

A Time Series Analysis of an Asset Class Returns Model for the German Capital Market

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Abstract

This contribution investigates time series data for the purpose of modelling asset class returns in the German capital markets. As a broader model class vector autoregressions with possible cointegration vectors are considered. Motivated by the dividend discount model on the market level cointegration of dividend yields, bond yields and inflation rates is tested. Both the methodology of *Johansen* and *Engle/Granger* is applied. As a consequence of these tests two different models are estimated: Model I is a pure vector autoregressive model and Model II incorporates the possible cointegration of bond yields and inflation rates.

Résumé

L'exposé présent s'occupe d'une analyse des données type séries temporelles avec le but de modéliser les rendements des classes d'actifs financiers sur le marché financier allemand. Les modèles d'autorégression vectorielles avec cointégration sont considérés comme une classe de modèles assez généraux. Inspiré par le DDM (dividend discount model) sur le niveau du marché totale des actions allemandes, cointégration entre la rentabilité des dividendes, la rentabilité des obligations et les taux d'inflation est testé aussi bien avec les méthodes de *Johansen* qu'avec la méthode d'*Engle/Granger*. En conséquence nous estimons deux types de modèles différents: Modèle I est un modèle d'autorégression vectorielle pure et Modèle II tient compte de cointégration possible entre la rentabilité des obligations et le taux d'inflation.

Keywords

Time series, analysis, asset allocation, vector autoregression, cointegration.

A. Econometric Framework

As a starting point for the empirical investigations that follow we look at vectorautoregressive (VAR) models. Cointegrating vectors are not ruled out a priori and will be tested for. The further exposition necessitates the introduction of some definitions.

For a p-dimensional VAR model of order k we will use the following representation:

$$Y_t = \Pi_1 Y_{t-1} + \dots + \Pi_k Y_{t-k} + \mu + \varepsilon_t, \quad t = 1, \dots, T, \quad (1)$$

where $Y_t = (y_{1t}, \dots, y_{pt})'$ is an p by 1 vector, Π_i ($i = 1, \dots, k$) are p by p matrices and $\mu = (\mu_1, \dots, \mu_p)'$ is a vector of constants, $\varepsilon_t = (\varepsilon_{1t}, \dots, \varepsilon_{pt})'$ is a vector of independent random disturbances with $E(\varepsilon_t) = 0$ and $E(\varepsilon_t \varepsilon_t') = \Sigma_\varepsilon$.

For an understanding of the problem of unit roots the next representation is useful. A VAR(k) model like (1) can be reformulated as a VAR(1) model¹:

$$Y_t^* = \mu^* + AY_{t-1}^* + U_t \quad (2)$$

with

$$Y_t^* := \begin{pmatrix} Y_t \\ Y_{t-1} \\ \vdots \\ Y_{t-k+1} \end{pmatrix}, \mu^* := \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \end{pmatrix}, A := \begin{pmatrix} \Pi_1 & \Pi_2 & \dots & \Pi_{k-1} & \Pi_k \\ I_p & 0 & \dots & 0 & 0 \\ 0 & I_p & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & I_p & 0 \end{pmatrix}, U_t := \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (3)$$

Y_t^* , μ^* and U_t are p·k - dimensional column vectors and I_p denotes an p by p-identity matrix. This process is stable if

$$\det(A - zI_{p \cdot k}) = 0 \quad (4)$$

This means that the characteristic roots have to be inside the unity circle².

Integration is defined as follows³:

Definition Integration:

An univariate process is called **integrated of order d** (short: I(d)) if the characteristic equation has d unit roots.

Based on the definition of integration we can define cointegration as follows⁴:

Definition Cointegration:

The components of the vector Y_t are said to be **cointegrated of order (d,b)** with $0 < b \leq d$, if all components are integrated of order d and if there exists a linear combination $\beta' Y_t$ ($\beta' = (\beta_1, \dots, \beta_p) \neq 0$) that is integrated of order (d-b). The vector β is called cointegrating vector.

Between p components there can exist up to (p-1) linearly independent cointegrating vectors. The number of linearly independent cointegrating vectors is called cointegration rank⁵.

Now we will focus on p-dimensional VAR processes, where the components of Y_t are integrated of order 1. The model

$$Y_t = \Pi_1 Y_{t-1} + \dots + \Pi_k Y_{t-k} + \mu + \varepsilon_t, \quad t = 1, \dots, T \quad (5)$$

with $\varepsilon_1, \dots, \varepsilon_T$ iid can be written as⁶

$$\begin{aligned} \Delta Y_t &= \Gamma_1 \Delta Y_{t-1} + \dots + \Gamma_{k-1} \Delta Y_{t-k+1} + \Pi Y_{t-k} + \mu + \varepsilon_t \\ \text{with } \Gamma_i &= - (I - \Pi_1 - \dots - \Pi_i), \quad i = 1, \dots, k-1 \\ \Pi &= - (I - \Pi_1 - \dots - \Pi_k). \end{aligned} \quad (6)$$

As the components of Y_t are integrated of order 1, ΔY_t and ΔY_{t-1} are stationary; the same is true for μ and ε_t . Y_{t-k} like Y_t is not stationary; as on both sides of the equation there must be a stationary process we can draw certain conclusions concerning the matrix Π :

- 1) The matrix Π is the null matrix with rank 0. In this case our model corresponds to a traditional differenced time series model.
- 2) The matrix Π has rank r with $0 < r < p$. We can write Π as the product of two p by r - matrices α and β : $\Pi = \alpha\beta'$. This implies that there exist r linearly

independent cointegrating vectors (the rows of β'), guaranteeing that $\beta'Y_t$ is stationary.

3) Π has full rank ($r = p$). In order to rule out an inconsistency, not only ΔY_t but also Y_t (and therefore $Y_{t,k}$) must be stationary.

Based on this reasoning, hypotheses concerning the cointegration of the components of Y_t can be tested by investigation of the estimated matrix Π in a model of the form (6).

In order to illustrate the role of the matrices α and β we want to stress the close relationship between the concept of cointegration and error correction models.

As can easily be seen we can rewrite (6) in error correction form⁷:

$$\begin{aligned} \Delta Y_t &= B_1 \Delta Y_{t-1} + \dots + B_{k-1} \Delta Y_{t-k+1} + \alpha \beta' Y_{t-1} + \mu + \varepsilon_t \\ \text{with } B_i &= -(\Pi_{i+1} + \dots + \Pi_k), \quad i = 1, \dots, k-1 \\ \alpha \beta' &= \Pi = -(I - \Pi_1 - \dots - \Pi_k). \end{aligned} \quad (7)$$

The economic intuition behind error correction models is that there is a long-run equilibrium which states that:

$$\beta' Y = 0. \quad (8)$$

According to the number of rows of the r by p -matrix β' there are r independent relationships between the components of Y . Temporary deviations are possible but lead to an correction towards the equilibrium via the matrix α .

In general economic theory should give hints which cointegrating vectors can be expected. As the choice of time series is crucial for finding cointegration, in the next chapter possible relationships between the variables that are of interest for modelling asset class returns are discussed.

B. Potential cointegrating vectors

1. Cointegration of dividend yields, bond yields and inflation rates

In order to model long-run economic relationships between the equity and bond markets it could be useful to decompose the performance of the equity market into the price and the dividend component. Let P_t denote the price index in time t and D_t denote the aggregated dividend payments during the last dividend period up to t . We can expect both variables to be integrated of order 1. Which cointegrating relationships may exist ?

One widely used valuation model for equities is the dividend discount model⁸ in numerous variations that assume different growth patterns for the dividend stream. Let I_t^P denote an index of consumer prices. The equation that determines the price of an equity in time $t = 0$ is⁹:

$$\frac{P_0}{I_0} = \sum_{t=1}^{\infty} E \left(\frac{D_t}{I_t^P} \right) \cdot (1+R_0^*)^{-t} \quad . \quad (9)$$

This representation centers around real dividends and real equity prices which means that R_0^* is to be interpreted as a real discount rate. Sometimes - in contrast to this - nominal prices and dividends are used¹⁰, which leads us to testing both variations.

The constant growth model assumes a constant rate of dividend growth g_0 . For the variant with real dividends and prices we have

$$E \left(\frac{D_t}{I_t^P} \right) = E \left(\frac{D_{t-1}}{I_{t-1}^P} \right) \cdot (1+g_0) \quad (10)$$

which yields:

$$\frac{P_0}{I_0^P} = \frac{\frac{D_0}{I_0} \cdot (1+g_0)}{r_0 - g_0} \quad . \quad (11)$$

As we see, the index of consumer prices cancels itself out and we get the following expression for the real discount rate R_0^* (or the internal rate of return of the dividend stream), which is to be applied for the equity in time $t = 0$:

$$R_0^* = \frac{D_0 \cdot (1+g_0)}{P_0} + g_0 \quad (12)$$

The nominal variant of (10)¹¹ leads to

$$R_0 = \frac{D_0 \cdot (1+g_0)}{P_0} + g_0 \quad (13)$$

for the nominal discount rate R_0 .

So far the classical dividend discount model for single stocks. It is a natural idea to apply this model to the valuation of an equity market price index¹². In the following the dividends, prices, growth rates and discount rates are used as aggregated quantities.

The discount rate R_0 that is applied to the asset class "equities" should be related to the return of long-term government bonds. As the cash flow of an equity investment is subject to higher risks than the cash flow of a government bond investment the investors will demand a risk premium. In the following we will assume that the dividend discount model holds in real terms. As a consequence we have to determine the expected real yield of a bond investment. As a proxy for this we can choose the current yield to maturity U_0 of a homogeneous class of debt instruments with long term to maturity. The "expected"¹³ real yield U_0^* can be represented using \tilde{I}_0 as an expression for the expected average rate of inflation during that period as

$$U_0^* = \frac{1+U_0}{1+\tilde{I}_0} - 1 = \frac{U_0 - \tilde{I}_0}{1+\tilde{I}_0} \approx U_0 - \tilde{I}_0 \quad (14)$$

The expected average rate of inflation will not necessarily be identical with the

current rate of inflation I_0 , although the latter will be a starting point for the forming of expectations. Assuming that the expected inflation rate can be written as a weighted average of the current rate of inflation and a level of inflation i that is considered "normal" in the long run we have:

$$\bar{I}_0 = \rho \cdot I_0 + (1 - \rho) \cdot i \quad . \quad (15)$$

In consideration of an additive risk premium E_0 the real yield that investors demand for an investment in equities is:

$$R_0 = U_0 - \rho \cdot I_0 - (1 - \rho) \cdot i + E_0 \quad . \quad (16)$$

Together with (12) this yields:

$$U_0 - \rho \cdot I_0 - (1 - \rho) \cdot i + E_0 - g_0 = \frac{D_0}{P_0} \cdot (1 + g_0) \quad (17)$$

If we assume the dividend discount model in nominal terms we can derive the following equation:

$$U_0 + E_0 - g_0 = \frac{D_0}{P_0} \cdot (1 + g_0) \quad (18)$$

If we want to analyse the situation in time t we have to exchange the subscript 0 with a t . With the assumption of a constant risk premium ($E_t = e$)¹⁴ and a constant growth rate of dividends ($g_t = g$)¹⁵ we can write:

$$\frac{D_t}{P_t} \cdot (1 + g) = U_t - \rho \cdot I_t + a \quad (19)$$

$$\text{with} \quad a = - (1 - \rho) \cdot i + e - g \quad .$$

This means that the dividend yield could be cointegrated with the bond yield and the inflation rate. In the case of the dividend discount model in nominal terms we would expect a cointegrating relationship between the dividend yield and the bond yield.

A very similar relationship was found by *Mills*¹⁶ for the British capital market:

$$U_t = c \cdot \frac{D_t}{P_t} \quad (20)$$

As a theoretical backing *Mills* refers primarily to Ellinger (1971), where c is called "confidence factor". As the above analysis shows, the relationship between dividend yields and bond yields can also be explained in terms of the dividend discount model. The assumptions of constant g and e could prove problematic. If we drop these the expression "a" in (19) is not necessarily stationary and there is no cointegration. This means we have to check whether the results of *Mills* are valid for the German capital market, too.

2. Cointegration of bond yields and inflation rates

In the light of *Fisher's* hypothesis¹⁷, that ex ante **real** bond yields are uncorrelated with the expected inflation rate, we would suspect that **nominal** bond yields are cointegrated with the expected average rate of inflation (proxied by a simple function of the current inflation rate). Intuitively this means that the real yield is stationary and will be drawn back to an equilibrium value u^* . If we model the expected average inflation rate as in (15), we have:

$$\begin{aligned} U_t - \bar{I}_t &= u^* \rightsquigarrow \\ U_t - \rho \cdot I_t - (1 - \rho) \cdot i &= u^* \rightsquigarrow \\ U_t - \rho \cdot I_t &= u^* + (1 - \rho) \cdot i \end{aligned} \quad (21)$$

As ρ should have a positive sign, the coefficient for the inflation rate in the cointegrating vector should be negative.

Each of the cointegrating relationships presented above can exist according to the definition only if the relevant variables (dividend yields, nominal bond yields and inflation rates) are integrated of order 1.

C. Data used in the empirical investigations

The ingredients for the empirical investigations are an performance index for equities and likewise for bonds as well as a proxy for the yield to maturity of "Schuldscheindarlehen". We used time series of the indices DAFOX and REXP and the yields to maturity of the fictitious 10-year title with a coupon of 7.5% which is part of the calculations leading to the REXP.

For the analysis of potential cointegration relationships between dividend yields, bond yields and (maybe) inflation rates we further need an index of consumer prices as well as an index of dividends and a price index for stocks (or a time series of dividend yields).

In the series 2 (equity markets) of the "Fachserie" 9 ("Geld und Kredit") the German federal bureau of statistics publishes the average dividend yields for the exchange listed German stocks based on month end values.

As an indicator of inflation we use the consumer price index for all private households in the seasonally adjusted form as it is published in the "Statistische Beihefte zu den Monatsberichten der Deutschen Bundesbank".

Using these index values we calculate the annualised rate of inflation i_t in month t as

$$i_t = \left(\frac{I^P_t}{I^P_{t-1}} \right)^{12} - 1 \quad . \quad (22)$$

As these calculated rates show a certain amount of erratic behaviour we used 4- and 12-month geometric moving averages of the inflation rate.

D. Survey of the econometric methods applied

The tests for cointegration in one of the approaches assume that the time series are integrated of order 1. Beginning with the seminal work of *Fuller*¹⁸ and *Dickey/Fuller*¹⁹ the statistical literature developed various unit root tests. These

tests typically start from the assumption that there is at most one unit root in the characteristic equation of the process. The definition of integrated processes yields that a process integrated of order 2 and higher contains more than one unit root. In practice this problem is usually circumvented by repeating the unit root test with the differenced time series, if there seems at least one unit root left.

The methodology described above is not fully satisfactory from a theoretical point of view. In monte carlo simulation studies it can be shown that *Dickey-Fuller*-tests declare stationarity in a higher percentage of cases than the level of significance suggests when in fact second or third differencing is needed to achieve stationarity²⁰. Therefore a sequential test procedure is used in the following time series investigations that was developed by *Dickey/Pantula*²¹.

For the tests and estimations of cointegrated systems I used the approaches of *Johansen*²² and *Engle/Granger*²³. The latter methodology is a two-step procedure. First the cointegration vector is estimated via ordinary least squares which necessitates the decision which of the variables should enter as the dependent. *Engle/Granger* point out different types of tests that use the residuals of the cointegration regression and recommend tests of the *Dickey-Fuller* or *Augmented-Dickey-Fuller* type. These tests were originally devised for testing integration in univariate times series. It is important to see that cointegration means that the residuals of the cointegration regression should not be integrated.

If the tests suggest that we can assume cointegration the residuals of the cointegration regression are used for the second step of the *Engle/Granger* methodology that determines the dynamics of the process.

Johansen starts with a p -dimensional process as in (5) and the additional assumption of normal distributed vektors $\varepsilon_1 \dots \varepsilon_T$ with mean zero and variance matrix Λ . *Johansen* now uses the representation (6) of this process. The hy-

pothesis of cointegration is identical with the restriction $\text{rank}(\Pi) =: r < p$ in the general model with full rank of Π ²⁴. In principal a likelihood-ratio test is applied that uses the ratio of the maximum²⁵ of the likelihood function in the restricted model and the maximum in the unrestricted model.

The estimation of a pure VAR-model without cointegration is relatively straight-forward. A VAR-model can be written in such a way that the generalized least squares methodology²⁶ is applicable which leads in this case to the estimation of the single equations by ordinary least squares.

E. Results of the tests for integration²⁷

First we will investigate the stationarity of dividend yields, bond yields and inflation rates by analysing monthly data from January 1967 to October 1993. As far as dividend yields and bond yields are concerned, the hypothesis of exactly one unit root cannot be rejected at the 95% significance level, whereas this hypothesis can be dismissed for the 4-month moving averages of inflation rates. The latter fact suggests that the inflation rates are stationary. Motivated by measurement errors in the inflation data that could easily hide a unit root in the actual rates of inflations we also looked at 12-month moving averages of inflation. Here the hypothesis of one unit root and likewise instationarity cannot be rejected.

F. Results of the tests for cointegration²⁸

1. Cointegration of dividend yields and bond yields

According to the findings of *Mills* (1991) for the British capital market we concentrate for the time being on tests for a cointegrating relationship between dividend yields and bond yields.

Following the *Johansen*-approach, we can conclude from the trace statistic that the hypothesis of no cointegration cannot be rejected at the 95% significance level which leads us to a traditional VAR model of the differenced time series.

The distribution of the test statistic has been tabulated under the assumption of the alternative hypothesis that the rank of Π is p . This assumption means that the VAR model is stationary in the levels which is rather heroic after our results of the unit root tests. Therefore we want to test if the hypothesis of no cointegration cannot be rejected against the hypothesis of cointegration of order 1. The appropriate test statistic in this case is the maximum eigenvalue statistic. The result of no cointegration is confirmed but it must be conceded that at the 90% level stationarity cannot be rejected.

According to the *Engle/Granger*-methodology first we regress dividend yields on bond yields and a constant. The residuals of this regression are used as an input for *Dickey-Fuller*- and *Augmented-Dickey-Fuller*-tests. Since the *Durbin-Watson*-statistic of the *Dickey-Fuller*-regression (1,5119) diverges rather drastically from the ideal value of 2, an *Augmented-Dickey-Fuller*-test seems appropriate. The inclusion of the once lagged differences is already sufficient for generating a *Durbin-Watson*-statistic of 1,997. The "t-statistic" for the *Augmented-Dickey-Fuller*-test with one lag is -2,876197, which must be compared with a critical value of -3,25 for the 95% level²⁹. Again this leads us to a negative result concerning the cointegration of dividend yields and bond yields.

As a consequence both the tests according to the *Johansen*- and *Engle/Granger*-methodology suggest that the hypothesis of a cointegrating relationship between dividend yields and long-term bond yields can be rejected for the German capital market.

2. Cointegration of dividend yields, bond yields and inflation rates

Now we have to ask whether our preliminary results change if we introduce the inflation rates into the analysis. The critical point here is the measurement error in inflation rates. The seasonally adjusted change of the consumer price index as it is published by the German Federal Statistical Bureau is not a very reliable indicator for inflation. While we have good reasons to believe that the "true" rate of inflation does not change very much in subsequent months (this would point to a variable integrated of order one) the annualised inflation rates vary rather wildly. In order to dampen this effect the German Federal Statistical Bureau calculates multi-period averages of these rates. As the tests for integration showed even 4-month averages show stationarity which can be interpreted as a consequence of the measurement errors pointed out above we also used 12-month averages.

The (perhaps artificial) stationarity of 4-month averages of inflation rates entails the *Engle/Granger*-tests for cointegration not being applicable because these tests assume that each variable is integrated of order one. Therefore only the tests according to *Johansen* remain which are still valid³⁰. The rank of cointegration is increased by one per each stationary variable included³¹.

Using 4-month averages of inflation rates, dividend yields and bond yields, the hypothesis of cointegration rank 1 can be rejected at the 95% level with respect to the trace- as well as the maximum-eigenvalue statistic. As we have included one stationary variable this result points to the fact that there is no cointegration of dividend yields, bond yields and inflation rates.

If we focus on 12-month averages of inflation rates, this notion is confirmed by the *Johansen*-tests and the tests according to *Engle/Granger* which are now applicable.

3. Cointegration of bond yields and inflation rates

The results of the last chapter give rise to the suspicion that there is no cointegration of bond yields and inflation rates since this is a special case of the hypothesis which was already rejected. On the other hand it is conceivable that by the (erroneous) inclusion of dividend yields, the cointegration tests were loaded with too much noise. Therefore the bivariate cointegration tests could in fact discover a cointegration vector for bond yields and inflation rates.

First the results according to *Engle/Granger*:

If we choose the bond yield as dependent variable, the hypothesis of no cointegration can be rejected at least at the 90% level. Since the cointegration of bond yields and inflation rates is economically very intuitive we now estimate an error correction model that includes the deviation from the long-term equilibrium as additional regressor. In this connection we have to face the following problem: As usual the results of the cointegration regression differ from the ones of the reverse regression. The first cointegration regression yields the following equilibrium relationship:

$$U_t = 0,06043358 + 0,4384696 \cdot I_t , \quad (23)$$

from the reverse regression follows:

$$\begin{aligned} I_t &= -0,06102332 + 1,278543 \cdot U_t \quad \leftarrow \\ U_t &= 0,0477288 + 0,782140 \cdot I_t . \end{aligned} \quad (24)$$

In general it cannot be decided which regression should be preferred. On the other hand the tests on the residuals and the economic intuition³² suggest, that the regression with the bond yields as dependent variable should return the "true" cointegrating vector. The above mentioned problem is nonexistent in the *Johansen* approach. Moreover, as we will see later on, the estimated parameters according to *Johansen-Verfahren* are very close to the results of the first cointegration regression. Therefore we use the residuals of the first cointegra-

tion regression for the estimation of the error correction model. We get:

$$\begin{pmatrix} \Delta U_t \\ \Delta I_t \end{pmatrix} = \begin{pmatrix} 0,0968659 & 0,1083536 \\ 0,0597997 & 0,1253459 \end{pmatrix} \begin{pmatrix} \Delta U_{t-1} \\ \Delta I_{t-1} \end{pmatrix} + \begin{pmatrix} -0,0670338 \\ 0,013366 \end{pmatrix} \begin{pmatrix} -0,06043358 & 1 & -0,4384696 \end{pmatrix} \begin{pmatrix} 1 \\ U_{t-1} \\ I_{t-1} \end{pmatrix} + \begin{pmatrix} \eta_{1t} \\ \eta_{2t} \end{pmatrix}. \quad (25)$$

In order to provide output that can be compared to the *Johansen* one we estimate the following model:

$$\begin{pmatrix} \Delta U_t \\ \Delta I_t \end{pmatrix} = \begin{pmatrix} 0,0298321 & 0,1377459 \\ 0,0731656 & 0,1194853 \end{pmatrix} \begin{pmatrix} \Delta U_{t-1} \\ \Delta I_{t-1} \end{pmatrix} + \begin{pmatrix} -0,0670338 \\ 0,013366 \end{pmatrix} \begin{pmatrix} -0,06043358 & 1 & -0,4384696 \end{pmatrix} \begin{pmatrix} 1 \\ U_{t-2} \\ I_{t-2} \end{pmatrix} + \begin{pmatrix} \eta_{1t} \\ \eta_{2t} \end{pmatrix}. \quad (26)$$

Turning to the cointegration tests according to *Johansen*, the hypothesis of the cointegration of bond yields and inflation rates is not confirmed further. Nevertheless, in order to draw comparisons, the vectors α and β are estimated according to the *Johansen*-approach; we find:

$$\hat{\alpha} = \begin{pmatrix} -0,065 \\ 0,019 \end{pmatrix}, \quad \hat{\beta} = \begin{pmatrix} -0,058 \\ 1 \\ -0,484 \end{pmatrix}.$$

These values are very similar to the ones we estimated with *Engle/Granger*.

As the question whether there is cointegration or the variables U_t and I_t are stationary could not be clearly be answered, in the following we will pursue both alternatives further: a pure VAR-model and then a VAR subsystem of the clearly not cointegrated variables are estimated.

G. Results of the estimations of VAR-models

First we will estimate a VAR-model which describes the evolution of the natural logarithms of the equity performance index and the bond performance index, the 10-year bond yields and the 12-month geometric means of inflation rates.

The econometric methods for estimating a VAR model that were referred to in chapter D do not assume that unit roots are absent. Therefore we have the choice whether we want to include the time series in differenced or undifferenced form. Since the differences of logarithms of the performance indices are good approximations of the monthly returns of these asset classes we will use the logarithms of the DAFOX and the REXP in differenced form. The bond yields and the inflation rates enter in the levels, however. This has several reasons: Whereas the differences of the logarithms of the performance indices have a positive mean, the differences of bond yields and inflation rates should have zero mean in the long run. In a VAR model which contains the bond yields and the inflation rates in differenced form there is no meaningful parameter restriction that guarantees such a zero mean. Moreover, the results of the integration and cointegration tests show that the hypothesis of one unit root in the time series of bond yields and inflation rates is much more debatable than with respect to the other two time series. In case the estimated VAR model with the levels of bond yields and inflation rates is stable this would imply that bond yields and inflation rates gravitate to a long-term mean, which is a very intuitive notion.

Let $DAFOX_t$ denote the value of the performance index DAFOX at time t . We define:

$$y_{1t} := \Delta \ln(DAFOX)_t = \ln(DAFOX)_t - \ln(DAFOX)_{t-1} \quad . \quad (28)$$

Likewise we define y_{2t} with respect to the bond performance index REXP. The components y_{3t} and y_{4t} denote the 10-year bond yields and inflation rates, respectively.

The estimated model with lag order 1 is³³ (**Modell I**):

$$\begin{pmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \\ y_{4t} \end{pmatrix} = \begin{pmatrix} 0.03466149 \\ -0.01100296 \\ 0.004046133 \\ 0.0004230372 \end{pmatrix} + \Pi_1 \begin{pmatrix} y_{1(t-1)} \\ y_{2(t-1)} \\ y_{3(t-1)} \\ y_{4(t-1)} \end{pmatrix} + \begin{pmatrix} e_{1t} \\ e_{2t} \\ e_{3t} \\ e_{4t} \end{pmatrix} \quad \text{with} \quad (29)$$

$$\Pi_1 := \begin{pmatrix} 0.07811434 & 0.2397818 & -0.4273814 & 0.09694949 \\ -0.04240155 & 0.3024532 & 0.2529554 & -0.1007636 \\ 0.007994871 & -0.04985398 & 0.9352191 & 0.03037756 \\ -0.006283244 & -0.01939144 & 0.004542831 & 0.9848420 \end{pmatrix} .$$

This process is stable since each eigenvalue is inside the unit circle.

In order to determine the equilibrium vector of a stable vectorautoregressive process we use the following relationship concerning the expected values³⁴:

$$E(Y)_t = \Pi_1 E(Y)_{t-1} + \dots + \Pi_k E(Y)_{t-k} + c \quad . \quad (30)$$

In the equilibrium we have:

$$E(Y)_t = E(Y)_{t-1} = \dots = E(Y)_{t-k} := \mu \quad . \quad (31)$$

This yields the equilibrium vector μ :

$$\begin{aligned} (I - \Pi_1 - \dots - \Pi_k)\mu &= c \Leftrightarrow \\ \mu &= (I - \Pi_1 - \dots - \Pi_k)^{-1}c \quad . \end{aligned} \quad (32)$$

For our estimated model ($k = 1$) we calculate:

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} = \begin{pmatrix} 0.0074515749 \\ 0.0060884222 \\ 0.077582266 \\ 0.040282160 \end{pmatrix}. \quad (33)$$

It should be remembered that the third and fourth component already represent expected values for an annualised yield (or inflation rate) whereas the first two components are monthly expected values.

As an alternative one can use **Modell II**, which contains as sub-systems the error correction model (25) of the last chapter and a VAR model for y_{1t} and y_{2t} . For the VAR sub-system we get:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} 0.005311839 \\ 0.004794468 \end{pmatrix} + \begin{pmatrix} 0.08971997 & 0.2694552 \\ -0.04792808 & 0.2818662 \end{pmatrix} \begin{pmatrix} y_{1(t-1)} \\ y_{2(t-1)} \end{pmatrix} + \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix}. \quad (34)$$

This process is also stable; for the expected values we find:

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0.0076603251 \\ 0.0061650396 \end{pmatrix}. \quad (35)$$

H. Conclusion

The cointegration between dividend yields and bond yields, which has been discovered by *Mills* for the UK capital market, could not be shown for the German market. Whereas the hypothesis of cointegration between dividend yields, bond yields and inflation rates, which is economically quite intuitive, is also not supported by the data, cointegration between bond yields and inflation rates cannot unequivocally be ruled out. As a consequence, two different models are estimated, of which one is a pure vectorautoregressive model.

Endnotes

1. See Lütkepohl (1991), p. 11.
2. See for example Fuller (1976), p. 47-52.
3. See Lütkepohl (1991), p. 350.
4. See Engle/Granger (1987), p. 253.
5. See Engle/Granger (1987), p. 254.
6. See Johansen/Juselius (1990), p. 170.
7. In most cases error correction models are represented in such a way that $\alpha\beta'Y_t$ is subtracted. The representation of the text was chosen in order to be consistent with the notation of *Johansen*.
8. For an overview see for example Farrell (1985) or Sharpe/Alexander (1990), pp. 463 - 472 and the literature quoted there. The application of the Dividend Discount Model to the whole asset class "equities" was suggested in Einhorn/Shangquan (1984) among others.
9. See Shiller (1981), p. 424. This formula assumes a constant discount factor which is theoretically not satisfactory but for our empirical purposes clearly sufficient.
10. See for example Sharpe/Alexander (1990), chapter 16.
11. This means that expected growth rates are independent of the expected inflation rates.
12. See for example Shiller (1981) or Einhorn/Shanquan (1984).
13. This is statistically not correct but can be assumed to be a good approximation of the calculus of the market participants.
14. In reality e_t will be influenced by certain factors (for example the expectations concerning the volatility of equity and bond returns. But even in case of e_t not being a constant, the other variables may be cointegrated as long as e_t is stationary.
15. In Germany there is a high continuity concerning dividends, which justifies such an approximation.
16. See Mills (1991), p. 242-251.
17. See Fisher (1930).
18. See Fuller (1976), p. 366 - 382.
19. See Dickey/Fuller (1979) and Dickey/Fuller (1981).
20. See Dickey/Pantula (1987), p. 455 f.
21. In Dickey/Pantula (1987) two methods are presented that circumvent the above mentioned problems. Simulation studies point to the superiority of the sequential t-test outlined in the text versus the sequential F-test, see Dickey/Pantula (1987), p. 459 f.

22. This approach was first presented in the seminal work Johansen (1988). Generalizations and applications are discussed in Johansen/Juselius (1990), Johansen (1991a) and Johansen (1991b).
23. Although the asymptotic properties of the estimator proposed in Engle/Granger (1987) were derived in Stock (1984) and Stock (1987), the estimation and test procedure is usually referred to as *Engle/Granger* approach. See for example Rüdell (1989), p. 51 or Perman (1991), p. 14.
24. Note that we can write $\Pi = \alpha\beta'$ with $\text{rank}(\alpha) = \text{rank}(\beta) = \text{rank}(\Pi)$ for $r = 0$ and $r = p$, too. A model with $\text{rank}(\Pi) = r_0$ can be represented as a restricted example of the more general model with $\text{rank}(\Pi) = r_1$ and $r_1 > r_0$.
25. The value of the likelihood function for the **estimated** parameter θ is the maximum likelihood value by definition with respect to θ .
26. For the method of Generalized Least Squares (GLS) see for example Greene (1993), p. 358 - 366.
27. Please refer to Stephan (1995), p. 243-244 and Appendix 5.
28. Please refer to Stephan (1995), p. 245-259.
29. The critical value for the 90% level is 2,98 and can be found in table 3 of Engle/Yoo (1991), p. 127.
30. See Juselius (1991), p. 3.
31. See Juselius (1991), p. 3.
32. It is much more intuitive to conceive of the bond yields being influenced by the inflation rates than vice versa.
33. At the lag order $k = 1$ the maximum value of the *Hannan-Quinn*-criterion is reached.
34. This follows from the general representation of a VAR(k)-process. Please note, that the vector of constants, which was denoted in (1) by μ , is now denoted by c in order to be distinguished from the vector of expectations.

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