An Application of the Vector Autoregressive Model with a Markov Regime to Inflation Rates

Lasse Koskinen[1] and Tarmo Pukkila[2]

Abstract

In this paper we consider a vector autoregressive model where the coefficients change according to a Markov chain mechanism. This model is applied to time series consisting of inflation rates of certain OECD countries. By incorporating a regime shift we analyze the behavior of inflation over a long and short horizon. The main objective of the approach used is to utilize the correlation between the inflation rates and, at the same time, model the local nonlinearity of each inflation series by changing the parameters.

Résumé

Dans cet article nous considérons un modèle autoregressif vectoriel où les coefficients changent selon un mécanisme Markovien. Ce modèle est appliqué à la série chronologique des rations d'inflation de certains pays de l'O.C.D.E. En y incorporant le changement de régime, nous analysons le comportement de l'inflation à un horizon et long et court. L'objectif principal de l'approche choisie est d'utiliser la corrélation entre les ratios d'inflation et, au même temps, de modeler la nonlinéarieté de chaque série d'inflation par le changement des paramètres.

Keywords

Vector autoregressive model, Markov regime, inflation rates.

1) Department of Mathematical Sciences, University of Tampere, P.O. Box 607, 33101 Tampere (Finland); Fax: + 358-31-215 6157.

2) Insurance Department, Ministry of Social Affairs and Health, P.O. Box 267, 00171 Helsinki (Finland); Fax: + 358-0-160 4168.

1. Introduction

The effects of inflation can be seen in many areas of insurance business. Changes in monetary value affect *e.g.* premiums, claims, and salaries. Assumptions about the future inflation are also necessary when assessing funding rates. So it is essential to take notice of the potential impact of inflation. For some purposes it is essential to develop a stochastic model of inflation. A stochastic model aims to produce realistic future inflation scenarios across the whole range of what might be regarded as possible, at the same time reflecting their respective probabilities of occurrence (see *e.g.* Daykin *et al.* 1994).

Different types of inflation models have been introduced by using different techniques. Wilkie (1984,1986) made use of the first order autoregressive process to model the dynamics of the consumer price index. This approach is intuitively appealing since it can be expected in general that the rate of inflation will not change very rapidly as a result of the inertia of economic phenomena. Thus, the level of inflation in the previous year will exert a strong influence.

Even though the Wilkie model is a useful description of inflation, in a review of the model, Geoghegan *et al.* (1992) identified certain areas of concern regarding the model. Firstly, the existence of a burst of inflation; there is a tendency for upward trends to continue. Secondly, there exist large irrecular shocks, such as those in the mid-1970s. Thirdly, there is the possible skewness of residuals.

Clarkson (1991) has suggested a nonlinear model for inflation. His model is more general than the Wilkie model, but its problem is the lack of a good estimation procedure. In 1988 Hamilton proposed a novel maximum likelihood estimation method for a nonlinear autoregressive model whose parameters change according to a hidden Markov process. This model can take into account irregular features of the data, like those appearing in inflation series. Kim (1993) and Brunner *et al.* (1993) used the Hamilton method when they studied the relationship between the inflation uncertainty and its level. Our purpose is to combine the Wilkie model with the Hamilton method in modelling the inflation of certain OECD countries. Especially we apply a first-order autoregressive model with a change in regime to consumer price index series.

2. An autoregressive model with a Markov regime

The studied model is discussed below. A more extensive treatment can be found in Hamilton (1988,1989,1994).

Let us consider a first-order vector autoregression (AR(1)) in which both the error vector ϵ_t and the autoregressive coefficient matrix Ψ might be different for different subsamples. The regime (state) of the process is denoted by s_t. We assume that s_t is a not observable N state Markov chain with s_t independent of ϵ_{t+k} for all t and k. Thus, the parameters of the process varies with a Markov chain:

$$\mathbf{y}_t - \mu_j = \Psi_j\big(\mathbf{y}_{t-1} - \mu_j\big) + \epsilon_t^j \tag{1}$$

where the state $j \in \{1, \ldots, N\}$ is the value of the process s_t at date t and noise ϵ_t^j is multivariate white noise with zero mean and covariance matrix Σ_j *i.e.* $E[\epsilon_t^j \epsilon_t^j] = 0$ and $E[\epsilon_t^j \epsilon_{t+k}^j] = \Sigma_j$ for $k \neq 0$.

In the model investigated $\mathbf{Y}_t = (\mathbf{y}'_t, \mathbf{y}'_{t-1}, \ldots, \mathbf{y}'_0)$ is a vector containing all observations obtained through date t. If the process is governed by regime $s_t = j$ at date t, then the conditional density of $\mathbf{y}_t$ is assumed to be given by

$$f(\mathbf{y}_t \mid s_t = j, \mathbf{Y}_t; \alpha) \tag{2}$$

where α is a vector of parameters characterizing the conditional density. These vectors are collected in an $N \times 1$ vector η.

The population parameters that describe a time series governed by (1) consist of α and N transition probabilities $\mathbf{p}$ of a Markov chain. From α and $\mathbf{p}$ form the combined parameter vector $\theta = (\alpha, \mathbf{p})$. Collect conditional probabilities $P\{s_t = j \mid \mathbf{Y}_t, \theta\}$ for $j = 1, \ldots, N$ in $N \times 1$ vector $\zeta_{t,t}$ and $P\{s_{t+1} = j \mid \mathbf{Y}_t, \theta\}$ for $j =$

$1, \ldots, N$ in $N \times 1$ vector $\zeta_{t+1,t}$. Given a starting value, the optimal inferences and estimates for each date t can be found by iterating the following pair of equations:

$$\zeta_{t,t} = \frac{\zeta_{t,t-1} \odot \eta_t}{\mathbf{1}' \left(\zeta_{t,t-1} \odot \eta_t\right)} \tag{3}$$

$$\zeta_{t+1,t} = \mathbf{P}\ \zeta_{t,t} \tag{4}$$

Here η_t represents a $N \times 1$ vector whose j^{th} element is the conditional density in (2), $\mathbf{P}$ represents the Markov matrix whose element p_{ij} gives the probabilities that the state j will be followed by state i , $\mathbf{1}$ represents an $N \times 1$ vector of 1s, and $\odot$ denotes element by element multiplication.

The log-likelihood function $L(\theta)$ for the observed data $\mathbf{Y_t}$ evaluated at the value of θ that was used to perform the iterations can also be calculated as a by-product of this algorithm from

$$L(\theta) = \sum_{t=1}^{T} \log f(\mathbf{y}|\mathbf{Y}_t;\theta) \tag{5}$$

where

$$f(\mathbf{y}|\mathbf{Y}_t;\theta) = \mathbf{1}' \left(\zeta_{t,t-1} \odot \eta_t\right) \tag{6}$$

To estimate the parameters of the model we can maximize the log-likelihood function (5) with respect to θ using a nonlinear optimization procedure. Probabilities for each state at each time point are given by (3) and (4). Thus, when we use this algorithm we do not optimize isolated parts of system, like Markov processes and autoregressive processes, separately, but we obtain the optimum of the whole system.

3. A regime shift extension of the Wilkie model of inflation and one-dimensional applications

In this paper we only consider consumer price inflation series where the logarithmic transformation and differencing has been performed and the mean has been subtracted. Following the idea of Wilkie (1984,1986) we assume that first-order autoregressive models are a suitable model class for the transformed inflation series. Thus, in the Hamilton method first-order autoregressive processes can be applied at each state. This leads to model (1). In order to specify our model in more detail we make some assumptions on inflation.

The inflation rates vary substantially from country to country and from time to time. An essential feature seems to be that in the long run inflation rates stay at moderate levels for a relatively long time but sudden increases occur. The Finnish Working Parties (Pentikäinen *et al.* 1982, 1989) used the term steady and shock inflation to distinguish between these two forms of inflation. So only two states are needed in (1) and it seems reasonable to set $\mu_1 < \mathbf{0}$ in the normal state (state 1) and $\mu_2 > \mathbf{0}$ in the shock state (state 2). Here '$<$' means the componentwise partial ordering of vectors. Moreover, according to Kim (1993) and Brunner *et al.* (1993) we suppose that higher inflation has higher variance. So we need different multivariate white noise processes ϵ_t^1 and ϵ_t^2 for different states. We now have a two state model that is an extension of the Wilkie model:

$$\mathbf{y}_t - \mu_1 = \Psi_1(\mathbf{y}_{t-1} - \mu_1) + \epsilon_t^1 \tag{7}$$

and

$$\mathbf{y}_t - \mu_2 = \Psi_2(\mathbf{y}_{t-1} - \mu_2) + \epsilon_t^2 \tag{8}$$

where the parameters to be specified are Ψ_1, Ψ_2, μ_1, μ_2, Σ_1 (covariance matrix of ϵ_t^1), Σ_2 (covariance matrix of ϵ_t^2), $p_1 = p_{1,1}$, and $p_2 = p_{2,2}$. A better fit for the data is possible to achieve by adding states to the model but it would make the model much more complicated. Natural measures of goodness-of-fit are used. First, we use the coefficient of determination

$$R^2 = 1 - SSE/SSY \tag{9}$$

where $SSE = \sum(y_i - \hat{y}_i)^2$ and $SSY = \sum(y_i - \overline{y})^2$. In the extended model at each date i inference about the estimate $\hat{y}_i$ is based on the more probable state. Second, we use log-likelihood $\log\big(L(\mathbf{X};\alpha)\big)$ of model, and third, the Akaike information criterion (AIC)

$$AIC = -2\,\log(L(\mathbf{X};\alpha)) + 2nk/(n-k-1) \tag{10}$$

where k is the number of parameters and n is the number of observations (see Brockwell *et al.* 1987). Note that it is not clear which information criteria should be used when we compare the linear and nonlinear models presented above.

Next we illustrate this method in the case of the USA consumer price inflation. The extended Wilkie model (7-8) has the form (11) for state 1 (normal state) ($p_1 = 0.9458$) and (12) for state 2 (shock state) ($p_2 = 0.3815$) ($\epsilon_t \sim N(0,1)$):

$$y_t + 0.01 = 0.63 \times (y_{t-1} + 0.01) + 0.013 \times \epsilon_t \tag{11}$$

and

$$y_t - 0.08 = 0.4 \times \big(y_{t-1} - 0.08\big) + 0.038 \times \epsilon_t \tag{12}$$

Respectively, the Wilkie model is

$$y_t = 0.7 \times y_{t-1} + 0.021 \times \epsilon_t \tag{13}$$

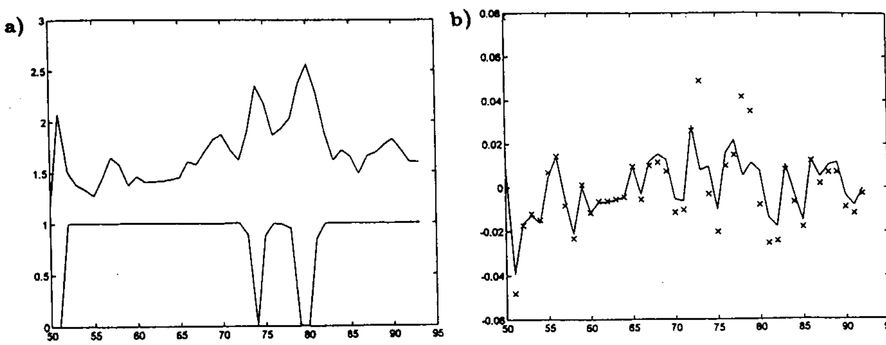

Figure.1. a) The upper curve denotes the consumer price inflation in USA (1949-1993), the lower curve denotes the probability of state 1 at each date (the curves are scaled), b) The solid line denotes the residuals of the extended Wilkie model (11-12), 'x' -marks denote the residuals of the Wilkie model (13).

We make decision that the model is in the shock state 2 at date t if probability $\{s_t = 2\}$ is greater than or equal to 0.5. From Figure 1 it can be seen that state has changed from 1 to 2 at the beginning of the Korean war in 1950 and during the oil crisis in the mid and late seventies. Also the residuals of the extended model at those points are smaller than those of the Wilkie model. At other time points only minor differences occur.

country	μ_1, μ_2	a_1, a_2	σ_1, σ_2	p_1, p_2	in shock state
USA	-0.01, 0.08	0.63, 0.40	0.013, 0.015	0.95, 0.38	50,51,74,79,80
England	-0.02, 0.15	0.61, 0.62	0.018, 0.024	0.93, 0.41	50,51,74,75,79,80
Germany	0, 0.0006	0.77, 0.25	0.01,0.046	0.99, 0.68	50,51,52

Table 1. The parameters of the extended Wilkie models (subscript 1 refers to the normal state).

Table 1 consists the parameters of the extended Wilkie models and Table 2 the goodness-of-fit measures for the models. It can be seen from Table 2 that the extended Wilkie model fits better to the data. For example the Wilkie model of USA has $R^2 = 0.82$, $\log(L) = 105.4$, and AIC = -206.6. The corresponding values for the extended Wilkie model are $R^2 = 0.93$, $\log(L) = 115.1$, and AIC = -210.6. As expected, the largest differences between the residuals of different models are during shock inflation periods for all countries. Table 1 shows that the extended Wilkie model (7-8) does not describe well the series in the case of Germany since both μ_1 and μ_2 are near zero. Moreover, the process is in the shock state only at the beginning of the fifties but not during the oil crises in the seventies. However, the goodness-of-fit measures of the extended model in Table 2 are better for Germany also. The oil shock states of Germany were of a longer duration and lower values than for example those of USA (see Figure 2). This might cause the problems in finding oil shock states for Germany.

country	model	R^2	log(L)	AIC	skewness
USA	Wilkie	0.82	105.4	-206.6	1.18
USA	Ext. W.	0.93	115.1	-210.6	-0.38
England	Wilkie	0.85	91.8	-179.3	0.94
England	Ext. W.	0.94	99.2	-178.2	0.31
Germany	Wilkie	0.75	104.4	-204,5	1.06
Germany	Ext. W.	0.81	130.4	-240.7	0.97

Table 2. The goodness-of-fit measures of the Wilkie model and its extension.

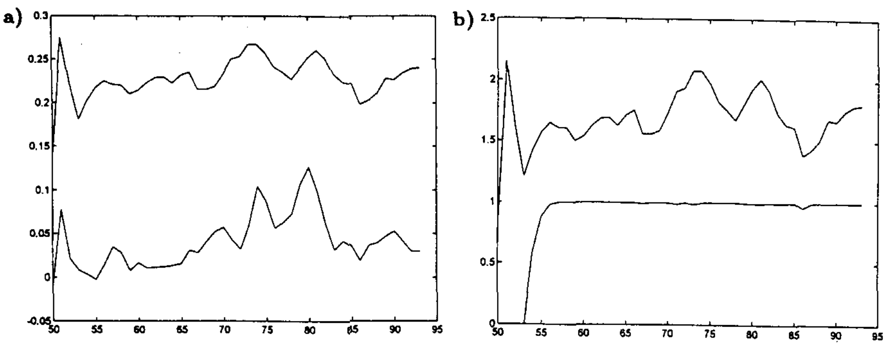

Figure 2. a) The upper curve denotes the consumer price inflation in Germany (1949-1993), the lower curve denotes the probability of state 1 at each date (the curves are scaled), b) The upper curve denotes the consumer price inflation in Germany (1949-1993), the lower curve denotes the consumer price inflation in USA (1949-1993)(the curves are scaled).

Remark. A major problem with the Wilkie model is the skewness of residuals. Table 2 shows that residuals of the extended Wilkie model (7-8) are less skew. The residuals of both models passed Portmanteau, Turning-point and Difference-sign tests of randomness at 95 % risk level. The only exception was the Wilkie model of USA that did not pass the Turning-point test.

4. A constrained vector model and its estimation procedure

For vector time series the optimization of likelihood (5) is a hard problem. Kim (1994) has proposed an approximative less complex algorithm for calculating (5). We now present a constrained version of the model (7-8) and its estimation procedure. The estimation procedure is a simple combination of standard vector autoregressive estimation and the Hamilton method. Vector valued Hamilton models are usually complicated. So there is a need to simplify them in order to get a better idea of the behavior of the models. Of course, simplifying serves numerical aspects too. The interpretation of the proposed model is easy since it has less parameters than the model (7-8).

The main idea of the estimation procedure is to apply the one-dimensional Hamilton method and a standard vector autoregressive procedure to the estimation of the majority of the parameters. Optimization of the vector form of (5) is only needed in finding the mean vector and the scaling parameter of covariance. For the considered inflation series the proposed model and estimation procedure give better results in terms of goodness-of-fit than the vector Wilkie model, and what is more important, the model finds shock inflation periods well. The vector model to be estimated is the following:

$$\mathbf{y}_t = \Psi \mathbf{y}_{t-1} + \epsilon_t \tag{14}$$

for a normal state and

$$\mathbf{y}_t - \mu = \Psi\big(\mathbf{y}_{t-1} - \mu\big) + \sigma\epsilon_t \tag{15}$$

for a shock state. Compared with (7-8) there are constraints $\Sigma = \Sigma_1 = \Sigma_2$ and $\Psi = \Psi_1 = \Psi_2$ and one additional covariance scaling parameter σ. The parameters to be estimated are $p_1, p_2, \Psi, \Sigma, \mu$ and σ. The autoregressive coefficient Ψ and white noise covariance Σ are found by a standard vector autoregressive method. The procedure consists of the following three steps.

Step 1. Apply the Hamilton method to each one-dimensional component series of a vector time series. Then set probabilities p_1 and p_2 of the model (14-15) equal to the averages of one dimensional models.

Step 2. Apply a standard estimation method of vector autoregression to the vector series. Then set the estimated parameters Σ and Ψ to model (14-15). (Normal state autoregression (14) is now completed.)

Step 3. Add the mean vector μ and the scaling parameter of noise σ to the shock state autoregression (15). Optimize the mean μ and the scaling parameter σ by applying the nonlinear optimization method to (14-15) (Here the optimization of (5) has to be performed with respect to μ and σ.)

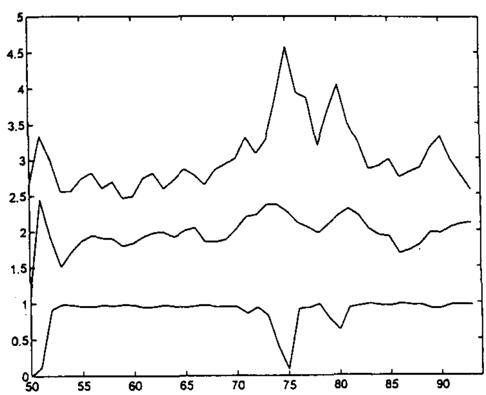

Figure 3. The upper curve denotes the consumer price inflation in England (1950-1993), the middle curve denotes the consumer price inflation in Germany (1950-1993), the lower curve denotes the probability of state 1 at each date when the vector model (14-15) has been applied to England-Germany vector series. (The curves are scaled.)

Again we make decision that the model is in the shock state 2 at date t if probability $\{s_t = 2\}$ is greater than or equal to 0.5. It can be seen from Figure 3 and Table 3 that the model (14-15) is in the shock state during the oil cricis in the mid seventies also when the vector series includes Germany as a component. Table 3 shows that the extension of the Wilkie model has larger log-likelihood, except in the case of USA-England combination whose likelihoods are the same.

country	model	log(L)	In shock state
USA-Germ.	Wilkie	227	-
USA-Germ.	Ext. W.	231	50,74
Engl.-Germ.	Wilkie	205	-
Engl.-Germ.	Ext. W.	210	50,51,74,75
USA-Engl.	Wilkie	212	-
USA-Engl.	Ext. W.	212	50,75, 79

Table 3. The log-likelihoods of the Wilkie model and its extension. The last column consists of the shock inflation years of the extended model.

5. Forecasts and simulations for the extended Wilkie model

Assumptions about the future rates of inflation are of great importance to insurance business. The developed models can be used in making scenarios of the future inflation rates. For short horizon decision making, exact forecast functions can be used but for a long horizon simulations are often needed. Both in forecasting and simulations the inference about shock inflation states is crucial. In the case of the model (7-8) the optimal k-period-ahead forecasts about the probabilities of states $\zeta_{t+k,t}$ are given by

$$\zeta_{t+k,t} = \mathbf{P}^k \zeta_{t,t} \tag{16}$$

where $\mathbf{P}$ is the transition matrix of the Markov chain and $\zeta_{t,t}$ is the probability state vector calculated by (3-4).

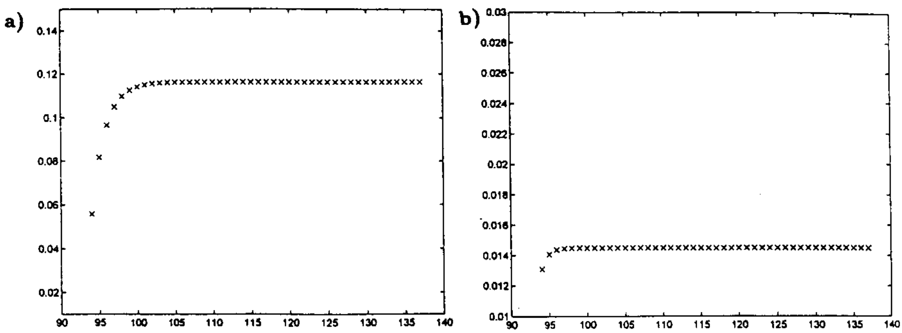

Figure 4. a) The optimal forecasts about the probabilities of shock inflation states for USA in 1994-2038, b) The optimal forecasts about the probabilities of shock inflation states for Germany in 1994-2038. Note that the scales of the axis are different in a) and b).

Based on one-dimensional models of Table 1, the forecasts about the states of USA and Germany are in Figure 4. As expected, USA has higher probabilities for the shock inflation state than Germany. The stationary level of the probability of shock inflation is about 10 times higher for USA. Simulated inflation rates corresponding to USA are in Figure 5. Simulations are based on the models (11-13). Naturally the outcomes show a higher degree of variability when the extended model is applied since shock inflation periods exist also in simulations.

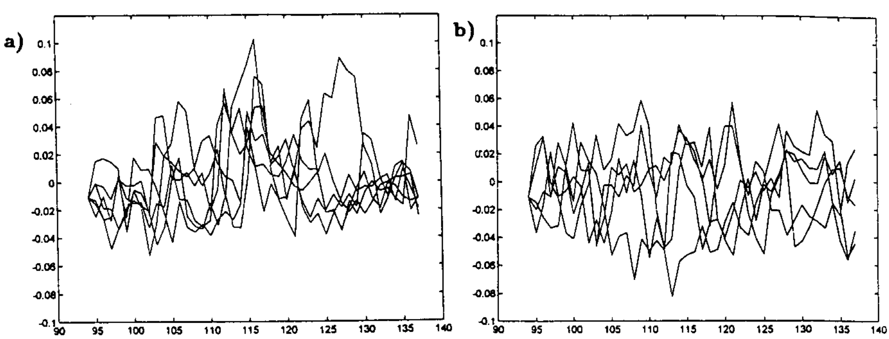

Figure 5. a) Inflation rates generated by the Wilkie model of USA, b) Inflation rates generated by the extended Wilkie model of USA.

Conclusions

In this paper we proposed an extension of the Wilkie AR model for the rates of inflation. In the long horizon the deviations from the average rate of inflation are asymmetric since shock inflation periods occur. The proposed extension of the Wilkie model is an AR model whose parameters change according to a two-regime (state) Markov chain. The change in regime makes it possible to introduce the asymmetry of inflation to this model. For vector inflation series we presented a simple constrained form model. The estimation of this model is easier to perform. The vector models were able to find shock inflation regimes even if one component was ill behaved. In general, the goodness-of-fit measures of the extended Wilkie model were better than those of the Wilkie model and the simulation results corresponded well to the fluctuations of inflation.

REFERENCES

P. Brockwell and R. Davis, "Time series: Theory and methods", Springer-Verlag, New York 1987.

A. Brunner and G. Hess, "Are higher levels of inflation less predictable? A state-dependent conditional heteroscedasticity approach, Journal of Businees and Economics Statistics, April 1993, Vol. 11, No 2.

C. Daykin, T. Pentikäinen, and M. Pesonen, "Practical Risk Theory for Actuaries", Chapman & Hall, 1994.

T. Geoghegan, R. Clarkson, and K. Feldman, "Report on the Wilkie Stochastic model, Journal of the Institute of Actuaries, 119, 1992.

R. Clarkson, "A nonlinear stochastic model for inflation", Proc. AFIR 3, 1991.

Hamilton J. D. "Rational-expectation econometric analysis of changes in regime", Journal of Economic Dynamics and Control 12, 1988.

Hamilton J. D. "A new approach to the economic analysis of nonstationary time series and the business cycle", Econometrica, Vol. 57, 1989.

Hamilton J. D. "Time Series Analysis", Princeton University Press, Princeton, 1994.

Kim C-J, "Unobserved-component time series models with Markov-switching heteroscedasticity: Changes in regime and the link between inflation rates and inflation uncertainty", Journal of Business and Economic Statistic, Vol. 11, No 3, 1993.

Kim C-J, "Dynamic linear models with Markov switching", Journal of Econometrics 60, 1994.

T. Pentikäinen and J. Rantala (ed.),"Solvency of insurers and equalization reserves, Vol. I and II, Finnish Insurance Training and Publishing Company, Helsinki, 1982.

T. Pentikäinen, H. Bonsdorf, M. Pesonen, J. Rantala, and M. Ruohonen, "Insurance solvency and financial strength", Finnish Insurance Training and Publishing Company, Helsinki, 1989.

A. Wilkie, "Steps towards a comprehensive stochastic investment model", Occasional Actuaries Research Discussion Paper No. 36, Institute of Actuaries, London, 1984

A. Wilkie, "A stochastic investment model for actuarial use", Transactions of the Faculty of Actuaries, 1986.