Examination of a Two-Factor Bond Option Valuation Model

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Abstract

In this paper, we examine a two-factor option pricing model, that could be used within a system to manage the total interest rate position of a financial institution. We deduce the theoretical features that are required for such a model and we discuss issues related to the implementation of the model. In particular, the problem of adapting the model to the current market information is addressed. The results of an empirical study within the market of German interest rate warrants show that the prediction quality of the model is considerable.

Résumé

Dans cet article, nous examinos un modèle à deux facteurs pour l'évaluation d'options, qui peut servir comme une partie d'un système pour contrôler le risque d'intérêts total d'une institution financière. Nous déduisons les qualités théoriques, qui sont nécessaires pour des modèles pareils et nous discutons des problèmes d'emploi. En particulier nous nous occupons de l'adaptation du modèle à la courbe des taux d'intérêts actuelle. Les résultats d'une étude empirique sur le marché allemand montrent la qualité du modèle.

Keywords

Capital markets, interest rate options, interest rate risk.

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I Introduction

Derivative instruments are currently the subject of controversial public and academic discussions. While derivatives are highly praised for their powerful ability to transfer and hedge risk, their complexity offsets this advantage because of the difficulty in precisely controlling risk associated with these instruments. Indeed, as a consequence of the interaction of the interest rate, foreign exchange, equity, and commodity markets, to name but a few of the most important factors, the assessment of market risk in option positions is quite complex. The management of the risk associated with options requires valuation models that provide current and future option values while incorporating a response function that adequately models the most important influencing factors. Whereas broadly accepted valuations models exist for equity and foreign exchange options, the valuation of interest rate options is a much more difficult problem. This is also reflected in the large number of different valuation approaches now available, which are employed simultaneously by market participants.

There are essentially three different approaches to the valuation of bond options. The first and most simplest approach follows closely the approach of Black and Scholes [1973] and uses the price of the underlying bond as an exogenous variable. Because of its easy applicability the model is often used as a benchmark. However, a consistent valuation of different options is not possible within this approach, because the model is based on the evolution of one single bond price rather than the whole term structure of interest rates.

The second, often called traditional approach to option valuation is to develop a model for a few exogenous factors,[1] treating bond price movements as a function of these factors. Derivatives (e.g. options on bonds) are then priced relative to the endogenous term structure of interest rates, with the

[1]In most cases the short rate is used as one of the factors.

assumption that the underlying bonds are correctly priced. Such models will not normally, however, generate a term structure consistent with market data.

The third approach is that of models which fully exploit the information available from the current term structure of interest rates. This approach consists of essentially two different subapproaches. The one pioneered by Ho and Lee [1986] and Heath, Jarrow, and Morton [1990, 1992] starts with the evolution of the entire zero-coupon price curve. The models within this approach are consistent with the current term structure by construction. A second subapproach, building on Black, Derman, and Toy [1990], Hull and White [1990, 1993], Black and Karasinski [1991], Jamshidian [1991], and Uhrig and Walter [1996], specifies the spot rate process and determines the parameters in such a way that the model is consistent with the current term structure.

Despite the variety of competing models for the valuation of interest rate contingent claims, there exist relatively few empirical studies which attempt to assess their empirical quality and suitability for practical use.[2]

This paper seeks to address this gap. Applying the popular traditional model proposed by Longstaff and Schwartz (1992) to the German market, I present the first empirical study that examines this model with respect to its practicality and its empirical quality.

II Important Features for models used as risk management tools

Models being applied within systems to manage the total interest rate position of a financial intermediary should possess at least the following features:

[2]Some of the rare examples are Amin and Morton [1994] or Dietrich-Campbell and Schwartz [1986]

- For the management of interest rate risks, consistent valuation models are needed because the rational control of these risks requires a consistent risk measurement. In order to achieve this goal the stochastic dynamics of the whole term structure of interest rates has to be modelled consistently. Thereby, the main problem is to obtain an arbitrage-free, feasible, and realistic description of how the term structure of interest rates evolves over time.
- In order to come up with realistic option values, the model should reflect the market actuality. Therefore it is necessary to use a model which takes into account the information about the current term structure of interest rates and the current term structure of volatilities.

Because the modified Black/Scholes approach as well as the traditional bond option pricing models (e.g. Vasicek [1977]) do not satisfy these two features, an application of these models within a risk management system is not advisable. In contrast, the highly sophisticated models of the third approach fullfil these requirements.

In addition to the aforementioned requirements, the model should be practical, which means that it should be easy to compute and to implement.

III The Model

The present study is based on a two-factor model proposed by Longstaff and Schwartz [1992], which uses the short-term interest rate and its volatility as exogenous stochastic variables. The Longstaff-Schwartz model has a number of attracting features:

1. Within this general equilibrium model, all interest rate derivatives can be valued consistently. As discussed above this is an essential prerequisite for a model being used as a risk management tool.

2. An extensive comparative static analysis showed that the two-factor model offers a reasonable description for the future interest rates and it allows for many observable term structure movements. In contrast to a one-factor model, the Longstaff-Schwartz model allows not only for shifts in the yield curve, it can also describe more complex movements, such as for example a twist of the yield curve.[3]

3. The model uses the level of interest rate volatility, which is a key variable in option pricing, explicitly as state variable.

4. Some empirical evidence was found that the short rate exhibits volatility clusters which can be modelled approximately by stochastic volatility.

5. The model endogenous volatility structure which is crucial for the pricing of interest rate options turned out to reflect the observed volatility structure for the German market quite successfully.

The reader is first provided with a brief formal description of the Longstaff-Schwartz model: The model starts from assumptions about the stochastic evolution of two independent factors, which both have an impact on the return of the production process in their economy. The dynamics of the two state variables are driven by the stochastic differential equations

$$\begin{aligned} dx &= \alpha_x(\gamma_x - x)dt + \sigma_x\sqrt{x}dz_1 \\ dy &= \alpha_y(\gamma_y - y)dt + \sigma_y\sqrt{y}dz_2, \end{aligned} \tag{1}$$

where α_x, γ_x, σ_x, α_y, γ_y, and σ_y are (positive) parameters. While both factors affect the mean of the instantaneous rate of return of the production process, only the second factor has an impact on its instantaneous variance and therefore risk due to the first factor is not priced in this economy.

[3]For a detailed comparative static analysis of the endogenous term structure of interest rates as well as of the endogenous volatility structure of the Longstaff-Schwartz model cf. Uhrig [1995].

Following the general equilibrium framework of Cox, Ingersoll, and Ross [1985a] Longstaff and Schwartz derive the fundamental partial differential equation

$$B_t + \alpha_x(\gamma_x - x)B_x + \big(\alpha_y(\gamma_y - y) - \lambda\sigma_y y\big)B_y + \tfrac{x}{2}\sigma_x^2 B_{xx} + \tfrac{y}{2}\sigma_y^2 B_{yy} = rB, \tag{2}$$

which is to be satisfied by the value of any derivative security $B(x, y, t)$. Here r is the short-term riskless rate and $\lambda\sqrt{y}$ is the market price of risk due to the factor y.[4]

The processes of the short-term riskless rate r and its instantaneous variance V are determined endogenously as part of the equilibrium:

$$\begin{aligned} r &= x + y \\ V &= \sigma_x^2 x + \sigma_y^2 y \end{aligned} \tag{3}$$

Using this system of linear equations the authors are able to express the valuation equation in terms of the new (observable) state variables r and V.

Within their model Longstaff and Schwartz derive closed form expressions for interest rate derivative securities and in particular for discount bonds. However, as in the Cox-Ingersoll-Ross model [1985b] and its predecessors by Vasicek [1977], Dothan [1978] and Brennan and Schwartz [1979], the risk-neutral processes of the state variables will not, in general, generate a term structure consistent with market data. But in order to come up with realistic option values it is very important to exploit the full information available in the current term structure interest rates.

In principle, this goal can be reached by an application of the approach proposed by Hull and White [1990], the key idea of which is to allow for a time-dependent parameter within the model. Longstaff and Schwartz [1993]

[4]Following Vasicek [1977] the market price of risk is defined here as the expected instantaneous excess return divided by the corresponding instantaneous risk.

touch on this subject for the Longstaff-Schwartz model. However, as the practical application of the Hull-White approach to the Longstaff-Schwartz model requires a concrete algorithm on the one hand and involves a number of problems on the other hand, it is worth discussing the adaption problem in detail for this model.

IV Implementation

In order to implement the extended Longstaff-Schwartz model the constant and time-dependent parameters of the model have to be estimated. I use the estimation procedure proposed by Longstaff and Schwartz [1993], which can be understood as a combined historical/cross sectional approach:

Estimation of the Volatility

In the first step the volatility of the short-term rate is estimated using the well-known GARCH-framework.[5] An Euler discretization of the interest rate process and a GARCH(1,1)-specification for the volatility results in the following econometric model:

$$\begin{aligned} r_t - r_{t-1} &= B_0 + B_1 r_{t-1} + B_2 V_t + \epsilon_t \\ \epsilon_t &\sim N(0, V_t) \\ V_t &= A_0 + A_1 r_{t-1} + A_2 V_{t-1} + A_3 \epsilon_{t-1}^2 \end{aligned} \tag{4}$$

$B_i, i = 0, ..., 2$ and $A_j, j = 0, ..., 3$ denote the parameters of the GARCH-model. Given the initial values V_0 and ϵ_0 as well as a time series of interest rates r_t, $t \in \{0, \ldots, T\}$, the parameters of the GARCH-model and a time

[5] See Bollerslev [1986].

series of volatilities V_t, $t \in \{1, \ldots, T\}$ can be obtained. The corresponding log likelihood function for a sample of T observations is

$$\ln L = -\frac{1}{2}\sum_{t=1}^{T}[\ln(2\pi) + \ln(V_t) + \frac{\epsilon_t^2}{V_t}]. \tag{5}$$

Estimation of the Parameters of the Stochastic Processes

The second step consists of the estimation of the parameters describing the movement of the short-term interest rate and its volatility. In estimating these parameters the historical time series of the short rate and the estimated volatilities is used. First, it is made use of the fact that the original state variables x and y are positive: To ensure this, the ratio of V and r is restricted to the interval of σ_y^2 and σ_x^2. This allows the somewhat arbitrary choice of the two diffusion parameters as the minimum and the maximum value of this ratio within the historical time series.

As r and V have long-run stationary unconditional distributions, their means Er_∞, EV_∞ and variances Vr_∞, VV_∞ can be expressed in terms of the parameters of the model (1). Therefore one can compute the means and variances of the historical time series of the interest rate and the volatility. By setting equal these empirical values to the theoretical ones, the remaining four parameters of the model can be calculated.[6]

Specifically, the six constant parameters result as the solution of the following system of equations:[7]

$$Er_\infty = \gamma_x + \gamma_y$$

[6] For a discussion of the feasibility of this estimation procedure, cf. Clewlow and Strickland [1994] and Longstaff and Schwartz [1994].

[7] I have assumed implicitly, that $\sigma_y < \sigma_x$. This assumption is discussed in Section VI.

$$\begin{aligned} Vr_\infty &= \frac{\sigma_x^2\gamma_x}{2\alpha_x} + \frac{\sigma_y^2\gamma_y}{2\alpha_y} \\ EV_\infty &= \sigma_x^2\gamma_x + \sigma_y^2\gamma_y \\ VV_\infty &= \sigma_x^6\frac{\gamma_x}{2\alpha_x} + \sigma_y^6\frac{\gamma_y}{2\alpha_y} \\ \min_t \frac{V(t)}{r(t)} &= \sigma_y^2 \\ \max_t \frac{V(t)}{r(t)} &= \sigma_x^2 \end{aligned} \tag{6}$$

Whereas the constant parameters are assumed to be positive in theory, this estimation procedure does not ensure that all parameter estimates are really positive. However, this empirical study always exhibited positive estimates.

Fitting the Model to the Initial Term Structure

In the final step the model is calibrated to the current term structure of interest rates. In order to achieve consistency with the current term structure, I extend the model and allow for a time-dependent risk parameter $\lambda(t)$. The problem in question is to determine the function $\lambda(t)$ in a way that the solutions $B(x_0, y_0, 0, T)$ of the partial differential equation (2) subject to the terminal conditions $B(x, y, T, T) = 1$ coincide with the current market prices of zero-coupon bonds $\hat{B}(T)$ for all maturities T. Here x_0 and y_0 denote the current values of the two state variables x and y.

With a separation of variables $B(x, y, t, T) = F(x, t, T)G(y, t, T)$ the partial differential equation (2) in two state variables can be decomposed in two partial differential equations in one state variable, respectively. Because the bond price component due to the factor x is unaffected by $\lambda(t)$, the adaption has to be carried out in function G. One way of determining the function $\lambda(t)$ is using the *"inverted implicit finite difference method"*.[8]

[8] For a detailed description of the method, in particular the composition of the tran-

Due to the separability of the partial differential equation for zero-coupon bonds, the adaption problem in this two-factor model basically corresponds to the adaption problem within the one-factor Cox-Ingersoll-Ross model. It has been argued by several authors, that the Cox-Ingersoll-Ross model cannot be brought in line with an arbitrary initial yield curve.[9] The argumentation is based on a condition due to Feller [1951], which ensures that zero is an unaccessible boundary for the interest rate process. The Feller condition, which is sufficient for the uniqueness of the endogenous zero coupon price curve, imposes restrictions on the parameters of the model. This is true for a time-dependent formulation of some drift parameter, but not for a time-dependent formulation of the market price of risk. In the latter case, the Feller condition produces no effect on the market price of risk. In other words, the standard condition of no-arbitrage models, that the risk-adjusted measure must be absolutely continuous with respect to the original measure, can be guaranteed when the market price of risk is made time-dependent, however the nature of the diffusion process could be affected by changing other parameters.

Therefore a time-dependent formulation of the market price of risk has the advantage that any initial discount function can be explained as long as the discount function is strictly decreasing and at least continuously differentiable. If the initial discount function is twice continuously differentiable, the resulting time-dependent function is a continuous function.

Consequently, in the case of the two-factor model, a necessary condition for the existence of such a (unique) function $\lambda(t)$ is that $\hat{G}(T) = \frac{\hat{B}(T)}{F(x_0,0,T)}$ is a strictly decreasing function. But due to the fact that is the ratio of two

sition matrices, the special treatment of the boundaries within the implicit scheme, and the λ-search, see Uhrig and Walter [1996]. This contribution includes also a discussion of the existence and uniqueness of such a function $\lambda(t)$, as well as a discussion of the possibility of using other parameters to fit an interest rate model to a given term structure of interest rates.

[9]Cf. for example Heath, Jarrow, and Morton [1992], p. 97.

strictly decreasing functions, namely the current discount function $\hat{B}(T)$ and the price component $F(x_0, 0, T)$ corresponding to the factor x, this condition is not necessaryly satisfied. The significance of this problem to the usefulness of the model is discussed in Section VI.

V Data

In this section, the basic characteristics of all data used within the empirical study is described.

German interest rate warrants began trading at the end of 1989. In this study I used all call and put options listed on the "Amtlicher Handel" and the "Geregelter Markt" of the Frankfurt Stock Exchange within the sample period January 1990 through December 1993. During this period, 19 different calls and 14 different puts traded on 13 different German government bonds. In Germany the government essentially issued bonds in two different time-to-maturity groups – the "Bundesanleihen" (BUND) with an initial time-to-maturity of about 10 years and the "Bundesobligationen"(BOBL) with an initial time-to-maturity of approximately 5 years. 10 of the 13 underlying bonds were from the first group, the remaining 3 from the second group. Exhibit 1 displays some details about the underlying bonds and the options written on each bond. During the sample period, the time-to-maturity of the bonds ranged from 8.1 to 9.9 years for the first group and from 3.4 to 3.8 years for the second group. The average time-to-maturity for the options was 1 year, the maximum being 2.9 years. With the exception of three European-type interest rate warrants, the options under consideration were American-type options. Weekly observations were used and the total number of option prices collected was 1826.

In order to estimate the current term structure of interest rates a homogeneous market segment with respect to bankruptcy, liquidity, and taxes is

EXHIBIT 1

Bond and option data

Bond			Bond price		Exercise price		calls	Obs.	puts	Obs.
			min	max	min	max				
BUND	7%	89/99	86.67	96.50	100	100	1	42	1	42
BUND	7.25%	90/00	87,85	96,93	99	100	2	105	1	49
BUND	7.75%	90/00	91.01	99.11	100	100	1	84	1	84
BUND	8.75%	90/00	98.05	103.15	100	100	1	74	1	74
BUND	9%	90/00	100.27	106.35	100	100	1	78	0	0
BOBL	8.75%	90/95	98.40	101.03	101	101	1	49	1	49
BUND	8,25%	91/01	99.57	105.23	100	100	1	49	1	49
BOBL	7.5%	92/97	102.73	108.60	102	103	2	98	2	98
BOBL	8%	92/97	98.28	107.08	99	99	1	58	1	58
BUND	8%	92/02	98.85	114.96	101	102	2	149	2	149
BUND	7.125%	92/02	101.70	109.74	103	104	2	78	2	78
BUND	8%	92/02	103.34	115.46	100	102	3	204	0	0
BUND	6.5%	93/03	102.93	105.81	103	103	1	14	1	14
Together:							19	1082	14	744

needed. These requirements are fulfilled by the straight bonds issued by the German government, a subsample of which coincides with the bonds underlying of the aforesaid interest rate warrants.[10] Within the estimation all noncallable bonds with a time to maturity between 0.5 and 10 years were included.

Besides this German money market rates were used for two purposes: Firstly, the one-month money market rate was used as a proxy for the short rate and secondly, the money market rates with a time to maturity up to 6 months were used to uphold the yield curve estimation for short maturities.

[10]As a range of German government bonds are deliverable into futures contracts (BUND and BOBL Future), some differences in liquidity between the cheapest to deliver bonds (often on the run bonds), other deliverable bonds, and non-deliverable bonds remain.

The data is available from the German Financial Data Base Mannheim/Karlsruhe.[11]

VI Practicality and Empirical Quality

The problem of valuing the interest rate warrants within the extended Longstaff-Schwartz model consists of three main subproblems:

- estimation of the current term structure of interest rates
- parameter estimation
- computation of the warrant's prices

The Role of the Current Term Structure

The estimation of the current term structure of interest rates involves some difficult trade-offs. The theoretical prices of the bonds underlying the options are determined by the current term structure of interest rates. As deviations between observed and theoretical bond prices transfer to differences between the observed and theoretical option values, an estimation procedure resulting in term structure estimates that explain the current bond prices with high accuracy is desirable. However, the estimation of zero bond prices with maximum accuracy fully transfers noise in the data of coupon bonds to the term structure of interest rates and leads to term structures which often are fairly erratic. The attempt to calibrate the Longstaff-Schwartz model to such yield curves resulted in time-dependent market prices of risk which

[11]This data base was established under the research program "Empirical Capital Market Research", supported by the German National Science Foundation.

were extremely variable. In these cases the empirical quality of the valuation model turned out to be very low. Therefore an acceptable compromise between accuracy and smoothness has to be found.

I apply a two-step procedure in order to achieve this compromise. In the first step, a discrete discount function is determined using a quadratic-linear programming approach. This results in term structure estimates with high accuracy in explaining observed bond prices. In the second step, the corresponding term structure is approximated with cubic splines.[12]

For each of the 209 valuation days, the current term structure of interest rates was estimated from prices of German government bonds. Each estimate used all currently traded German government bonds with a time to maturity of between 0.5 and 10 years, the average number of bonds being 100. By smoothing the term structure of interest rates for the research period from 1990 to 1993, the average absolute deviation of prices increased from DM 0.07 to DM 0.15 per DM 100 face value.

Estimation Results

Estimated volatility of the short-term rate within the GARCH-model described in Section IV was based on weekly data of one-month money market rates based on the average of bid and ask rates within the sample period from January 1970 to December 1989. The money market rates of the first three months were used to estimate an initial volatility V_0. The remaining interest rates were used as input for the GARCH-estimation. Exhibit 2 provides summary statistics about the one-month money market rates. The evolution of the money market rates is displayed visually in Exhibit 3. Exhibit 4 presents the week-to-week changes in the money market rate.

The one-month money market rate shows a cyclic course with periods of high interest rates followed from periods of low interest rates. The sample mean

[12]For details to this estimation procedure, cf. Uhrig and Walter [1994].

EXHIBIT 2
Summary statistics

Money market rates and changes of the money market rate		
	r_t	$r_{t+1} - r_t$
Observations	1031	1030
Mean	0.064447	-0.000013
Std. dev.	0.026948	0.003923
Minimum	0.025000	-0.030000
Maximum	0.147500	0.026250
Skewness	0.865710	0.262330
Kurtosis	-0.158650	15.443790
Autocorrel.(1)	0.988635	0.072815
Autocorrel.(2)	0.975738	-0.091651
Autocorrel.(3)	0.964799	-0.140230
Autocorrel.(4)	0.956820	0.027699

EXHIBIT 3
Evolution of the one-month money market rate 1970 – 1993

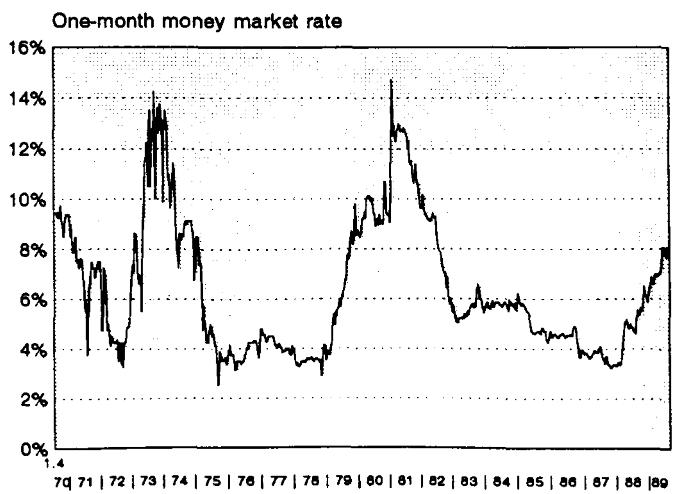
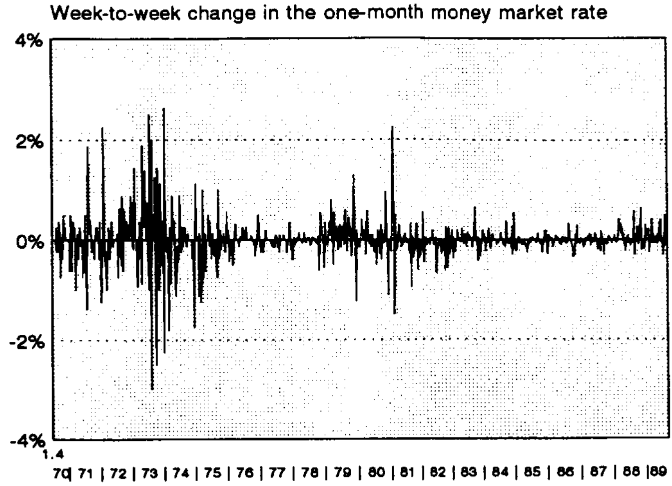


EXHIBIT 4



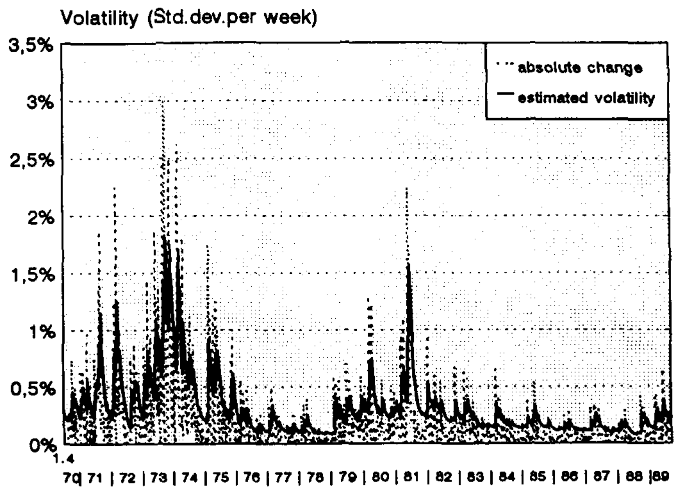
of 6.44% p. a. can be understood as a kind of long-term mean, where the interest rates are fluctuating around. A comparison of the levels in Exhibit 3 and the changes in levels in Exhibit 4 shows that the largest interest rate movements take place in periods of high interest rates around 1973/74 and 1980/81. Furthermore Exhibit 4 shows the clustering of volatility.

Exhibit 5 plots the estimated standard deviation per week (the solid line) and as an ex post measure of volatility the absolute changes in r. A comparison of these values shows that the estimated volatility captures the qualitative course of the absolute changes successfully.

With the historical time series of the two state variables, the constant parameters can be computed easily using the method described in Section IV. In order to estimate the respective required sample means and variances for the short-term rate, the time series from January 1970 to the current valuation day was employed. In contrast to this, the estimation of the moments for the volatility as well as for the ratio between the volatility and the short-

EXHIBIT 5

Estimated volatility



term rate is based on the previous 52 observations, e.g. the observations within one year prior to the current valuation day, respectively. Use of an asymmetrical treatment is supported by the fact that the short-term rate shows a mean-reverting feature with cycles lasting for some years, whereas the volatility is oscillating with quite short cycles. Although this procedure is not justified from a theoretical point of view, I believe that estimating the moments of the volatility from short periods allows to reflect the current market expectation in a better way.

Exhibit 6 reports the estimated parameter values.

The original state variable x shows a high mean-reverting coefficient α_x and a high volatility in comparison to the state variable y. In contrast, the long-term mean of the state variable x is lower than the long-term mean of the state variable y.

EXHIBIT 6
Parameter estimates 1990 - 1993

Parameter	Average	Minimum	Maximum	Standard deviation
α_x	1.035	0.539	1.546	0.270
γ_x	0.026	0.017	0.039	0.005
σ_x	0.105	0.082	0.125	0.015
α_y	0.081	0.054	0.111	0.015
γ_y	0.041	0.027	0.048	0.006
σ_y	0.048	0.045	0.052	0.002

Parameter Estimation and its Impact on the Fitting Ability

The derivation of the system of equations (6) made implicit use of the assumption that $\sigma_y \leq \sigma_x$. It its worth noting that one cannot arbitrarily choose one parameter as the lower, because this assumption has far-reaching consequences: The market price of risk of x is zero while the market price of risk due to the second factor is used to calibrate the model to the current term structure of interest rates. If one chooses $\sigma_x \leq \sigma_y$ the roles of the factors reverse. Therefore the decision to choose one particular parameter as the lower one is also a question of deciding whether risk of a highly fluctuating factor is compensated for or the risk of a factor with a rather low variation is priced within the economy. From an economical point of view there is no compelling reason for the one or the other choice as long as the factors remain unspecified. However, there is a technical reason that forces the choice of $\sigma_y < \sigma_x$: In order to calibrate the model to the current term structure, the function $\hat{G}(T)$ has to be strictly decreasing. On the one hand, the course of $\hat{G}(T)$ is dependent on the current discount function $\hat{B}(T)$, on the other hand it depends on the price component $F(x_0, 0, T)$ due to the factor x and therefore on the estimated parameters. An inspection of the functions $\hat{G}(T)$ shows that the functions are strictly decreasing within the whole sample period where $\sigma_y < \sigma_x$, but they are strictly decreasing in only 175 of the 209 valuation days where $\sigma_x < \sigma_y$.

Valuation of Interest Rates Warrants

Finally, I valued the interest rate warrants for each valuation day by computing the theoretical option values using the alternating direction implicit method.[13] For the American options, the numerical algorithm was modified by merely taking the value of the American option to be the larger of its exercised and its unexercised value at each step.

In the following, a comparison of the theoretical option prices is made with their empirical counterparts. It should be noticed at this point that the theoretical options values are based only on informations that are available from the cash market as no implied volatilities from the option market were used. This way of implementing the model allows to test the expanatory power of the model in its original sense without using input values that are themselves dependent of the quality of the model.

Exhibit 7 summarizes the pricing errors. For the whole sample, the mean option price (column 3) was DM 3.21. Columns 4 to 6 report the average absolute pricing errors, the average pricing errors (defined as model value minus market value), and the average relative absolute pricing errors.[14] Column 7 gives the standard deviation of the absolute pricing error. For the whole sample the mean absolute deviation was DM 0.30. As the 5th column indicates, on average the model underpriced both the calls and the

[13]Cf. McKee and Mitchell [1970]. In contrast to the valuation of zero coupon bonds, the valuation of options on coupon bonds makes is necessary to solve a partial differential equation in two state variables, because the terminal condition cannot be separated.

[14]To calculate the relative pricing errors all observations were removed where the market price of the option was less than DM 0.10. The total number of observations eliminated was 253 of which 81 were calls and 172 were puts.

EXHIBIT 7
Deviation between the model values and market values of the warrants

Aggregate pricing errors						
1	2	3	4	5	6	7
	Obs.	Av. Price [DM]	Mean abs. dev. [DM]	Mean dev. [DM]	Mean rel. abs. Dev	St. dev. [DM]
All	1826	3.2138	0.2990	-0.0949	22.24%	0.4077
Calls	1082	3.7277	0.2765	-0.0364	16.79%	0.3292
Puts	744	2.4665	0.3316	-0.1798	31.77%	0.4985

Pricing errors by moneyness of the option – calls						
Moneyness	Obs.	Av. Price [DM]	Mean abs. dev. [DM]	Mean dev. [DM]	Mean rel. abs. Dev	St. dev. [DM]
0.85 - 0.98	253	0.5921	0.3487	- 0.3116	55.95%	0.4916
0.98 - 1.02	271	1.9118	0.2701	- 0.0229	14.71%	0.3000
1.02 - 1.10	558	6.0313	0.2469	0.0817	5.21%	0.2315
Moneyness is defined as bond price divided by exercise price.						

Pricing errors by moneyness of the option – puts						
Moneyness	Obs.	Av. Price [DM]	Mean abs. dev. [DM]	Mean dev. [DM]	Mean rel. abs. Dev	St. dev. [DM]
0.85 - 0.98	287	0.1883	0.1189	- 0.0936	66.17%	0.2105
0.98 - 1.02	275	1.3939	0.3641	- 0.1610	30.80%	0.5179
1.02 - 1.10	182	7.6798	0.6181	- 0.3443	9.74%	0.6213
Moneyness is defined as exercise price divided by bond price.						

Pricing errors by moneyness of the option – calls + puts						
Moneyness	Obs.	Av. Price [DM]	Mean abs. dev. [DM]	Mean dev. [DM]	Mean rel. abs. Dev	St. dev. [DM]
0.85 - 0.98	540	0.3775	0.2266	- 0.1958	60.13%	0.3872
0.98 - 1.02	546	1.6510	0.3174	- 0.0924	22.79%	0.4266
1.02 - 1.10	740	6.4367	0.3382	- 0.0230	6.33%	0.4011

EXHIBIT 7 cont.
Deviation between the model values and market values of the warrants

Pricing errors by time to maturity – calls						
Time to maturity Months	Obs.	Av. Price [DM]	Mean abs. dev. [DM]	Mean dev. [DM]	Mean rel. abs. Dev	St. dev. [DM]
0 - 6	337	3.1374	0.1253	0.0142	12.45%	0.1372
6 - 12	384	2.4306	0.2828	-0.0832	22.49%	0.3435
12 - 18	124	2.8227	0.4682	-0.2058	22.80%	0.4639
18 - 24	93	8.6416	0.3933	-0.3454	10.58%	0.3176
24 - 30	78	7.5553	0.1500	0.0442	2.11%	0.0966
30 - 36	66	4.5412	0.6372	0.6352	15.24%	0.2868

Pricing errors by time to maturity – puts						
Time to maturity Months	Obs.	Av. Price [DM]	Mean abs. dev. [DM]	Mean dev. [DM]	Mean rel. abs. Dev	St. dev. [DM]
0 - 6	288	2.2501	0.1431	0.0154	33.39%	0.1879
6 - 12	332	2.2297	0.3027	-0.1952	32.50%	0.3303
12 - 18	93	3.6641	0.8527	-0.6735	30.26%	0.8527
18 - 24	31	3.4206	0.8308	-0.3485	21.15%	0.9403

Pricing errors by time to maturity – calls + puts						
Time to maturity Months	Obs.	Av. Price [DM]	Mean abs. dev. [DM]	Mean dev. [DM]	Mean rel. abs. Dev	St. dev. [DM]
0 - 6	625	2.7285	0.1335	0.0148	20.28%	0.1628
6 - 12	716	2.3375	0.2920	-0.1351	26.86%	0.3376
12 - 18	217	3.1833	0.6330	-0.4062	25.94%	0.6862
18 - 24	124	7.3362	0.5027	-0.3462	13.22%	0.5767
24 - 30	78	7.5553	0.1500	0.0442	2.11%	0.0966
30 - 36	66	4.5412	0.6372	0.6352	15.24%	0.2868

puts. However, the average pricing error of DM -0.04 and DM -0.18 were rather small. The average absolute relative pricing error was 22% for the whole sample. The mean absolute deviation as a percent of the average option value is much smaller than the absolute relative pricing error. This indicates an observation typical of empirical studies on option pricing, where absolute relative pricing errors are often found to be higher for out-of-the-money options. A comparison of the aggregate results for calls and puts shows that the pricing errors where consistently lower for the call values than for the put values. A reason for this might be that the calls' average price of DM 3.73 is higher than the puts' average price of DM 2.47.

Exhibit 7 also contains the pricing errors by the moneyness of the options. As already mentioned, the relative absolute pricing error of out-of-the-money options was much higher than the corresponding error of in-the-money options. But for the calls even the absolute pricing error increased when the moneyness decreased. Whereas column 5 reports low pricing errors on average for at- and in-the-money calls, there seems to be a systematic pricing error for out-of-the-money calls. Here the model underpriced the calls with a pricing error of DM -0.31 by an average option price of DM 0.59.

The examination of the pricing errors by time to maturity shows that the options with a very short time to maturity ($\leq$ 6 months) were slightly overpriced by the model whereas options with a maturity from 6 to 24 months were underpriced by the model on average. Calls with a very long time to maturity (over 30 months) were overpriced by the model significantly.

Exhibit 8 reports the pricing error over time.

The inspection of the pricing error over time offers some interesting insights. Whereas the absolute pricing error was DM 0.60 in 1990, it reduced to DM 0.26 and DM 0.31 in 1991 and 1992. In 1993 the absolute pricing error was just DM 0.16 on average. The relative absolute pricing error decreased from 36% in 1990 to 17% in 1993.

EXHIBIT 8
Pricing Error over time

	1990	1991	1992	1993
Obs.	376	330	360	760
Av. Price [DM]	3.8995	1.9907	2.3394	3.8199
Mean abs. dev. [DM]	0.6025	0.2572	0.3051	0.1640
Mean dev. [DM]	-0,5003	-0.0444	0.2006	-0.0562
Mean rel. abs. dev.	35.56%	21.62%	19.78%	17.08%
St. dev. [DM]	0.6998	0.3103	0.2664	0.1548
Mean rel. abs. dev. (at-the-money)	28.09%	25.87%	21.81%	18.96%

The evolution of the average pricing error is also instructive: In 1990 nearly all options were underpriced significantly by the model. However underpricing reduced over time and in 1993 the pricing error was on average close to zero. With these results there are good reasons for the supposition that the issuers of the warrants quoted (too) high prices in the beginning of the sample period, which also coincides with the beginning of trading in interest rate warrants, and the market was willing to accept such high prices at first.

The empirical studies by Trautmann [1990] and by Dietrich-Campbell and Schwartz [1986] provide a useful point of reference for the results found here. Trautmann examined deviations between Black-Scholes values and market prices of calls on German stocks for the German floor market from 1983 to 1987 using historical volatilities. He found relative deviations of 13% and relative absolute deviations of 29%.[15] Dietrich-Campbell and Schwartz compared theoretical prices for American call and put options on US government bonds with the corresponding market prices using the Brennan-

[15]Cf. Trautmann [1990], p. 96.

Schwartz two-factor model.[16] They report an average pricing error of \$ 0.33 for calls and \$ 0.30 for puts with a mean option price of \$ 2.21. These results are clearly worse than those of the present study (DM -0.04 for calls and DM -0.17 for puts with an average option value of DM 3.21). As absolute error measure they used the root mean square pricing error with values of \$ 0.67 for calls and \$ 0.68 for puts. In comparison, in the present study the root mean square pricing error was DM 0.43 for calls and DM 0.60 for puts.[17]

Due to the differences in the market segments these results cannot be compared directly. However, it shows on the one hand that the empirical quality of the extended Longstaff-Schwartz model for valuing bond options is at least comparable to the Black-Scholes model for valuing stock options for the German market, on the other hand the empirical results are superior to the results within the study of Dietrich-Campbell and Schwartz.

VII Conclusion

In this paper I have applied the extended Longstaff-Schwartz model for valuing German interest rate warrants. The model allows for a consistent valuation of interest rate derivatives and has the property that it uses all information on the current term structure. Additional to these theoretical qualities the empirical results show that the prediction quality of the model is considerable.

However, the model should not only be evaluated by its empirical quality but also by its ease of use and by its practicality. The parameter estimation is quite costly because the state variable "volatility" is not directly observable

[16] Cf. Dietrich-Campbell and Schwartz [1986] and Brennan and Schwartz [1982, 1983].

[17] The root mean square pricing error can be easily computed from the values of Exhibit 7.

and because six parameters have to be estimated from one time series. Therefore, the volatility of the short-term rate must first be estimated before the six parameters of the model can be obtained by solving a simple system of equations. However, one cannot make sure that the system of equations always results in reasonable parameter estimates, because one can imagine circumstances where some of the parameters become negative. This would be of course in contrast to the formulation of the processes in (1). Further, there is no guarantee that the model can always be fitted to market data. However, none of these problems surfaced in this empirical study.

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