Z_V(t_i) \equiv 0$, which means that it is not necessary at all points of time for *both* fixed and floaating rate payments to be made.

Taking the point of view of the fixed-rate payer and defining

$$Z_S(t_i) := Z_V(t_i) - Z_F(t_i) \quad i=1,\ldots,n \quad , \tag{9}$$

then the interest rate swap is completely characterized by the following stream of payments

$$\{Z_S(t_1), Z_S(t_2), \ldots, Z_S(t_n)\} \quad . \tag{10}$$

This payment stream is a *stochastic* one as the amounts of the floating-rate payments are normally unknown at t_0, their realizations depending on the development of the term structure of interest rates.

Let at time $t_0 = 0$ the existing (time-continuous) term structure be given by $r(t) := r(0,t)$. That means the present value P_S of the payment stream (10) is given by

$$P_S = \sum_{i=1}^{n} Z_S(t_i) e^{-t_i r(t_i)} \quad . \tag{11}$$

In a similar manner, we define the present values P_F and P_V of the payment streams $\{Z_F(t_1), \ldots, Z_F(t_n)+N\}$ and $\{Z(t_1), \ldots, Z_F(t_n)+N\}$ respectively. Clearly, that means that we have

$$P_S = P_V - P_F \quad . \tag{12}$$

In addition, we will use the term structure $r(\tau,t)$ as existing at a point of time $t_0 < \tau < t_1$ and will analyze the present value $P_S(\tau)$ of payment stream (10) at time τ which is given by

$$P_S(\tau) = \sum_{i=1}^{n} Z_S(t_i)\exp\left[-\int_{\tau}^{t_i} i(\tau,u)\,du\right] \tag{13}$$
$$= \sum_{i=1}^{n} Z_S(t_i)\, e^{-(t_i-\tau)r(\tau,t_i-\tau)} \quad .$$

Let us now take a closer look at the structure of the payment streams $\{Z_F(t_i)\}$ and $\{Z_V(t_i)\}$. The former is based on fixed interest payments. Let i_0 denote the corresponding coupon rate defined for a time period of length h. Then, assuming N=1 we have

$$Z_F(t_i) = i_0 \quad \text{for all } i=1,\dots,n \quad . \tag{14}$$

Additionally, we have assumed here that fixed interest rate payments are made at every point of time t_i.

We now have to pay attention to a convention in the swap markets[1]. Usually the quotation of the entire swap transaction is given as a constant increment (*swap spread*) over the yield to maturity of the appropriate fixed-rate bond, e. g. the yield of the most recently issued Treasury bonds with maturity t_n ("spread over treasury"). Within our model, this can be formalized by using an *effective* fixed swap rate, which means that $Z_F(t_i)$ has the form

1) For the conventions in the swap markets see for example BROWN/SMITH (1991).

$$Z_F(t_i) = i_0 + M \quad , \; i = 1,..,n \; , \tag{15}$$

where M is a constant margin which specifies the swap spread.

Now let us turn to the floating-rate part of the swap. As we have seen, the floating interest rate payments are of stochastic nature. To be able to resolve this uncertainty and to value the interest rate swap one usually assumes that all forward accumulation factors, f(s,t,v), implied by the term structure, r(s,t), at time s are identical to the realized future accumulation factors, q(t,v). This is the standard procedure for the valuation of floating-rate notes (FRN) and the floating-rate part of a swap, cf. CHANCE (1983), FINNERTY (1989, especially pp. 90-91), MORGAN (1986) and BECKSTROM (1990). On that basis for resolving the interest rate uncertainty problem, it is now possible to proceed as in the case of *deterministic* payment streams when valuing floating-rate notes and interest rate swaps.

Finally, we have to pay attention to another convention in the swap markets concerning the floating-rate schedule. This schedule depends on an index, usually LIBOR for floating-rate interest rate payments. It is now usual convention[2] to fix LIBOR one settlement date *ahead* of the payment date. That is, the floating-rate payment at point of time t_i is identical to the LIBOR (with maturity h) that prevailed at time t_{i-1}. Within the framework of our model, that convention can be incorporated as follows:

2) See BROWN/SMITH (1991, p. 71).

$$\begin{aligned} Z_V(t_i) &= f(0,t_{i-1},t_i) - 1 \\ &= \exp\left[t_i r(0,t_i) - t_{i-1} r(0,t_{i-1})\right] - 1 \ . \end{aligned} \tag{16}$$

Now, both the fixed-rate and floating-rate parts of an interest rate swap are specified. This allows us to state the following conclusion about the margin M: M can be considered fair, if the present values of the fixed and the floating-rate payment stream at initiation date are identical. Taking the point of view of the fixed-rate payer, that means the present value (11) must equal zero, i.e.

$$\sum_{i=1}^{n} [f(0,t_{i-1},t_i) - 1 - (i_0 + M)]\, e^{-t_i r(t_i)} = 0 \ . \tag{17}$$

For the sake of simplicity, we have assumed that at each payment date there are fixed- and floating-rate payments. Solving for M yields

$$M = \frac{\sum_{i=1}^{n} [f(0,t_{i-1},t_i) - (1 + i_0)]\, e^{-t_i r(t_i)}}{\sum_{i=1}^{n} e^{-t_i r(t_i)}} \ . \tag{18}$$

In reality, the margin will be unfair as a rule, letting the present value P_S differ from zero. In addition, even a margin which is fair at the outset won't be fair after the first change of the term structure of interest rates.

If we want to analyze the interest rate swap not at initiation date but at a point of time $0 < \tau < t_1$, we have to modify our approach. The floating-rate payment due at t_1 is already fixed at $t=0$ and it shall be denoted F_0. The following floating-rate payments must be modeled according to the term structure of interest rates prevailing at τ. Thus, for the analysis of an interest rate swap at the point of time $0 < \tau < t_1$, the structure of the floating-rate payments may be characterised by

$$\begin{aligned} Z_V(t_1) &= F_0 \\ Z_V(t_i) &= f(\tau, t_{i-1}, t_i) - 1 \quad i = 2, \ldots, n \ . \end{aligned} \tag{19}$$

The present value $P_V(\tau)$ of the floating-rate part of an interest rate swap is found to be

$$\begin{aligned} P_V(\tau) &= F_0 e^{-(t_1-\tau)r(\tau,t_1-\tau)} \\ &+ \sum_{i=2}^{n} [e^{(t_i-\tau)r(\tau,t_i-\tau)-(t_{i-1}-\tau)r(\tau,t_{i-1}-\tau)} - 1] e^{-(t_i-\tau)r(\tau,t_i-\tau)} \\ &+ e^{-(t_n-\tau)r(\tau,t_n-\tau)} \\ &= F_0 e^{-(t_1-\tau)r(\tau,t_1-\tau)} \\ &+ \sum_{i=2}^{n} [e^{-(t_{i-1}-\tau)r(\tau,t_{i-1}-\tau)} - e^{-(t_i-\tau)r(\tau,t_i-\tau)}] \\ &+ e^{-(t_n-\tau)r(\tau,t_n-\tau)} \ . \end{aligned}$$

This yields (remember $t_i = h\, i$)

$$P_V(\tau) = (F_0 + 1) e^{-(h-\tau)r(\tau,h-\tau)} \ . \tag{20}$$

4. Derivation of Duration Measures in Single-factor Duration Models

In the following, we will consider the payment stream $\{Z(t_1), \ldots, Z(t_n)\}$ and analyze the consequences of a change in the term structure of interest rates at time τ ($0 \leq \tau < t_1$). Let $r(\tau,t) := r(\tau,t,x_0)$ denote the initial term structure prevailing in time τ and $r(\tau,t,x)$ denote the term structure observed after an instant shift in τ. The change of the one-dimensional real parameter, x, is assumed to describe the instant interest rate "shock". This way of modelling interest rate risk in SFDMs follows BIERWAG (1987, pp. 312 - 314). The general approach to derive duration measures, D, for different interest rate shocks is based on the immunizing property of these duration measures. The different duration definitions must satisfy the condition that in time $s = D$ ($> \tau$), the value of the payment stream, $\{Z(t_1), \ldots, Z(t_n)\}$, is immunized against the initial interest rate shock in τ. In order to state this condition more precisely, let us first define the present value of our payment stream in time τ corresponding to the term structure, $r(\tau,t,x)$, as

$$P(\tau,x) = \sum_{i=1}^{n} Z(t_i)e^{-(t_i-\tau)r(\tau,t_i-\tau,x)} \quad . \tag{21}$$

Assuming the term structure, $r(\tau,t,x)$, the capital value, $K(s,x)$, of our payment stream in any time $s > \tau$ is

$$K(s,x) = e^{(s-\tau)r(\tau,s-\tau,x)}P(\tau,x) \quad . \tag{22}$$

Immunization of the value in time s is achieved if the following condition is satisfied:

$$K(s,x) \geq K(s,x_0) \quad \text{for all } x \ . \tag{23}$$

That is, regardless of direction and amount of the interest rate shock, the value of the payment stream in time s at least equals the value which would have been achieved had the initial term stucture prevailed until s. The function of the capital value in s has a global minimum for the initial term structure. The duration (more precisely: the immunizing duration) of the payment stream corresponding to the initial term structure and to the interest rate shock assumed is to be defined in such a way that D equals time s fulfilling the condition (23). With $K'(s,x) := \partial K(s,x)/\partial x$ we have:

$$K'(D,x) = 0 \ . \tag{24}$$

The statement in (24) contains only the necessary condition for the duration, but since the function (22) is convex as a rule [for special cases of payment streams, where this needs not be the case, cf. BIERWAG (1987, p. 113 f.)], this condition normally is sufficient. Let us furthermore define $r'(\tau,t,x) := \partial r(\tau,t,x)/\partial x$ and $P'(\tau,x) := \partial P(\tau,x)/\partial x$ as well as $r(\tau,t) := r(\tau,t,x_0)$ and $P(\tau) := P(\tau,x_0)$. Then from (22), it follows that

$$K'(s,x_0) = (s-\tau)\,r'(\tau,s-\tau,x_0)\,e^{(s-\tau)r(\tau,s-\tau)}P(\tau) + e^{(s-\tau)r(\tau,s-\tau)}P'(\tau,x_0) \ .$$

Thus conditition (24) can be restated as

$$(D-\tau)\,r'(\tau,D-\tau,x_0) = -\frac{P'(\tau,x_o)}{P(\tau)} \ , \tag{25}$$

with

$$P'(\tau,x_0) = -\sum_{i=1}^{n}(t_i-\tau)Z(t_i)e^{-(t_i-\tau)r(\tau,t_i-\tau)}r'(\tau,t_i-\tau,x_0) \ . \tag{26}$$

The condition in (25) is the suitable general definition of duration in a SFDM. Note however, that this duration measure depends on the initial term structure prevailing in time τ but not on τ itself, because D is defined as the *point of time* where immunization is achieved. The (remaining) duration D*, i.e. the *period of time* from τ to D is

$$D^*(\tau) = D - \tau \quad . \tag{27}$$

Since the duration measure $D^*(\tau)$ seems better suited to reflect the dynamics of the situation, we will concentrate in the following on this measure, the general definition of which is as follows:

$$D^*(\tau) r'(\tau, D^*(\tau), x_0) = - \frac{P'(\tau, x_0)}{P(\tau)} \quad . \tag{28}$$

If $r'(\tau,t,x_0) = g(\tau)$, i.e. independent from t, we can obtain an explicit solution for the (remaining) duration D*:

$$D^*(\tau) = - \frac{P'(\tau, x_0)}{g(\tau) P(\tau)} \quad . \tag{29}$$

With (26), this can be put in concrete terms as follows:

$$D^*(\tau) = \frac{1}{P(\tau)} \sum_{i=1}^{n} (t_i - \tau) Z(t_i) e^{-(t_i - \tau) r(\tau, t_i - \tau)} \quad . \tag{30}$$

Example 2: Macaulay-Duration

Let us return to the case of a flat term structure of interest rates as in example 1. We assume that the prevailing term structure in τ is flat, i.e., $r^*(\tau,t) \equiv r_\tau$ or $r(\tau,t) \equiv \ln(1+r_\tau)$, and that this term structure is affected by an instant shock in τ, which in the discrete time model is additive, i.e. $r_\tau \rightarrow$

$r_\tau + x$. In the framework of our general analysis we have (with $x_0 = 0$)

$$r(\tau,t,x) = \ln(1 + r_\tau + x) \quad , \tag{31}$$

leading to $r'(\tau,t,x_0) = 1/\ (1+r_\tau)$. Thus the condition for (30) being valid is met and we have

$$D^*(\tau) = \frac{\sum_{i=1}^{n} (t_i - \tau) Z(t_i)(1 + r_\tau)^{-(t_i - \tau)}}{\sum_{i=1}^{n} Z(t_i)(1 + r_\tau)^{-(t_i - \tau)}}$$

and thus

$$D^*(\tau) = \frac{1}{P(\tau)} \sum_{i=1}^{n} t_i Z(t_i)(1 + r_\tau)^{-(t_i - \tau)} - \tau \quad . \tag{32}$$

Example 3: Fischer/Weil (1971)

FISHER/WEIL (1971) assume, in a continuous time framework, an interest rate process of the type ($x_0 = 0$)

$$r(\tau,t,x) = x + r(\tau,t) \quad , \tag{33}$$

i.e. an additive (in a discrete time framework a multiplicative) shock. It follows that $r'(\tau,t,x_0) \equiv 1$ and therefore from (30),

$$D^*(\tau) = \frac{\sum_{i=1}^{n} (t_i - \tau) Z(t_i) e^{-(t_i-\tau) r(\tau, t_i-\tau)}}{\sum_{i=1}^{n} Z(t_i) e^{-(t_i-\tau) r(\tau, t_i-\tau)}} \tag{34}$$

$$= \frac{1}{P(\tau)} \sum_{i=1}^{n} t_i Z(t_i) e^{-(t_i-\tau) r(\tau, t_i-\tau)} - \tau \quad .$$

Example 4: Khang (1979)

KHANG (1979) considers the following interest rate process ($x_0 = 1$):

$$r(\tau,t,x) = \frac{x \ln(1+\alpha t)}{\alpha t} + r(\tau,t) \quad . \tag{35}$$

This shock was designed to capture the higher fluctuations of short-term rates compared with long-term rates, the parameter, α, controls the ratio of the shifts in the long versus the short range of the term stucture. The statement in (35) yields $r'(\tau,t,x_0) = \ln(1+\alpha t) / \alpha t$, and we have from (28)

$$D^*(\tau) \frac{\ln(1+\alpha D^*(\tau))}{\alpha D^*(\tau)} =$$

$$\frac{1}{P(\tau)} \sum_{i=1}^{n} (t_i - \tau) Z(t_i) e^{-(t_i-\tau) r(\tau, t_i-\tau)} \frac{\ln[1+\alpha(t_i-\tau)]}{\alpha(t_i-\tau)} \quad .$$

This yields

$$\ln[1+\alpha D^*(\tau)] = \frac{1}{P(\tau)} \sum_{i=1}^{n} \ln[1+\alpha(t_i-\tau)]Z(t_i)e^{-(t_i-\tau)r(\tau,t_i-\tau)}, \quad (36)$$

which can be trivially solved for $D^*(\tau)$.

5. The Immunizing Duration of the Floating-Rate Part of a Swap

According to (19), the floating-rate side of an interest rate swap can be represented as a deterministic payment stream. Thus, the results of the last section are directly applicable. From (28), the (remaining) duration D_V^* of the floating-rate part at time τ is

$$D_V^*(\tau)r'(\tau,D_V^*(\tau),x_0) = -\frac{P_V'(\tau,x_o)}{P_V(\tau)} . \quad (37)$$

If $r'(\tau,t,x_0) = g(\tau)$, we can obtain the following explicit solution analogous to (29) for the duration of the floating-rate side of the swap :

$$D_V^*(\tau) = -\frac{P_V'(\tau,x_0)}{g(\tau)P_V(\tau)} . \quad (38)$$

In (20) we have already found a general expression for the present value of the floating-rate part of a swap (including the notional principal N=1). This yields

$$P_V'(\tau) = -(F_0+1)(h-\tau)r'(\tau,h-\tau)e^{-(h-\tau)r(\tau,h-\tau)} \quad (39)$$

and with $t_1 = h$ we obtain from (38):

$$D_V^*(\tau) = t_1 - \tau \quad . \tag{40}$$

In the case where $r'(\tau,t,x_0) = g(\tau)$, we have the intuitive property

$$\lim_{\tau \to t_1} D_V^*(\tau) = 0 \quad , \tag{41}$$

i.e. directly *before* a determination date of the floating rate the duration is zero. The condition $r'(\tau,t,x_0) = g(\tau)$ is - as shown in examples 2 and 3 - satisfied in the case of the Macaulay- and Fisher/Weil-duration. Therefore, our result in (41) can be compared with the calculations of CHANCE (1983) and MORGAN (1986). The comparison shows that Chance's approach is erroneous whereas Morgan's results, as far as they concern a floating rate note of the type considered by Chance, are verified.

Moreover, it can be shown that the property in (40) is common to any single-factor duration model. According to (20), we have

$$-\frac{P_V'(\tau,x_0)}{P_V(\tau)} = (t_1 - \tau) r'(\tau, t_1 - \tau, x_0)$$

and therefore with (37), it follows that

$$D_V^*(\tau) r'(\tau, D_V^*(\tau), x_0) = (t_1 - \tau) r'(\tau, t_1 - \tau, x_0) \quad . \tag{42}$$

Obviously, $D_V^* = t_1 - \tau$ is always a solution of (42). In the case that the function $H(z) := z\, r'(\tau,z,x_0)$ possesses an inverse function $z = H^{-1}$ the solution will be *unique*. A sufficient condition is that $H(z)$ is strictly monotonically increasing or decreasing, i.e. (in the case of differentiability) $H'(z) > 0$ or $H'(z) < 0$. This is equivalent to the conditions $x\, r''(\tau,z,x_0) \,/\, r'(\tau,z,x_0) > -1$ or < -1, which are restrictions on the *elasticity* of the function $r'(\tau,z,x_0)$. In the case of example 4, the model of KHANG (1979), we have $H(z) = \ln(1+\alpha z) / \alpha$ and thus an invertible function.

6. The Immunizing Duration of an Interest Rate Swap

According to section 3, an interest rate swap can be represented as a (deterministic) payment stream $\{Z_S(t_1), \ldots, Z_S(t_n)\}$. Therefore the (remaining) duration D_S^* of a swap at time τ is, from (28), implicitly defined as

$$D_S^*(\tau)\, r'(\tau, D_S^*(\tau), x_0) = -\frac{P_S'(\tau,x_o)}{P_S(\tau)} . \tag{43}$$

In the case when $r'(\tau,t,x_0) = g(\tau)$ we obtain in analogy to (29) the following definition of the duration in an explicit form:

$$D_S^*(\tau) = -\frac{P_S'(\tau,x_o)}{g(\tau)P_S(\tau)} . \tag{44}$$

As we have $P_S(\tau) = P_V(\tau) - P_F(\tau)$ according to (12), it follows from (44) that

$$D_S^*(\tau) = -\left[\frac{P_V(\tau)}{P_S(\tau)}\frac{P_V'(\tau,x_o)}{g(\tau)P_V(\tau)} - \frac{P_F(\tau)}{P_S(\tau)}\frac{P_F'(\tau,x_o)}{g(\tau)P_F(\tau)}\right] . \tag{45}$$

With $D_F^*(\tau) := -P_F'(\tau,x_0) / g(\tau)\, P_F(\tau)$ we have

$$D_S^*(\tau) = \frac{P_V(\tau)D_V^*(\tau) - P_F(\tau)D_V^*(\tau)}{P_S(\tau)} \quad . \tag{46}$$

Expression (46) gives a functional decomposition of the (immunizing) duration of an interest rate swap into the durations of the fixed- and the floating-rate part of the swap. The result in (46), in particular shows that the definition of the (modified) duration of an interest rate swap given by GOODMAN (1991, p. 309), namely $D_S = D_V - D_F$, must be flawed.

The case $r'(\tau,t,x_0) = g(\tau)$ covers, according to examples 2 and 3, in particular the cases of the Macaulay-Duration and the Fisher/Weil-Duration. The duration of the entire swap can be calculated on the basis of (46) in connection with (20), (40) and (32) or (34) respectively. A general decomposition for the general duration definition (43) along the lines of (46) is

$$D_S^* r'(\tau,D_S) = \frac{P_V}{P_S} D_V^* r'(\tau,D_F) - \frac{P_F}{P_S} D_F^* r'(\tau,D_V) \; . \tag{47}$$

In the case of example 4, this can be simplified to

$$\ln(1+\alpha D_S^*) = \frac{P_V}{P_S}\ln(1+\alpha D_V^*) - \frac{P_F}{P_S}\ln(1+\alpha D_F^*)$$

and

$$D_S^* = \left[\left(\frac{(1+\alpha D_V)^{P_V}}{(1+\alpha D_F)^{P_F}}\right)^{\frac{1}{P_S}} - 1\right] / \alpha \quad . \tag{48}$$

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