# Volatility Forecasting with Nonlinear and Linear Time Series Models: A Comparison

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Abstract

In recent years empirical research on financial markets has focused more and more on modelling anomalies of expected asset returns' probability distributions. While it is widely undisputed that the volatility clustering phenomenon can be fitted quite well by ARCH/GARCH type models, the out-of-sample forecasting performance of those models needs further scrutiny. In this paper, the quality of out of sample forecasts obtained from a variety of nonlinear volatility models which have been recently introduced into the literature is evaluated and compared to the forecasting performance of older nonlinear models and a standard linear model.

Résumé

Au cours de ces dernières années, la recherche empirique du marché financier s'est de plus en plus concentré sur le modellage d'anomalies dans le calcul de probabilité des revenus de l'actif financier. Tandis qu'il es en principe incontesté que le "volatility clustering" peut clairement être expliqué à l'aide de modelles du type ARCH/GARCH, la probabilité de prévision en ce qui concerne le genre des modelles est en principe encore inéxplorée. Dans ce rapport, la possibilité de pronostics de nouveau modelle non linéaire de volatilité est comparée avec la qualité des pronostics de modelles de progression temporelle, traditionels linéaires et anciens non linéaires.

Keywords

ARCH/GARCH models, forecasting with nonlinear volatility models.

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## Volatility Forecasting with Nonlinear Time Series Models: A Comparison

Until about 10 years ago, empirical research on financial markets focused on explaining and predicting the expected return on an asset. These studies generally assumed that price changes of financial assets are normally distributed. Yet as some researchers (Mandelbrot (1963), Fama (1965)) pointed out early on, the empirical probability distribution of asset returns exhibits some anomalies that are at odds with the assumption of a normal distribution. For instance, the distribution of returns often exhibits "lepto-kurtosis", meaning that small as well as large upward or downward fluctuations of returns occur more frequently than would be the case under a normal distribution. In particular, their distribution exhibits fat tails. For forecasting purposes, this property of asset returns would be quite uninteresting if large positive or negative fluctuations occurred at random over time. Yet this is not the case. Rather small and big swings in asset returns occur in clusters. Thus, in a period of high market volatility, future returns are expected to also exhibit above-average volatility. As a consequence, absolute values of asset returns should be autocorrelated. This hypothesis forms the starting point for the time series approaches to predicting volatility that are discussed below.

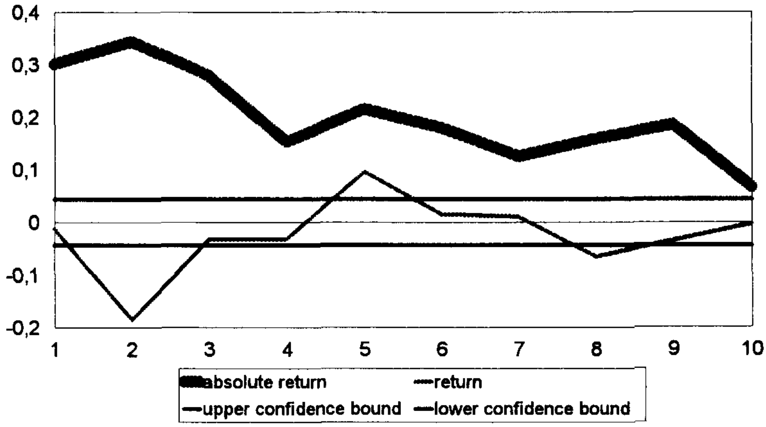
How can volatility forecasts be of help to investors managing their assets? They are useful in at least two ways. First, prices of stock and interest rate derivatives depend on the future volatility of the underlying instrument. Therefore, a sound volatility forecast is a valuable input in option pricing models (Schmitt 1995). Second, reliable volatility forecasts can be used to construct improved interval forecasts of asset returns.

### Volatility clustering: Empirical evidence for Germany and the United States

As has been argued above, the phenomenon of volatility clustering can be illustrated by computing autocorrelations of absolute returns. For the S&P 500-Future, autocorrelation coefficients of order 1 to 10 and their 95%-significance band are displayed in Chart 1.

## Chart 1: Autocorrelation of Returns

(S&P 500-FUTURE)

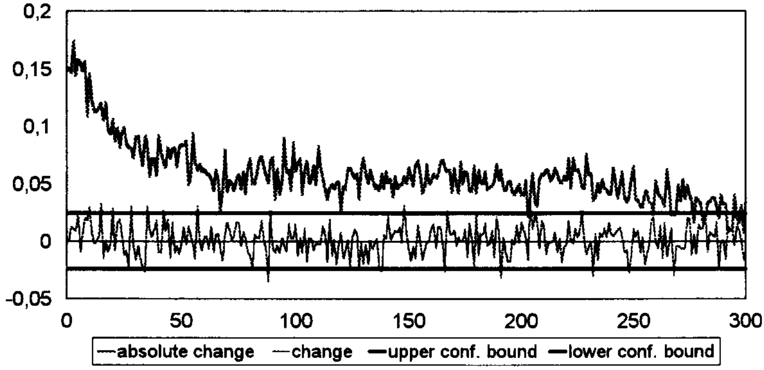



Clearly, the null of absence of autocorrelation among current *absolute* returns has to be rejected for all orders up to 10. By contrast, 7 out of 10 autocorrelation coefficients for the corresponding return series lie within the 95%-confidence interval. For the DAX-Future, almost the same result obtains: while 9 out of 10 autocorrelation coefficients for absolute returns are outside the confidence band, returns themselves cannot be said to display serial correlation of any order at the 95% confidence level. The D-mark/U.S. dollar exchange rate exhibits the same "anomaly" to an even higher degree. Chart 2 shows that the autocorrelation coefficients of order 1 to 250 lie outside the 95% interval, implying that absolute returns follow a process with a very long memory (see also Ding, Granger and Engle (1993)).

Ideally, a statistical model should reflect all of the above-mentioned properties of the data: lepto-kurtosis, uncorrelated returns *and* serially dependent absolute returns. Unlike traditional models, nonlinear time series models claim to take all of these factors into account. To find out whether the new approaches are superior in forecasting financial-market volatility, we will first examine the forecasting accuracy of a standard linear model. This will serve as a point of reference for our subsequent analysis of nonlinear models.

Chart 2: Autocorrelation of Changes

(DM/USD Exchange Rate)




**Volatility forecasting with a simple linear time series model**

Given the above-mentioned autocorrelation between absolute returns, it seems natural to specify an ARMA-model of absolute returns, x. More specifically, on the basis of the usual correlation diagnostics, the following ARMA(3,0)-model was fitted to the data:

(1) $x(t)=\alpha+\beta x(t-1)+\gamma x(t-2)+\delta x(t-3)+\varepsilon(t)$.

The parameters of the model were repeatedly estimated on a moving time window of constant length using daily data for the DAX and the S&P 500 futures contract. After each reestimation, a 1-step-ahead out of sample forecast of the absolute return for the next day was obtained. The result were 300 forecasts. In order to evaluate the relative forecasting accuracy of the model, the mean square forecast error was divided by the mean square forecast error that would have occured if one had simply extrapolated the average absolute return from the past. If the ratio U is smaller than 1, the forecast based on the respective model is more accurate on average than one that predicts volatility (as measured by absolute return) not to change over time. It turns out that the linear time series model introduced above cannot achieve a significant outperformance of a naive constant volatility forecast for the DAX-Future (U=1.0), whereas a 22% outperformance (U=0.78) results for the S&P 500 Future. However, it should not go unmentioned that the good relative performance of the model in the U.S. market is in part attributable to the stock market crash in October 1987.

**Volatility forecasting with nonlinear time series models**

In the last years, a special class of nonlinear models has been developed that mimic the typical distribution patterns of financial returns mentioned above. These models are called autoregressive conditional heteroskedasticity (ARCH) models. The basic ARCH model as introduced by Engle (1982) has the following basic structure:

$$(2)\ r(t) = \mu + \varepsilon(t),$$

where it is assumed that

$$(3)\ \varepsilon(t) = v(t)\left[a + b\varepsilon(t-1)^2 + c\varepsilon(t-2)^2 + d\varepsilon(t-3)^2 + \ldots\right]^{1/2}$$

Here r(t) denotes the actual return on the respective asset at time t, $\mu$ the average return, $\varepsilon(t)$ the unexpected return determined by random factors, and $v(t)$ an i.i.d. standard normally distributed random variable. Parameters b,c,d,... are supposed to be non-negative, while $a>0$ is assumed. Based on these assumptions, it becomes transparent that the current conditional standard deviation s(t) of the return generically depends positively on squared random factors $\varepsilon(t-1)^2$, $\varepsilon(t-2)^2$, $\varepsilon(t-3)^2$... in the recent past. This phenomenon is described as autoregressive conditional heteroskedasticity which essentially means that volatility is high when it has also been high in the past and low if the absolute return has fluctuated minimally in the recent past. By contrast, if we assume that parameters, b,c,d..., in equation (3) are equal to zero, we obtain a traditional homoskedastic model with a time-independent (conditional) variance as a special case.

**Extensions of the ARCH-model**

Since the pioneering work by Engle, the above-described basic model has been modified and broadened to adapt it better to the characteristics of specific financial market data. Older extensions, which are well known by now, include the GARCH model proposed by Bollerlev (1986) , the ARCH in mean (ARCH-M) model by Engle, Lilien, and Robins (1987) and the Taylor (1986) & Schwert (1989) model (henceforth: T/S model).

For a discussion of those models we refer the reader to the literature (e.g. Hamilton (1994). We want to concentrate on three models that have been introduced more recently into the literature:

(i) the asymmetric generalized **ARCH** (A-GARCH) model (see Nelson (1991)),

(ii) the asymmetric power **ARCH** (A-PARCH) model ( see Ding, Granger, and Engle (1993)), and

(iii) the autoregressive volatility (AV) model (Hsieh (1995)).

**1) The A-GARCH model:**

This model expands the standard approach in two directions. First, the conditional variance of the return at time t, $s(t)^2$, *explicitly* depends upon the lagged value $s(t-1)^2$. Second, the model takes into account that returns tend to react differently to good and bad news.

This asymmetric pattern is captured by adding positive past return surprises to the volatility equation, while negative surprises are neglected. Formally, we obtain the following structure:

$$(4)\ s(t)^2 = a + b\,\varepsilon(t-1)^2 + c\,s(t-1)^2 + d\,\chi(t-1)\varepsilon(t-1),$$

where, as before, $a>0$ and $b,c,d\geq 0$. In addition, $\chi(t-1)=1$ if $\varepsilon(t-1)>0$ and $\chi(t-1)=0$ otherwise.

**2) The A-PARCH model:**

This approach is more general than the A-GARCH model insofar as all lagged variables do not appear with a fixed exponent, but with an arbitrary exponent, $\delta$, which is subsequently estimated (see Ding, Engle and Granger (1993)). Thus, the volatility equation of the A-PARCH model is given by the following expression (where $|\bullet|$ denotes absolute values and all parameters are restricted as before except that $-1<\beta<1$ and $\delta\geq 0$ is imposed):

$$(5)\ s(t)^{\delta} = a + b(|\varepsilon(t-1)| - \beta\varepsilon(t-1))^{\delta} + d\,s(t-1)^{\delta}$$

**3) The AV model:**

In contrast to results obtained from applications of GARCH-type models, Hsieh (1995) presents evidence that volatility of the DM/USD exchange rate is strongly mean-reverting. In order to capture this effect, the following multiplicative volatility equation is specified in the AV model (see Hsieh (1995, p. 60)):

$$(6)\ s(t) = \exp\{a\}\exp\{\varepsilon(t)\}s(t-1)^{b}s(t-2)^{c}s(t-3)^{d}..$$

To evaluate the forecasting quality of all three models, we first used the Berndt-Hall-Hall-Hausman algorithm to estimate the parameters of the log-likelihood function on the data set described above. As for the linear AR-model, we then computed the U-ratio for 300 out of sample one-day-ahead predictions. Table 1 compares the performance of the models with that of the older nonlinear time series models mentioned before.

**Table1: Performance of Nonlinear Volatility Models**

| model | coefficients stable over time ? | | value of log-likelihood | | U-ratio | |
|---|---|---|---|---|---|---|
| | DAX-Future | S&P500-Future | DAX-Future | S&P500-Future | DAX-Future | S&P500-Future |
| ARCH | yes | yes | 5062 | 8484 | 0.96 | 0.64 |
| ARCH-M | no | yes | 5046 | 8486 | 0.98 | 0.64 |
| GARCH | yes | no | 5058 | 8533 | 0.98 | 0.62 |
| T/S | yes | yes | 5050 | 8524 | 0.98 | 0.63 |
| A-GARCH | yes | yes | 5067 | 8547 | 0.98 | 0.62 |
| A-PARCH | no | no | 5055 | 8572 | 0.97 | 0.69 |
| AV | yes | no | 5027 | 8470 | 0.99 | 0.65 |

The first two columns of Table 1 indicate that especially the A-PARCH model suffers from parameter instability. Therefore, it does not seem to be well suited for forecasting purposes, although its explanatory power (as measured by the value of the maximized likilihood) is high especially for S&P 500 returns.

As far as the U-ratio criterion is concerned, it is immediately apparent that all models deliver better forecasts on average than (a) the naive constant volatility model and (b) the linear AR-model discussed above. In particular, the outperformance is significant for S&P 500 Future data. Thus, volatility clustering seems to be more pronounced in U.S. than German data.

In order to explore whether the results above critically depend on the specific data that have been used and/or the chosen forecast horizon, the analysis was extended in both directions. Table 2 summarizes the relative forecasting accuracy of the A-GARCH and the AV models over a period from 1 to 5 days on data sets including return data on the BUND and US Bond Future as well as day to day changes of the DM/USD exchange rate. The A-PARCH model has been excluded due to parameter instability.

**Table 2: Forecasting performance of nonlinear volatility models (forecasting horizon: 1 to 5 days)**

| | U-ratio | | | | |
|---|---|---|---|---|---|
| model/fore-casting horizon | DAX-Future | S&P500-Future | BUND-FUTURE | FUTURE US-BONDS | DM/$ exchange rate |
| A-GARCH | | | | | |
| 1 day | 0.98 | 0.61 | 0.97 | 1 | 1 |
| 2 days | 0.96 | 0.61 | 0.96 | 0.99 | 1 |
| 3 days | 0.95 | 0.60 | 0.96 | 0.99 | 0.99 |
| 4 days | 0.96 | 0.59 | 0.97 | 0.99 | 0.99 |
| 5 days | 0.96 | 0.59 | 0.98 | 0.99 | 1 |
| AV | | | | | |
| 1 day | 0.99 | 0.65 | 1 | 1 | 1 |
| 2 days | 0.99 | 0.65 | 1 | 1 | 1 |
| 3 days | 0.99 | 0.65 | 1 | 1 | 1 |
| 4 days | 0.99 | 0.65 | 1 | 1 | 1 |
| 5 days | 0.99 | 0.65 | 1 | 1 | 1 |

It seems fair to say that the results so far are confirmed in table 2. Overall, the evidence presented here supports the tentative conclusion that one should not be overly optimistic regarding the out of sample forecasting performance of ARCH/GARCH type volatility models.

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