Optimal Optioned Portfolios with Confidence Limits on Shortfall Constraints

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Abstract

In this paper we examine the problem of managing portfolios consisting of both, stocks and options. Due to the resulting asymmetric portfolio return distribution we do not use mean variance analysis but represent the preferences of the investors in terms of confidence limits on down side risk measures. For the simultaneous optimization of the stock and option positions we derive portfolios with a maximum expected return under a given preference structure expressed by shortfall constraints. To identify the optimal optioned portfolio we derive an approximation of the return distribution. The solution identified by this procedure will dominate comparable portfolios derived by using mean variance analysis. On the basis of Monte Carlo Simulations we will illustrate our results and demonstrate the stochastic dominance of these solutions.

Résumé

Dans ce papier nous examinons le problème d'arranger les portefeuilles qui portent des actions et des options. La rentabilité espérée résultante de la distribution de la portefeuille asymétrique nous n'utilisons pas l'analyse moyenne-variance. Pour l'optimisation simultanée des positions d'action et d'option nous dérivons des portefeuilles avec une rentabilité espérée maximale sous une structure préferée exprimée par des restrictions des pertes. Pour identifier l'option-portefeuille optimale nous dérivons une approximation de la distribution de rentabilité. La solution identifiée par cette procédure dominera des portefeuilles comparables dérivées en utilisant l'analyse moyenne-variance. Nous illustrons ces résultats avec des simulations de Monte Carlo et nous démontrerons la dominance stochastique de ces solutions.

Keywords

Portfolio management, options, stochastic dominance, Monte Carlo simulation.

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1 Introduction

Common characteristics of option positions within the portfolio management context are their ability to adjust the risk position of a portfolio according to an individual level desired by the investor. The virtually unlimited ways to combine options with other financial instruments allow investors also to bet directly on individual estimations about the price behavior of the underlying instruments. In this sense options expand the investment spectrum of financial markets and allow a more efficient allocation of risk according to individual characteristics and risk preferences of different market participants. Thus, the effectiveness of those markets can be increased by options.

Traditional portfolio selection is based on mean variance analysis. Specifying only mean and variance of a return distribution function of an underlying instrument, however, is not sufficient to model the characteristics of the asymmetric return distribution resulting for optioned portfolios. Investors' risk preferences are therefore not correctly expressed within a mean variance framework.

The following article will look at a procedure to optimize portfolios consisting of both, options and stocks. The investors' risk preferences are represented in terms of downside risk measures which provide a more general concept of risk than the traditional notion of volatility. In our case individual risk-return preferences are expressed by limits on shortfall constraints to an investor specific benchmark.

To circumvent the technical optimization problems arising from stochastic constraints we use an approximation of the return distribution under which the complex optimization problem can be transformed into a linear problem being comparably easy to solve. The model approach used here builds up on some previous work presented by BOOKSTABER/CLARKE, (1983), PELSSER/VORST (1990) and SCHEUENSTUHL/ZAGST (1995).

The results of this optimization procedure are illustrated using an empirical example based on the German stock market. We will compare this optimal optioned portfolio with portfolio compositions based on traditional mean variance models. In case of strongly asymmetric return distributions, it is possible that efficient portfolios in the mean variance approach fail to be an optimal solution in the sense that it presents a solution which is dominated by our optimization result according to the concept of stochastic dominance.

The paper is organized as follows:

In the following section 2 we will outline the portfolio selection problem with options and introduce the theoretical framework upon we will model the optimization problem.

Using a case study and data material from actual financial markets we illustrate the optimization procedure and analyze the solution in section 3. Monte Carlo simulations are used to characterize the return distribution of the optimal optioned portfolio.

In the final part 4 we will compare this portfolio composition to portfolios selected on a comparable mean variance optimization basis. A summary completes the article.

2 Portfolio Optimization with Options

Motivations to add options to a portfolio are manifold and option based strategies are mostly tailored to individual circumstances. ARDITT/LEVY (1975), BRENNAN/SOLANKI (1981), MERTON/SCHOLES/GLADSTONE (1978) illustrate how options can improve the portfolio decision situation of an investor and how an investor can implement his or her specific market expectations directly into a trading strategy.

Option Induced Asymmetric Return Distributions

Figure 1 illustrates the well known risk reducing and risk controlling effect of option positions. The upper left part of the figure shows the downside protection of put options and their profit and loss effects. The upper right part of the picture shows the profit and loss effects of a more complex butterfly option position limiting the loss side and trying to benefit from an individual expectation on the price behavior of the underlying stock.

The lower left graph of figure 1 displays return density functions of a portfolio with reduced downside risk through the use of a 0%, 25%, 50%, and 75% fraction of (at the money) put options for each stock in the portfolio. Similarly, the graphs on the lower right side illustrates the resulting return density functions when we use both, long put and short call positions in connection with the underlying stock position. The option positions used in this example are symmetric and at the money.

These examples highlight two characteristic features of option positions: on the one hand, the reduction of the downside potential of the portfolio, and on the other hand, the more skewed density functions for optioned portfolios generated by the skewed chance-risk profile of these position in contrast to the typical symmetric return distribution of stock only portfolios. Because it is practically impossible to calculate the resulting portfolio return distribution in an explicit mathematical way (BOOKSTABER/CLARKE (1983)) the distribution functions of the above examples were calculated using Monte Carlo Simulation techniques.

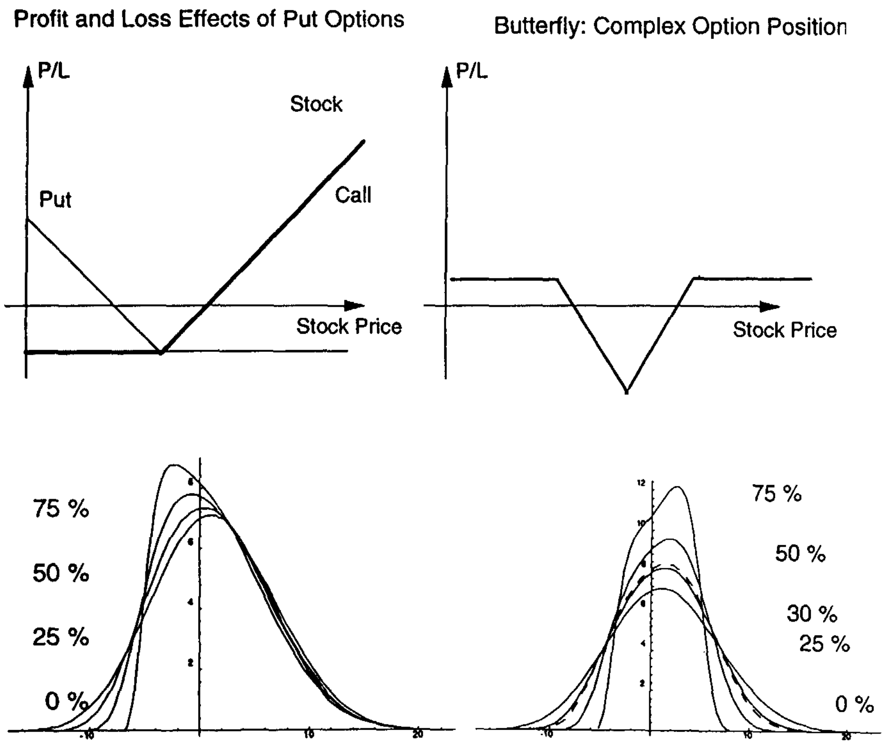

Figure 1: Profit and loss effects of two option positions and their corresponding return density functions

Mean Variance Analysis versus General Downside Risk Approach

The selection of an optimal portfolio is equivalent to choosing a distribution function which in principle is meant to maximize the utility function of the investor. The traditional portfolio theory introduced by MARKOWITZ (1952) and SHARPE (1970) is based on the concept of mean-variance analysis where only two parameters, the mean and the variance of the portfolio return determine the optimal portfolio choice. For symmetric return distributions this reduced characterization of the stochastic nature of the portfolio result might be a sufficient approximation. However, in the case of options this is typically not the case when already a small number of options added to a portfolio leads to a strongly asymmetric return distribution.

The so-called *'Mean-Variance-Paradox'* illustrates that under such asymmetric conditions mean-variance based decision criteria may fail to come up with an utility maximizing solution. They may even violate basic criteria such as first and second order stochastic dominance criteria (COPELAND/WESTON (1991), p. 276). Especially, the variance or standard deviation will not provide sufficient statistical information about what investors conceive to be risk. Decision criteria based solely on expected return and standard deviation may not be compatible with the notion of the Bernoulli Principle and rational decision making (see BAMBERG/COENENBERG (1987)).

To resolve the problem we suggest a more general decision criteria based on the notion of *lower partial moments* (LPM). This concept reaches back to the beginning of portfolio selection (ROY, (1952)) and has been applied in other investment decision situations (LEIBOWITZ / KOGELMAN (1987, 1989, 1996), HARLOW (1992)). According to HARLOW (1992) we will define the lower partial moment of order n and corresponding to a investor given benchmark B as:

$$LPM_n(B) \quad = \quad \int_{-\infty}^{B} (B - r_P)^n dF_{R_P}(r_P) \tag{1}$$

where F_{R_P} is the distribution function of the return R_P of portfolio P.[1] Depending on the parameter n, different aspects of the downside risk of a portfolio can be expressed. For orders $n = 0, 1, 2, 3$ these number have an intuitive meaning (HARLOW, (1992) p.23).

By looking only at portfolio returns falling short of a given minimal investment target, the lower partial moments define risk measures which are intuitively appealing to the risk perception of an investor (danger of loss). Deviations exceeding the minimal return goal are typically not experienced as risk in the sense of an unpleasant event. Specific risk attitudes of investors can be expressed by using different orders n and/or different benchmarks B, respectively. Simple shortfall conditions addressing only the probability of falling below the benchmark are described by the LPM of order 0.

The shaded area under the density function in figure 2 represents a graphical illustration of the shortfall risk α given a certain benchmark B. For the cumulative distribution function of a portfolio return this type of preference constraint is illustrated in the lower picture of figure 2. Only those portfolios with return distribution function lying below the shaded area which is determined by a set of three different shortfall constraints satisfy the preferences of the investor.

Graphical illustration of the shortfall risk and shortfall constraints given a certain benchmark B and a shortfall probability α.

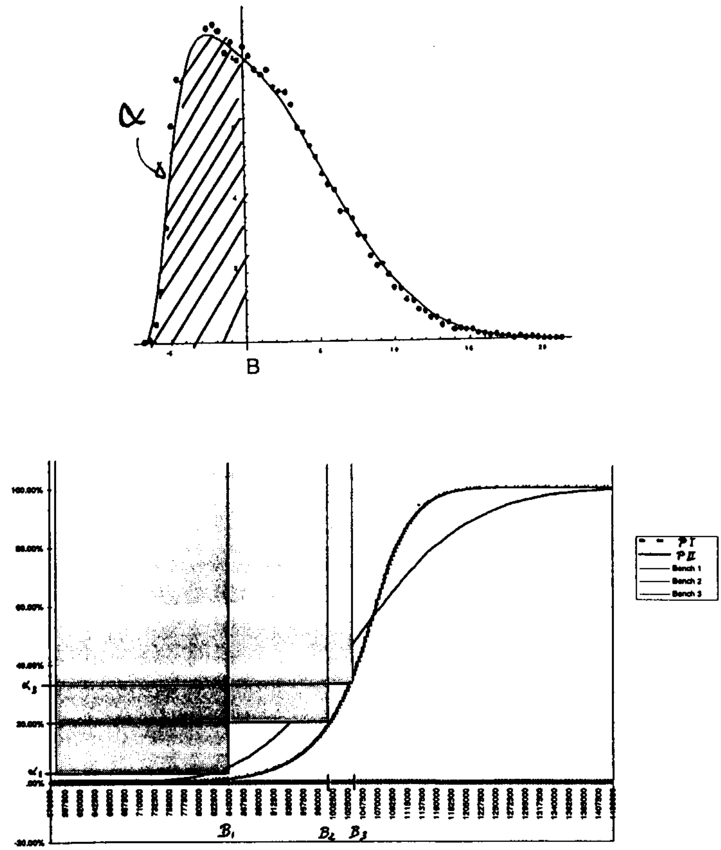

Figure 2: Shortfall Constraints and Return Distribution Function

This concept implicitly assumes that investors are able to express their preferences in terms of a minimal portfolio return they want to achieve and which can be lower only with a predefined shortfall probability α. For example, an investor wants to invest her money only in a portfolio requiring that the chance (probability) of the realized return R_P at the end of the investment horizon being below her predefined goal of $B = 3\%$ is less than a shortfall probability of $\alpha = 10\%$. The corresponding formal shortfall constraint for the portfolio selection would then be:

$$Prob\{R_P \leq 3\%\} \;\; \leq \;\; 10\%.$$

Shortfall Concept and Value-at-Risk: At this point we want to mention that the concept of lower partial moments is closely related to the notion of *Value-at-Risk* (VaR) which, currently, is discussed extensively among bank regulators and risk managers in financial institutions. For a market position or a trading book with return R_P and shortfall probability α, the Value-at-Risk number is just the maximal B with $LPM_0(B)$ being less or equal than α.

The Optioned Portfolio Optimization Problem

Considering the problems arising from asymmetric return distributions, our approach of portfolio optimization with options is based on the risk concept of shortfall constraints. The model assumes that investment decisions are based on a one period investment horizon $[0, T]$ and no changes of the portfolio structure are made within that holding period.[2]

Our goal is to maximize the expected return of the total portfolio value (PortVal(T)) satisfying the investor's risk preferences and respecting his investment framework:

- The investor has given a budget of b units of money which is to be invested in financial securities.
- The capital market under consideration allows us to invest in $n \in \mathbb{N}$ different stocks. In addition to that, for each stock $i \in \{1, \ldots, n\}$ there exist different call and put option series with exercise prices K_{ij}, $j \in \{1, \ldots, n_i\}$ and maturity dates T_{ijk}, $k \in \{1, \ldots, m_i\}$.
- Furthermore, this includes the possibility to deposit money with a bank account and receive a risk free rate of interest of $r_f > 0$ during the planning horizon.[3]

Then the value $PortVal(t)$ of a portfolio of these investment alternatives at time $t \in [0, T]$, is given by

$$PortVal(t) \;\; = \;\; \sum_{i=1}^{n} \left\{ \alpha_i S_i(t) + \sum_{j=1}^{n_i} \sum_{k=1}^{m_i} \{\beta_{ijk} C_{ijk}(t, T_{ijk}) + \gamma_{ijk} P_{ijk}(t, T_{ijk})\} \right\}, \quad (2)$$

where α_i, β_{ijk}, and γ_{ijk} denote the (decision) variables about the number of stocks, calls, and puts which should be held in the portfolio.

The risk preferences of the investor are specified in terms of shortfall constraints of the following form:

$$\text{(S)} \qquad Prob\{PortVal(T) \geq B(\alpha)\} \;\; \geq \;\; 1 - \alpha \, ,$$

with $B(\alpha)$ being a benchmark or threshold return explicitly given by the investor and the portfolio return which might not fall short of this benchmark with a probability larger than α.

From this framework the following portfolio optimization problem (OP) results which represents the basic type of problem we will discuss in this paper:

$$(OP) \begin{cases} \max\limits_{\alpha_i, \beta_{ijk}, \gamma_{ijk}} \; E[PortVal(T)] & \text{(Z) target function} \\ \text{subject to the following constraints:} & \\ PortVal(0) \leq b & \text{(B)udget constraint} \\ P(PortVal(T) \geq B(\alpha)) \geq 1 - \alpha, & \text{(S)hortfall constraint(s)} \end{cases}$$

To solve the (OP) we need to have some knowledge about the stochastic properties of the portfolio return and thereby about the stochastic properties of single investment alternatives and their cross-correlation among each other. Especially, the stochastic constraint (S) complicates the optimization substantially and we seek to replace it by an approximation which is easier to handle.

Modeling the Decision Situation

Market prices of financial securities follow a geometric Brownian motion. This is a quite common assumption about the stochastic behavior of stock prices and is also a basic assumption about the pricing of options with the BLACK/SCHOLES pricing framework (HULL (1993), Ch. 10).

To consider portfolio effects we model an index linked stock price behavior and express stochastic relationships between securities on the basis of a common stock index (see SHARPE, 1964). Using this approach we do not need to specify all correlation between each of the single securities which would require a vast amount of data material.

The general trend of the market is described by stock index S_M. The price process of this market index follows an Ito-Process governed by the following dynamic

$$dS_M \;\; = \;\; \mu_M S_M dt + \sigma_M S_M dZ_M \tag{3}$$

where:

μ_M instantaneous drift rate of the market index
σ_M^2 instantaneous variance rate of the market index

Z_M standard Brownian motion.

For the stock prices we assume that they follow an index linked continuos price process of the following kind where their relationship to the market is explicitly considered:

$$dS_i = \mu_i S_i dt + \eta_{iM} S_i dZ_M + \eta_{ii} S_i dZ_i \tag{4}$$

where:

μ_i	instantaneous drift rate of stock i
η_{iM}	relation of the instantaneous covariance-rate c_{iM} of stock price S_i and market index S_M and the volatility σ_M of the market index
Z_i	standard Brownian motion

We assume that the parameters $\mu_M, \sigma_M, \eta_{ii}, \eta_{iM}$ are constant over the planning horizon T and that the processes Z_M and Z_i are independent Brownian motions for all $i = 1, \ldots, n$. Therefore the stock prices follow a geometric Brownian motion with instantaneous variance rate $\eta_{iM}^2 + \eta_{ii}^2$. The volatility of a single stock price is therefore given by $\sigma_i := \sqrt{\eta_{iM}^2 + \eta_{ii}^2}$ and the covariance rate between two stock prices S_i and S_j can be calculated using their respective relation to the market index.

In a continuos Capital Asset Pricing Model we have the following relation between the market index S_M and the stock price S_i of a single stock according to MERTON (1972):

$$\mu_i - r = \frac{\eta_{iM}}{\sigma_M}(\mu_M - r) \tag{5}$$

where r denotes the riskless interest rate. The expression

$$\beta_i := \frac{\eta_{iM}}{\sigma_M} = \frac{c_{iM}}{\sigma_M^2} = \frac{\rho_{iM}\,\sigma_i}{\sigma_M}$$

is usually called the beta of stock i and provides a measure of the sensitivity of the stock price with respect to movements of the market index. The correlation coefficient between the stock price S_i and the market index S_M is denoted by:

$$\rho_{iM} := \frac{c_{iM}}{\sigma_i\,\sigma_M}$$

Modeling price dynamics in this (standard) way allows us to make use of some helpful stochastic properties which will be employed later in several steps (HULL (1992), p.207).

Approximation of the Final Value Distribution

As a basic requirement for this approximation to be accomplished we have to assume that the respective portfolio is well diversified. With this assumption the portfolio contains only systematic risk while the unsystematic risk is supposed to be practically zero.[4] Formally, we therefore require the following Approximation Assumption of the portfolio:

$$\text{(AA)} \qquad Var(PortVal(T)|X_M(T) = x) \quad \approx \quad 0 \qquad \text{for all} \quad x \in \mathbb{R}.$$

This assumption clearly illustrates that the contingent distribution function of the final portfolio value for a given market index $S_M(T) = s$ or $X_M(T) = x$, respectively[5] is concentrated around the contingent expected portfolio value $E[PortVal(T)|X_M(T) = x]$. That is:

$$P(PortVal(T) \leq v|X_M(T) = x) \quad \approx \quad \begin{cases} 1, & E[PortVal(T)|X_M(T) = x] \leq v \\ 0, & otherwise. \end{cases} \tag{6}$$

The approximation we use for the optimization procedure is given in SCHEUENSTUHL/ZAGST (1995):

Approximation: Let $\alpha \in (0,1)$, x_α be the α-Fractile of the Standard Normal distribution with $P(X_M \leq x_\alpha) = \alpha$ and $B(\alpha)$ denotes a benchmark with respect to the final portfolio value. Furthermore, be

$$E[PortVal(T)|X_M(T) = x] \geq B(\alpha) \quad \text{for all} \quad x \geq x_\alpha.$$

Then for the final value of the portfolio we have:

$$P(PortVal(T) \leq B(\alpha)) \leq \alpha.$$

Monotonicity Constraints

Showing that $E[PortVal(T)|X_M(T) = x]$ is a monoton, increasing function in x would substantially reduce the computational efforts. With such a monotonicity property it would be sufficient to show that the condition holds for x_α. For all other $x \geq x_\alpha$ it would be satisfied implicitly.

Sufficient conditions for such a monotoniciy are given in SCHEUENSTUHL/ZAGST (1995): If the following conditions hold

$$\text{(M)} \qquad \alpha_i - \gamma_i + \sum_{j=1}^{J} \sum_{k=1}^{K} (\beta_{ijk} + \gamma_{ijk}) \geq 0\,,$$

for all $0 \leq J \leq n_i$, $0 \leq K \leq m_i$, and for all $1 \leq i \leq n$ where we assume that the exercise prices of the options are increasing in j and the time to maturity of the options is increasing in k (for each option), then the function

$$x \mapsto E[PortVal(T)|X_M(T) = x]$$

is increasing for all $x \geq x_S$, where the lower bound x_S for the simpler case of $T_{ijk} = T$ (i.e. all options have their maturity dates T_{ijk} at the planning horizon T) can be set to $x_S = -\infty$.

The Modified Optimization Problem

Combining both results, the approximation and the monotonicity conditions we have sufficient conditions under which the conditional expected final portfolio value does not fall short of a given benchmark $B(\alpha)$ for all realizations of the market index above a special s_α (i.e. for all realizations of $X_M(T)$ above a certain value of x_α) and we can substitute the original probability constraint (S) by a set of linear conditions (M) and a shortfall constraint on the conditional expectation ($\tilde{S}$).

The original optimization problem (OP) thus can be modified into a linear model (LP).

$$(LP) \begin{cases} \max\limits_{\alpha_i,\beta_{ijk},\gamma_{ijk}} \; E[PortVal(T)] & \text{(Z) target function} \\ \text{subject to the following constraints:} & \\ PortVal(0) \leq b & \text{(B)udget constraint} \\ \alpha_i - \gamma_i + \sum\limits_{j=1}^{J}\sum\limits_{k=1}^{K}(\beta_{ijk} + \gamma_{ijk}) \geq 0 & \text{(M)onotonicity constraints} \\ \text{for all} \quad i = 1,\ldots,n, \quad 0 \leq J \leq n_i, \quad 0 \leq K \leq m_i & \\ E[PortVal(T)|X_M(T) = x_\alpha] \geq B(\alpha). & (\tilde{S})\text{hortfall constraint(s)} \end{cases}$$

3 An Application to Optimal Optioned Portfolio Selection

In this section of the paper we discuss possible applications of the model introduced above on the basis of a case study using real financial data. The numerical examples also provide the basis for a comparison of the shortfall based optioned portfolio optimization with classical mean variance portfolio models presented in section 4.

Data Material

The investment spectrum under consideration includes financial instruments available in the German stock market which is represented by the German stock index (DAX) and the corresponding options which are traded at the German Futures and Option Board (DTB). The data on stocks, on put and on call options used in the examples are summarized in table 1 and table 2:

Market Index / Stock	Empirically detected parameters from market prices			Theoretically deduced parameters from our model		
	σ_i	ρ_{iM}	β_{iM}	$\eta_{i,M}$	$\eta_{i,i}$	μ_i^{CAPM}
DAX	0,1247	1,0000	1,0000	0,1247	—	0,1100
BASF	0,1843	0,7911	1,1692	0,1458	0,1127	0,1202
Daimler Benz	0,1582	0,7770	0,9857	0,1229	0,0996	0,1091
Dt. Bank	0,1623	0,7330	0,9540	0,1190	0,1104	0,1072
Kaufhof	0,2114	0,5750	0,9748	0,1216	0,1730	0,1085
VEBA	0,1565	0,7334	0,9330	0,1163	0,1047	0,1060

Data Source: Handelsblatt, July 2, 1996.

Table 1: Data basis for case study

Table 1 provides the basic parameters and stochastic properties of the investment alternatives which we need as inputs for our further calculations. The data material with respect to σ_i, ρ_{iM}, and β_{iM} is based on actual closing rates provided by Handelsblatt on July 2, 1996. The parameters in the second block, η_{iM} , η_{ii}, and μ_i^{CAPM} were deduced from our model introduced above .

For each stock we consider three call and three put options. The options are chosen such that for each stock always one put and one call is *out of the money*, one will be *at the money* , and the other one is *in the money* . The option prices we will use later on are theoretical BLACK/SCHOLES prices based on these parameters. To reduce unnecessary complexity of our calculations we will assume the basic situation where all options have the same maturity $T = T_{ijk}$, and T is the time horizon of our portfolio problem. For the successive calculations we assume $T = 0.5$ years.

The investor may also leave his money on a bank account or buy save government bonds. This investment would yield the current riskless rate of return of $r_f = 5\%$.

Stock i	Market Price $S_i(0)$	Exercise Prices K_{ij} for Puts and Calls		
BASF	43.50	40.00	42.50	45.00
Daimler Benz	80.90	75.00	80.00	85.00
Dt. Bank	72.20	70.00	72.50	75.00
Kaufhof	574.50	550.00	575.00	600.00
VEBA	80.90	75.00	80.00	85.00

Table 2: Available Option

There are, of course, no options on this type of investment available.

The individual situation of the investor under consideration is characterized by his initial budget and his risk preferences: We let $b = 1'000'000$ which is the amount to be (optimally) invested into securities of the above investment spectrum. The investor expressed his risk preferences in terms of the following shortfall constraints:

$$P\{PortVal(T) \geq 0,85\ PortVal(0)\} \geq 0,99 \qquad (S_1)$$
$$P\{PortVal(T) \geq 1,00\ PortVal(0)\} \geq 0,80 \qquad (S_2)$$
$$P\{PortVal(T) \geq 1,03\ PortVal(0)\} \geq 0,66 \qquad (S_3)$$

The example also indicates the different stages an investor goes through to express different aspects of his or her (risk) preferences: The selection of markets and instruments can be considered as a prior step which already reflects a certain attitude of the investor. In this selection we may find for example many ethical or religious based dimensions of the preference. The step of specifying shortfall constraints on the return distribution may be considered to be the more quantitative part of expressing ones preferences. In this sense in each stage of the process the investor expresses different dimensions of his or her individual values. Therefore, this process of stepwise identifying the motives and preferences of an investor seems to be more practically suitable than to assume a rather abstract utility function.

For the case study the investor states three different benchmarks ($-15\%, 0\%$ and 3%) expressing three different aspects of concern:

- With target $B(\alpha_1) = -15\%$ and $\alpha_1 = 1\%$ she defines a lower bound for the portfolio return. The portfolio should not loose more than 15% in all practical cases (i.e. in 97% of all outcomes). This worst case loss is comparable with a Value at Risk (VaR) number for the portfolio.
- With $B(\alpha_2) = 0\%$ the investor expresses her wish to preserve the nominal capital. Only in one fifth of all cases ($\alpha_2 = 20\%$) a nominal loss might occur.
- Furthermore, in at least one third ($\alpha_3 = 33\%$) of all cases the portfolio should produce a positive return of at least 3%, which might be the current rate of

inflation. So the investor's concern about real capital preservation is expressed with this third constraint.

The problem statement leads to the following linear program:

$$(LP)\begin{cases}
\max\limits_{\substack{\alpha_1,\dots,\alpha_5 \\ \beta_{1,1},\dots,\beta_{5,3};\ \gamma_{1,1},\dots,\gamma_{5,3}}} \quad E[PortVal(T)] & (Z) \\
\text{subject to the following constraints:} & \\
PortVal(0) \leq 1\,000\,000 & (B) \\
\alpha_i - \gamma_i + \sum_{j=1}^{J}(\beta_{ij} + \gamma_{ij}) \geq 0 & (M) \\
\text{for all } 1 \leq i \leq 5 \text{ and } 0 \leq J \leq 3, & \\
E[PortVal(T)|X_{DAX}(T) = -2.3263] \geq 850\,000 & (\tilde{S_1}) \\
E[PortVal(T)|X_{DAX}(T) = -0.8416] \geq 1\,000\,000 & (\tilde{S_2}) \\
E[PortVal(T)|X_{DAX}(T) = -0.4398] \geq 1\,030\,000 & (\tilde{S_3})
\end{cases}$$

The goal (Z) of the investment decision is to maximize the expected portfolio value at time T, the investment horizon. The limited budget, which will be fully invested in the optimal solution, presents the first constraint (B). The conditional expectations represent the stochastic preference constraints $(\tilde{S})$, and in combination with condition (M) we can express (S_1), (S_2) and (S_3) in this simple form $(\tilde{S_1})$, $(\tilde{S_2})$ and $(\tilde{S_3})$.

To ensure a minimum diversification of the portfolio (as the approximation assumption requires) and to avoid unrealistic extreme solutions which could not be realized in real markets, we add some volume constraints for all assets which are summarized in table 3.

Asset	General Volume Constraints Minimum	Maximum
Each Stock	10% Budget	100% Budget
All Stocks	10% Budget	100% Budget
Each Call	-12'500 Contracts	+12'500 Contracts
All Calls	-50% Budget	+100% Budget
Each Put	-12'500 Contracts	+12'500 Contracts
All Puts	-50% Budget	+100% Budget

Table 3: General Volume Constraints

OOP - PORTFOLIO Table 4: Optimal Optioned Portfolio for Case Study

STOCKS						
BASF.F	2500.0000	a	43.50	=	108750.00	(10.88%)
DAIG.F	1250.0000	a	80.90	=	101125.00	(10.11%)
DBKG.F	1400.0000	a	72.20	=	101080.00	(10.11%)
KFHG.F	175.0000	a	574.50	=	100537.50	(10.05%)
VEBG.F	7274.5060	a	80.90	=	588507.54	(58.85%)
CALLS						
BASF.F (40.00)	9646.7720	a	5.09	=	49128.80	(4.91%)
BASF.F (42.50)	-12500.0000	a	3.38	=	-42259.24	(-4.23%)
BASF.F (45.00)	353.2280	a	2.08	=	735.03	(0.07%)
DAIG.F (75.00)	4029.6190	a	8.61	=	34713.12	(3.47%)
DAIG.F (80.00)	7220.3810	a	5.17	=	37297.54	(3.73%)
DAIG.F (85.00)	-12500.0000	a	2.74	=	-34239.50	(-3.42%)
DBKG.F (70.00)	-1400.0000	a	5.55	=	-7772.83	(-0.78%)
DBKG.F (72.50)	-12500.0000	a	4.07	=	-50855.98	(-5.09%)
DBKG.F (75.00)	12500.0000	a	2.87	=	35910.07	(3.59%)
KFHG.F (550.00)	-12500.0000	a	55.55	=	-694361.70	(-69.44%)
KFHG.F (575.00)	12500.0000	a	41.11	=	513843.75	(51.38%)
KFHG.F (600.00)	-175.0000	a	29.47	=	-5156.44	(-0.52%)
VEBG.F (75.00)	-12500.0000	a	8.59	=	-107375.52	(-10.74%)
VEBG.F (80.00)	12500.0000	a	5.13	=	64118.87	(6.41%)
VEBG.F (85.00)	1153.5650	a	2.70	=	3115.64	(0.31%)
PUTS						
BASF.F (40.00)	-12500.0000	a	0.61	=	-7564.59	(-0.76%)
BASF.F (42.50)	-12500.0000	a	1.33	=	-16642.63	(-1.66%)
BASF.F (45.00)	-353.2280	a	2.47	=	-872.41	(-0.09%)
DAIG.F (75.00)	1250.0000	a	0.86	=	1078.42	(0.11%)
DAIG.F (80.00)	-12500.0000	a	2.29	=	-28629.80	(-2.86%)
DAIG.F (85.00)	12500.0000	a	4.74	=	59256.28	(5.93%)
DBKG.F (70.00)	1400.0000	a	1.62	=	2273.20	(0.23%)
DBKG.F (72.50)	12500.0000	a	2.58	=	32230.59	(3.22%)
DBKG.F (75.00)	-12500.0000	a	3.82	=	-47763.11	(-4.78%)
KFHG.F (550.00)	12500.0000	a	17.47	=	218367.34	(21.84%)
KFHG.F (575.00)	-12500.0000	a	27.41	=	-342633.74	(-34.26%)
KFHG.F (600.00)	175.0000	a	40.15	=	7026.48	(0.70%)
VEBG.F (75.00)	12500.0000	a	0.84	=	10478.56	(1.05%)
VEBG.F (80.00)	-12500.0000	a	2.25	=	-28178.78	(-2.82%)
VEBG.F (85.00)	7274.5060	a	4.70	=	34206.36	(3.42%)

YIELD - ANALYSIS			
budget:	1000000.00		
pf-value (2/7/96)	689473.80	rf-value (2/7/96)	310526.20
pf-value (31/12/96)	730753.58	rf-value(31/12/96)	318387.21
pf-yield	11.63%	rf-yield	5.00%
YIELD	***9.59%***		

Analyses of the Optimal Optioned Portfolio Selection

To solve (LP) we apply GAMS, a standard optimization software, which yields the following optimal optioned portfolio composition presented in table 4.

The following figure 3 illustrates how the return distribution of the optimal optioned portfolio and its approximation given by the conditional expectation meet the shortfall constraints stated by the investor. It also indicates the high accuracy by which the conditional expectation approximates the true underlying distribution function of the optioned portfolio value.[6] To obtain these distribution functions we applied Monte Carlo simulations with 50'000 runs to simulate possible portfolio results on the basis of the given portfolio structure and the stochastic properties of the assets.

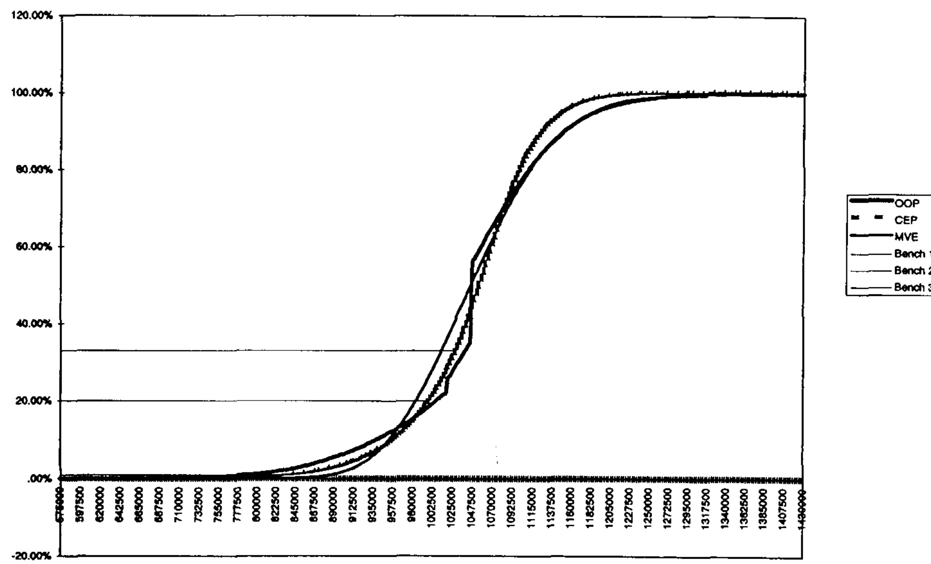

Figure 3: Shortfall Restrictions and Distribution Functions

The rather asymmetric return distribution is illustrated in figure 4.

4 OOP Optimization versus Mean Variance Portfolio Selection

Considering the problem that the optimization procedure given by (OP) is rather complex and even the approximation (LP) we used is still very extensive we want to

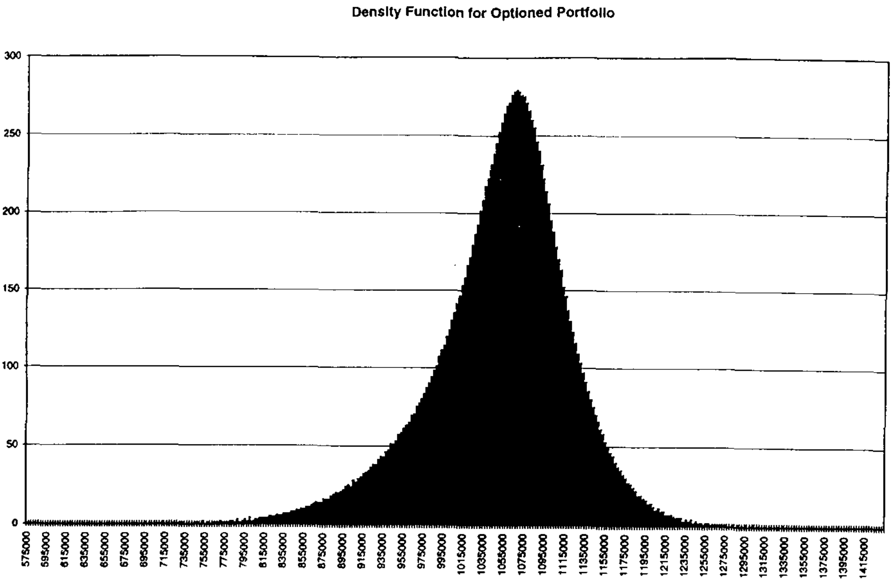

Figure 4: Asymmetric return Distribution of the Case Study

compare the solution for the case study with a portfolio composition selected on a comparable mean variance optimization basis.

As a comparable solution in the mean variance world we identify that mean variance efficient portfolio MVE on the efficient frontier which has the same mean as the optimized optioned Portfolio (OOP). Since both portfolios have the same mean (here $\mu_{MVO} = \mu_{OOP} = 9.59\%$) but MVE (with $\sigma_{MVE} = 15.37\%$) has a smaller volatility than OOP (with $\sigma_{OOP} = 17.26\%$) we would argue that MVE dominates OOP and no risk avers investor should prefer OOP to MVE. However, due to the typical asymmetric return distribution of optioned portfolios the mean variance solution may fail to come up with the utility maximizing solution. Using the concept of (first and second order) stochastic dominance we will investigate which portfolio solution dominates the other, if at all.

Looking at figure 3 with all distribution functions intersecting each other tells us that none of the portfolios dominates an other one in the sense of first degree stochastic dominance. The condition of first degree stochastic dominance for the comparison between OOP and MVE would be (Cox/Martin/MacMinn (1991), p.188) either

$$F_{OOP}(v) \leq F_{MVE}(v) \qquad \text{or} \qquad F_{OOP}(v) \geq F_{MVE}(v).$$

for all possible realizations v of the portfolio value. But the condition does not hold for any of the distribution functions $F_{.}$ in this case.

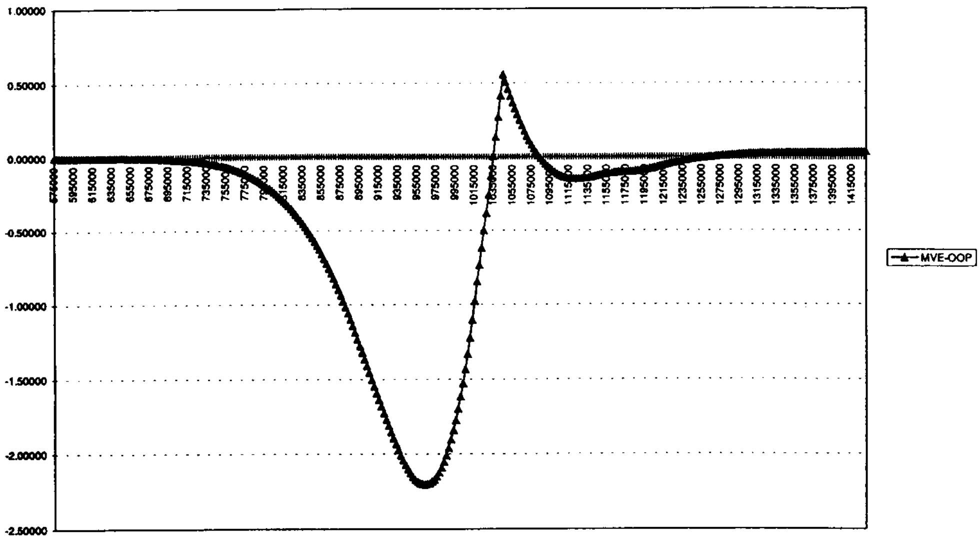

Figure 5: Cumulative Differences of the return distribution function of portfolio MVE and portfolio OOP

To get a clearer picture we want to look at the second degree stochastic dominance situation. Figure 5 shows the cumulative difference between the two distribution functions of MVE and OOP as a function of all possible portfolio results w:

$$\int_{-\infty}^{w} (F_{OOP}(v) \;-\; F_{MVO}(v))\, dv.$$

The corresponding condition for MVE being stochastic dominant of OOP in the second order sense would require that this difference is always positive. However, as the picture shows, this is not true and the dominance condition does not hold. Thus, the OOP is not dominated by such a comparable efficient mean variance portfolio and we can not give a definite answer to the question which portfolio is better. On the other hand, figure 3 clearly shows that the mean variance solution MVE does normally not satisfy the explicit preferences stated by the investor. In general we may expect that in such a "non-normal" framework, portfolio optimization on a traditional mean variance basis does not necessarily provide a better solution. In many realistic cases the OOP chosen by our optimization procedure portfolio will be the more appropriate choice.

[1]Equivalently, we could define lower partial moments using the expectation operator in the following way: $LPM_n(B) = E[1_{(-\infty,B)}(B - r_P)^n]$. with $1_{(-\infty,B)}$ being an index set where it equals 1, if the portfolio return scores under the critical benchmark B, and 0 otherwise. This definition emphasizes the character of a moment of the downside part of the distribution.

[2]Not allowing for continuos trading and no portfolio restructuring within the holding period implies that option positions can not be synthetically replicated by a permanently adjusted combination of stocks and bonds. In this sense, options are genuine asset positions and may influence a portfolio's return distribution in a way which can normally not be achieved by a pure investment in bonds and stocks.

[3]This bank account investment opportunity (w.l.o.g. let $i^* = n$) will be modeled like any other security or stock except that we assume that there is no systematic risk and no individual risk associated with this investment ($\eta_{n,n} = \eta_{n,M} = 0$). According to the CAPM it follows directly that the expected return $\mu_n = r$. Furthermore, we assume that there are no options traded to a riskless security and therefore $m_n = n_n = 0$. The current market price of this riskless security may be standardized to be $S_n(0) = 1$.

[4]Empirical studies support that assumption. They clearly illustrate that the portfolio risk of (even naiv) diversified portfolios is practically only systematic risk already with a small number of stocks. To neglect the unsystematic risk part for portfolios with more than five components causes only a very small error SOLNIK/NOETZLIN (1982).

[5]According to the model we introduced above the market index $S_M(T)$ and the stock prices $S_i(T)$ at maturity time T follow a Lognormal distribution. Therefore, we can express the continuos return of the index and the individual stocks in the following way: For the continuos rates of stock returns up to time T we get:

$$\ln\left(\frac{S_M(T)}{S_M(0)}\right) = (\mu_M - \frac{1}{2}\sigma_M^2)T + \sigma_M\sqrt{T}X_M \tag{7}$$

$$\ln\left(\frac{S_i(T)}{S_i(0)}\right) = (\mu_i - \frac{1}{2}\eta_{iM}^2 - \frac{1}{2}\eta_{ii}^2)T + \eta_{iM}\sqrt{T}X_M + \eta_{ii}\sqrt{T}X_i \tag{8}$$

where X_M and X_i, $i = 1, \ldots, n$ are independent random variables that follow a standard Normal distribution. This shows the direct link between $S_M(T)$ and X_M. Stochastic properties of $S_M(T)$ can, therefore, expressed in terms of X_M which is typically more convenient.

[6]The quality of the approximation can be assessed by using for example either the *average distance* or the *maximum distance* between the two distribution functions. With the *Euclidean* or the *Maximum* norm, respectively we use two well known distance measures. For the case study these two distances are: 0.027 on average and 0.122 as the maximum deviation.

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