Shortfall-Probability-Based Diagrams of Efficient Frontiers

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Abstract

This paper analyses different types of shortfall-risk-based diagrams of efficient frontiers and compares them with each other in terms of their advantages and disadvantages. In addition, a new type is developed, which is able to overcome most of the shortcomings of existing approaches. A scenario with assets having normally distributed returns is used to illustrate the mathematical analysis. Furthermore, two methods of solving the portfolio selection problem in the generalized case are developed.

Résumé

L´article analyse différents types de diagrammes de courbes d´efficacité basés sur le shortfall-risk et compare leurs avantages et désavantages. De plus, un nouveau type de diagramme est développé, lequel évite les désavantages des concepts existants. L´analyse mathématique est ensuite illustrée à l´aide d´un exemple comprenant différentes valeurs boursières dont les rendements suivent une loi normale. Enfin, deux méthodes sont présentées avec lesquelles le problème de sélection de portefeuilles peut être résolu pour n´importe quelle fonction de répartition.

Keywords

Portfolio optimization, shortfall risk, efficient frontier, lower partial moment.

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1. Introduction

It is well kown, that under particular assumptions referring to the shape of asset return distributions, portfolio optimization based on shortfall probabilities according to ROY (1952), leads to portfolios, which are positioned on the classical Efficient Frontier in the Mean-Variance-World of MARKOWITZ (1952).[2] This is especially true in the case of normally distributed returns. Therefore, a section of the μ-σ-Efficient-Frontier is also efficient in a *Mean-Shortfall-Probability-World.* To take not only analytically but also graphically account of this fact, several modified representations of the classical Efficient Frontier has been proposed in the literature. These diagrams may be used by institutional investment comittees as well as in consulting the private banker's customer and lead to a better understanding of the shortfall risk concept, in general. Moreover, performance results can be communicated ex post using these graphics.

The purpose of this paper is to discuss the different types of diagrams and to compare them which each other in terms of their advantages and disadvantages. In addition, a new type is developed, which is able to overcome most of the shortcomings of the existing approaches. The paper is structured as follows: Firstly, the conventional approaches are explained analytically and graphically in detail. This is the so-called *Mean-LPM$_0$-Efficient-Frontier* in the second section, the *Efficient Shortfall Frontier* in the third section and the *L-Efficient-Frontier* in the fourth section. An illustrative scenario used in each section may help to understand the different concepts. In the fifth section a new type of diagram, the *Efficient Shortfall Surface* is introduced. Whereas in the first five sections the existence of normally distributed returns is assumed, in the sixth section the consequences of a relaxation of this assumption are discussed. Two different methods of solving the generalized portfolio optimization problem are shown here. The final appendix contains data, which are used for the scenario.

2. The Mean-LPM$_0$-Efficient-Frontier

The first type of shortfall-probabitlity-based representation of portfolios presented here is the so-called *Mean-LPM$_0$-Efficient-Frontier*[3] by HARLOW (1991), who positions all feasible portfolios in a Mean-Shortfall-Probability-World[4] and derives a corresponding efficient frontier there. To illustrate the approach, consider a portfolio with a normally distributed return $R \sim N(\mu,\sigma)$. Specifying a target return t^*, the probability $p(t^*)$ of this portfolio of returning less than this threshold is

$$p(t^*) = \frac{1}{\sqrt{2\pi}\sigma} \cdot \int_{-\infty}^{t^*} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx \,. \tag{1}$$

Substituting $x \equiv \sigma \cdot z + \mu$, which represents the transition to a random variable $Z \sim N(0,1)$, equation (1) can be modified to

$$p(t^*) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\frac{t^*-\mu}{\sigma}} e^{-\frac{1}{2}z^2} \, dz \,. \tag{2}$$

Together with corresponding expected returns μ, all feasible portfolios can be now positioned in the μ-$p(t^*)$-World. HARLOW (1991) introduced the name *Mean-LPM$_0$-Efficient-Frontier* for the resulting efficient frontier.[5] Analytically this alternative efficient frontier can be derived by substituting the equation of the classical Mean-Variance Efficient Frontier[6,7]

$$\sigma = \sqrt{\frac{c\mu^2 - 2b\mu + a}{ac - b^2}} \tag{3}$$

into the equation of the shortfall probability (2):

$$p(t^*) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{-\sqrt{\frac{(ac-b^2)\cdot(\mu-t^*)^2}{c\mu^2-2b\mu+a}}} e^{-\frac{1}{2}z^2}\, dz\ . \tag{4}$$

Equation (4) shows for all portfolios positioned on the Markowitz Efficient Frontier the relationship between their shortfall probability $p(t^*)$ and their expected return μ, given the fixed target return t^*. Figure 1 displays the relationship graphically. In order to draw a concrete efficient frontier, a simple but realistic scenario has been used. The underlying assumptions and data can be found in the appendix.

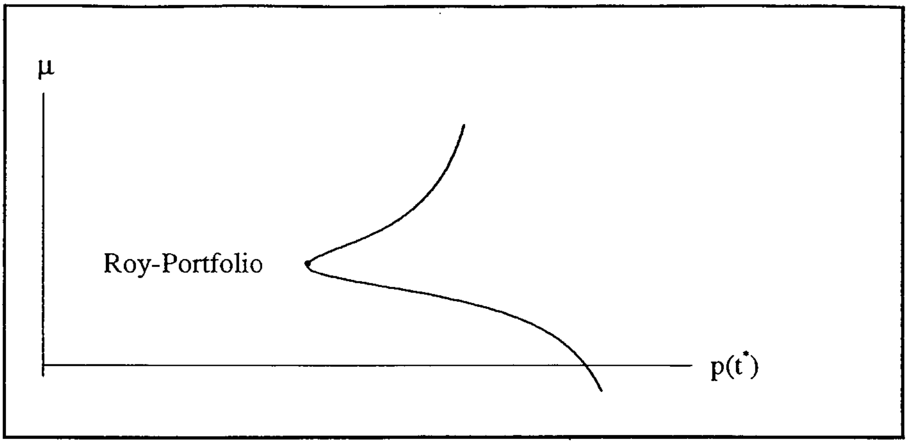

Figure 1: The Mean-$LPM_0(t^*)$-Efficient-Frontier.

Looking at Figure 1, several points deserve special attention: The global Minimum-Shortfall-Probability-Portfolio within the μ-$p(t^*)$-World is given by the so-called *Roy-Portfolio* or *Safety-First-Portfolio*. This portfolios is found by drawing a tangent to the classical Efficient Frontier [see Figure 5] through the target return t^* on the vertical axis. The tangent is termed *Shortfall Line*, because it connects all portfolios with equal shortfall probability referring to t^*. All portfolios, positioned in the μ-σ-World on the Efficient Frontier *above* the Roy-Portfolio are also efficient in the μ-$p(t^*)$-World. They can be found on the Mean-LPM_0-Efficient-Frontier *above* the Roy-Portfolio [see Figure 1]. Portfolios, which are positioned in the μ-σ-World on the Efficient Frontier but *under* the Roy-Portfolio, are inefficient in the μ-$p(t^*)$-World, because for each of these portfolios another portfolio with identical shortfall probability referring to t^* exists, but with a higher expected return. This is especially correct for the so-called Minimum-Standard-Deviation-Portfolio, which is the portfolio with global minimum standard deviation in the μ-σ-World [see Figure 5]. Only in (for practical purposes irrelevant) case of $t^* \rightarrow -\infty$, in which the Roy-Portfolio approaches the Minimum-Standard-Deviation-Portfolio, this portfolio becomes efficient in the μ-$p(t^*)$-World. Please note, that all other feasible portfolios in the μ-$p(t^*)$-World are positioned on the right of the Mean-$LPM_0(t^*)$-Efficient-Frontier in Figure 1.

The approach of HARLOW (1991) described so far, has a significant disadvantage: In order to derive a Mean-$LPM_0(t^*)$-Efficient-Frontier, a *fixed* target return must specified *in advance*. This requires, that the investor using HARLOW'S methodology must be able to specify a single, most relevant threshold for himself. But in practice, often the opposite may be observed: At the beginning of the asset allocation process, most investors[8] do not know what is a suitable target for them and therefore want to consider several or even better *all* feasible alternatives, first.[9] To implement such a procedure within the framework of HARLOW (1991), for each possible target return a *separate* efficient frontier would have to be calculated and plotted. Subsequently, all the resulting curves, which can *not* be drawn in one diagram, must be compared with each other.

3. The Efficient Shortfall Frontier

Is the investor unsure about the suitable target return he should employ, he alternatively can position all Safety-First-Portfolios in a *Threshold-Shortfall-Probability-World*[10]. In this world, for each feasible target return t the corresponding minimal shortfall probability p(t), which can be achieved, is displayed. The resulting efficient frontier is termed *Efficient Shortfall Frontier* [see JAEGER/RUDOLF/ZIMMERMANN (1995)]. Analytically, this Efficient Shortfall Frontier can be derived using the equation

$$\mu = \frac{(ac - b^2) \cdot \sigma_P}{c \cdot \mu_P - b} \cdot \sigma + \frac{b \cdot \mu_P - a}{c \cdot \mu_P - b} \qquad (5)$$

of a tangent to the Mean-Variance Efficient Frontier in point (σ_P, μ_P). Please note, that the Mean-Variance Efficient Frontier is a hyperbola.[11] For each point (σ_P,μ_P) on the efficient section of the hyperbola, intersection and slope of the corresponding tangent has to be calculated. The intersection equals the target return t, and with the slope of the tangent, the shortfall probability p(t) can be calculated. Substituting these results in (2), the sought-after functional relationship between target return t and corresponding shortfall probability p(t) is:

$$p(t) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{-\sqrt{c \cdot t^2 - 2bt + a}} e^{-\frac{1}{2}z^2} \, dz. \tag{6}$$

Please note also, that in equation (6) the restriction, that only target returns strictly below the expected return b/c of the Minimum-Standard-Deviation-Portfolio are allowed, have to be observed.[12] The following Figure 2 represents equation (6) graphically, using again the example from the appendix as underlying data:

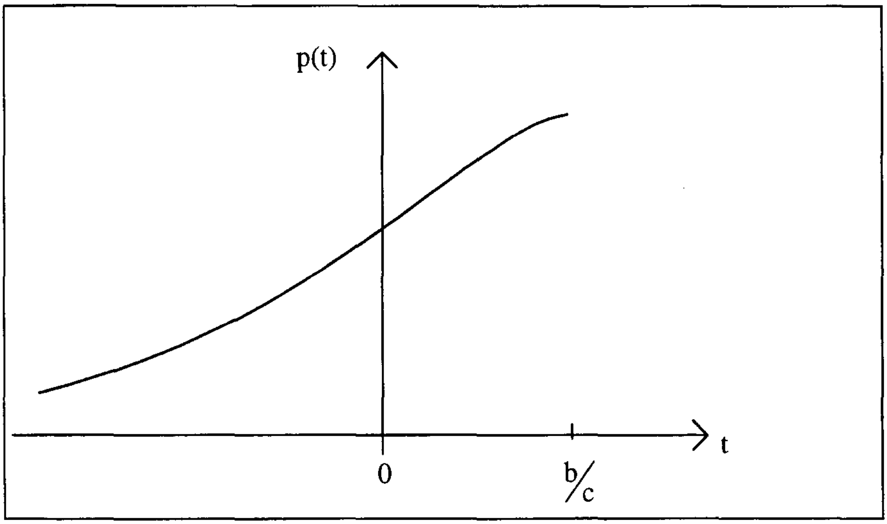

Figure 2: The Efficient Shortfall Frontier in the t-p(t)-World.

In addition, Figure 2 illustrates the situation for $t \to -\infty$. In this case p(t) asymptotically approaches zero, as it also may be seen analytically from (6). Interpreted economically this means, that an investor is able to meet his decreasing target return with increasing probability. Graphical correspondence of the process $t \to -\infty$ is the shift of the intercept of the Shortfall Line in the μ-σ-World downwards. In the limiting case, the Shortfall Line equals the vertical axis. The set of all feasible portfolios in Figure 3 is positioned *above* the Efficient Shortfall Frontier.

The approach of JAEGER/RUDOLF/ZIMMERMANN (1995) shows to those investors, who are still unsure about the target return they should select, *all* alternatives summarized in *one* diagram. So in this respect, the Efficient Shortfall Frontier has a clear advantage compared to Harlow's Mean-LPM_0-Efficient-Frontier discussed earlier. But on the other side, an important disadvantage of this type of graph must be considered: The t-p(t)-diagram does *not* provide any information about the expected return of the portfolios, which is an absolute essential characteristic of any portfolio. Therefore, this approach is also not able to meet our aim of representing portfolios in a Mean-Shortfall-Probability-World analogue to the Mean-Variance-World.

4. The L-Efficient-Frontier

The above discussed type of representation for portfolios in a t-p(t)-World is able to show shortfall-miminal portfolios for *all* feasible target returns, but offers no information about the corresponding expected returns of these portfolios. An improved approach by KADUFF/SPREMANN (1996), the so-called *L-Efficient-Frontier* merges both advantages: Without the need of specifying a fixed target return, all Safety-First-Portfolios are positioned in the μ-p(t)-World. Therefore, the expected return of each portfolio can be directly read from such a diagram. One should notice carefully, that the shortfall probability p(t) displayed here on the horizontal axis, is calculated for *varying* target returns, which separates the μ-p(t)-World strictly from Harlow's μ-p(t^*)-World dicussed in the second section of this paper.

Analytically, the functional relationship between expected returns and shortfall probabilities for varying targets of all Safety-First-Portfolios, can be found by substituting the equation for the hyperbola tangent

$$\mu(t)=\frac{a-b\cdot t}{b-c\cdot t} \tag{7}$$

in (6)[13,14]:

$$p(t)=\frac{1}{\sqrt{2\pi}}\cdot\int_{-\infty}^{\frac{\left(\frac{b\mu(t)-a}{c\mu(t)-b}-\mu(t)\right)\cdot\sqrt{ac-b^2}}{\sqrt{c\mu(t)^2-2b\mu(t)+a}}} e^{-\frac{1}{2}z^2}\,dz. \tag{8}$$

Figure 3 shows this relationship graphically. Again, in order to calculate a concrete L-Efficient-Frontier, the scenario from the appendix has been used. Several remarks to Figure 3 are necessary:

1. For $\mu \to {}^{b}\!/\!{}_{c}$ the L-Efficient-Frontier converges on the vertical axis.[15] This equals the limiting case $t \to -\infty$. In this situation, only relevant for theoretical considerations, the Minimum-Standard-Deviation-Portfolio is Safety First, with a shortfall probability of zero.

2. For $p(t) \to \frac{1}{\sqrt{2\pi}}\cdot\int_{-\infty}^{-\sqrt{\frac{ac-b^2}{c}}} e^{-\frac{1}{2}z^2}\,dz$ the L-Efficient-Frontier converges on infinity.

   In the μ-σ-World, this situation is represented by shifting the intersection t of the Shortfall Line to ${}^{b}\!/\!{}_{c}$ from below on the vertical axis. ${}^{b}\!/\!{}_{c}$ itsself is the origin of the asymptotes of the hyperbola on the vertical axis. The slope of the Shortfall Line then approaches the slope of the upper asymptote of the hyperbola, which equals $\sqrt{\frac{ac-b^2}{c}}$. Economically interpreted this means, that a portfolio can be identified, which offers an unlimited expected return, combined with a shortfall probability of at most $\frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{-\sqrt{\frac{ac-b^2}{c}}} e^{-\frac{1}{2}z^2}\, dz$, referring to a target return of b/c.

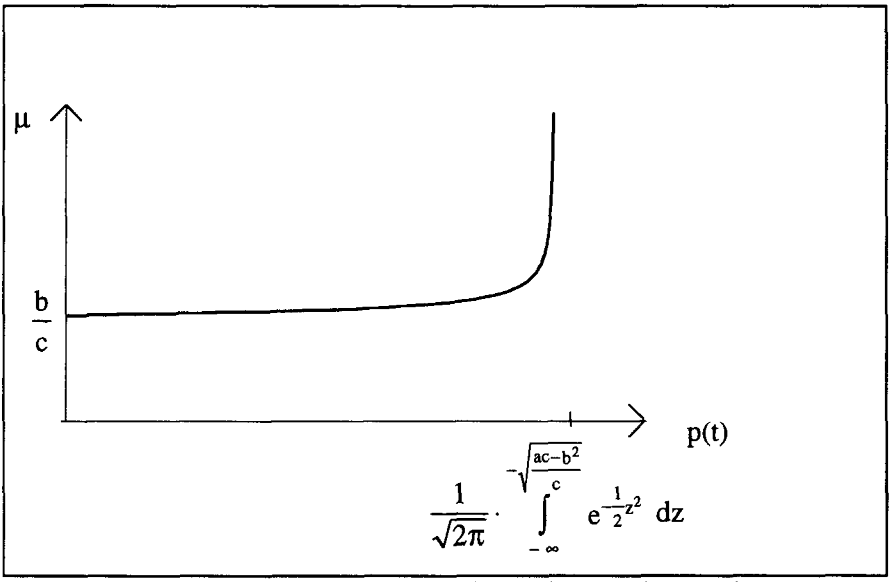

Figure 3: The L-Efficient-Frontier in the Mean-Shortfall-Probability-World.

The set of all feasible portfolios in Figure 3 is limited *upwards* by the L-Efficient-Frontier. The L-Efficient-Frontier allows the presentation of the expected returns of *all* Safety-First-Portfolios and their shortfall probabilities referring to *all* permissible target returns. But unfortunately, these target returns vary along the L-Efficient-Frontier, without being displayed directly in the μ-p(t)-World.

Ultimately, an investor, who has decided that shortfall probabilities are suitable risk measures for his purposes, has to add not only one but *two additional dimensions* to the expected returns, that is the shortfall probability *and* the corresponding target return, to which the shortfall probability is calculated. The logic consequence is the extension of the two-dimensional diagrams in a *third dimension.*

5. The Efficient Shortfall Surface

For it, in analogy to the μ-σ-World each feasible portfolio is to be positioned in a μ-t-p(t)-World, a procedure, which can be illustrated graphically very easily. Of special interest now are those portfolios, which yield the *maximum* expected return for a given target return/shortfall probability combination. This criteria of optimization equals the method of TELSER (1955)[16], with the difference, that not only one but all feasible t-p(t)-combinations are considered. Within the μ-t-p(t)-World, these portfolios generate a surface for which we introduce the name *Efficient Shortfall Surface.* For the corresponding efficiency property the term *Mean-Shortfall-Constraint-Efficiency*[17] will be used.

Analytically, the *Mean-Shortfall-Constraint-Efficient* portfolios can be derived by intersecting the so-called Telser-Line with the classical Efficient Frontier. The Telser-Line is the line identified in the Mean-Variance-World by selecting a target return and a shortfall probability. The upper intersection point between the Telser-Line and the Efficient Frontier identifies - if existent - the portfolio, which meets both restrictions referring to target return and shortfall probability and yields the maximum expected return. We will represent this maximum expected return through μ_S. Combined with the corresponding standard deviation σ_S, which can be easily calculated from equation (3) for all portfolios on the Efficient Frontier, μ_S is substituted in (2). This results in the equation[18]:

$$p(t) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{-\sqrt{\frac{(ac-b^2)\cdot(\mu_S - t)^2}{c\mu_S^2 - 2b\mu_S + a}}} e^{-\frac{1}{2}z^2}\, dz \,, \qquad (9)$$

which can be interpreted in the following way: Any two of the three parameters μ_S, t and p(t) can be chosen arbitrarily[19], fixing the third one unambiguously. This identifies a point on the three-dimensional Efficient Shortfall Surface in the μ-t-p(t)-World.[20] Figure 4 represents equation (9) graphically. Again, the example of the appendix has been used to generate this concrete surface:[21]

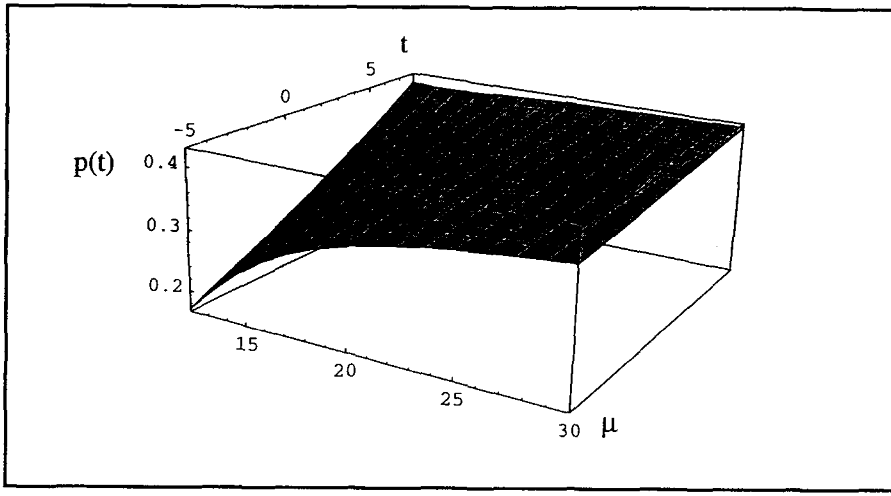

Figure 4: The Efficient Shortfall Surface in the μ-t-p(t)-World.

Several remarks to Figure 4 are in order now: In contrast to all previous diagrams, the scale of the axes have been plotted here, in order to ease the understanding of the diagram. Special attention should be paid to the scale of the z-axis, which shows the shortfall probabilities of the portfolios. The scale increases upwards, which means that the set of all feasible portfolios is positioned *above* the Efficient Shortfall Surface. In other words, the Efficient Shortfall Surface restricts the set of feasible portfolios *downwards*. Furthermore, the shape of the Efficient Shortfall Surface coincides with our intution: The acceptance of a decreasing target return is ceteris paribus[22] compensated with a decreasing probablity of failing to meet this target. It should be also stressed, that under the assumptions made, each point on the Efficient Shortfall Surface identifys a portfolio unambiguously [see endnote 20], but this relationship is not injective in a mathematical sense. This means, that the same portfolio can be plotted on different points on the Efficient Shortfall Frontier through different t-p(t)-combinations.

Figure 4 can also be employed to visualise the three classical portfolio optimization strategies based on shortfall probabilities by ROY (1952), KATAOKA (1963) and TELSER (1955):

1. The Safety-First-Rule of ROY can be illustrated by fitting in a vertical surface, parallel to the surface spanned by the x- and z-axis, runing through the prespecified target return. Intersecting this surface with the Efficient Shortfall Surface yields an intersection line in the three-dimensional space. The point on this line with the minimum height referring to the z-axis is the Roy-Portfolio.

2. The Kataoka-Portfolio[23] can be identified by intersecting a horizontal surface through the prespecified shortfall probability with the Efficient Shortfall Surface. Again, a intersection line in the three-dimensional space results. The point with the maximum y-scale value on this line is the Kataoka-Portfolio.

3. The method of TELSER equals graphically the intersection of a line, generated through the specification of a target return and corresponding shortfall probability[24], with the Efficient Shortfall Surface. An intersection point results which represents the Telser-Portfolio.

6. Extensions

The previous analysis is, as mentioned, based on the assumption of normally distributed portfolio returns.[25] In general, there are two possibilities of treating situations, in which this assumption is violated:

1. The first possibility is to use the inequality of CHEBYSHEV, which requires no information about the shape of the probability distribututions at all. According to this inequality, the following relationship for the shortfall probability p(t) referring to a

target return t holds:

$$p(t) \leq \frac{\sigma(R)^2}{[\mu(t) - t]^2}\ . \tag{10}$$

Of course, with this approach no statements about the actual shortfall probabilities of portfolios can be made, but at least *upper bounds* for the shortfall probabilities can be calculated. In this way, all previous derived relationships can be generalized very easily. Also, the corresponding diagrams may be modified in this manner: In the μ-$p(t^*)$-World the shortfall-probability-scale on the x-axis is substituted by an upper-bound-estimation-scale according to CHEBYSHEV. Therefore, all portfolios are now positioned in a world, which could be termed ***Mean-Shortfall-Probability-Bound-World***, with a ***Mean-Shortfall-Probability-Bound-Efficient-Frontier***. In the t-p(t)-World the y-axis has to be transformed, yielding a ***Threshold-Shortfall-Probability-Bound-World***, with an ***Efficient-Shortfall-Bound-Frontier***. The L-Efficient-Frontier of the μ-p(t)-World can be transformed into a ***Generalized-Distribution-L-Efficient-Frontier***. The three-dimensional μ-t-p(t)-World can be modified to a ***Mean-Threshold-Shortfall-Probability-Bound-World***, with an ***Efficient-Shortfall-Probability-Bound-Surface***.

Instead of using no information about the probability distributions of portfolios[26], the investor may decide - probably because of empirical investigations - to assume a concrete type of not normally distributed asset return. This could be an alternative ***continuous distribution***, for example the lognormal distribution, or a ***discontinuous one***, generated out of the histogram of historical return realisations. In both cases, no complete equation for the shortfall probabilities of efficient portfolios can be derived in general, but nevertheless, shortfall probabilities can be calculated using numerical approximations. The numerical methods that may be employed can guarantee a sufficient precise analysis for practical purposes. Because of this, the use of alternative distribution assumptions admittedly involves extended computing time, but is no problem from a mathematical point of view.[27]

[1] Dipl. math. oec. Jochen V. Kaduff, Swiss Institute of Banking and Finance. The author wishes to thank the Gottlieb Daimler- und Karl Benz-Stiftung for financial support of his research projekt.
[2] See KADUFF/SPREMANN (1996).
[3] The name explains itsself from the fact, that the shortfall probability is exactly the zero-degree Lower Partial Moment LPM_0.
[4] Hereafter, also the term *μ-p(t*)-World* is used. The „*"-Index at t indicates, that the target return t, referring to which the shortfall probability is calculated, has to be chosen fix.
[5] HARLOW (1991) analysed n-degree Lower Partial Moments in general. He especially calculated scenarios for first-degree and second-degree Lower Partial Moments. He subordinates zero-degree Lower Partial Moment, because of its inability to yield information about how severe a possible shortfall of the target might be.
[6] This assumes implicitly the presence of an optimization problem according to the Black Model [see MARKOWITZ (1987)], but other modellings of the restrictions in the optimization program can be treated analogously.
[7] σ is the minimal standard deviation of all portfolios with parametric mean μ. V indicates the variance-covariance matrix of the assets, and V^{-1} the inverse to it. In addition there is $a=\mu^T V^{-1}\mu$, $b=\mu^T V^{-1}e$ and $c=e^T V^{-1}e$, with the unity vector e of corresponding dimension.
[8] This is the case for private as well as institutional investors.
[9] Theoretically, the investor has an unlimited number of alternatives to select his target return from. For practical applications this is of course not true, but still the number of alternatives can be high.
[10] In order to short notation the name *t-p(t)-World* is going to be used hereafter.
[11] This assumes implicitly that an optimization problem according to the Black-Model is used. Other formulations can be treated analogously .
[12] Only this guarantees the existence of tangential points on the *efficient section* of the hyperbola.
[13] Again, this assumes implicitly, that an optimization problem according to the Black-Model is used.
[14] Under the assumption of normally distributed returns Safety-First-Portfolios can be found only on the upper section of the hyperbola. Therefore, μ has to satisfy the condition: $\mu > b/c$ in equation (8).
[15] The term b/c equals the expected return of the Minimum-Standard-Deviation-Portfolio.
[16] According to TELSER (1955), an investor should specify in advance an individual target return as well as an acceptable probability of failing to reach that target, and then select from the set of feasible portfolios meeting both conditions simultaneously the portfolio with the maximum expected return.
[17] This term expresses the fact, that both risk dimensions, target return and corresponding shortfall probability are covered parametrically.
[18] Again, we implicitly assume an optimization problem according to the Black-Model here.
[19] In choosing μ_S the restriction $\mu_S > b/c$, in choosing t the restriction $t < b/c$ and in choosing p(t) the restriction $0 < p(t) < 1$ has to be respected.
[20] The selection of the two parameters in the μ-σ-World is nothing more than the identification of a Shortfall Line, because a line can be identified unambiguously by either fixing two points (this means choosing μ_S and t) or fixing one point and the slope (this means choosing p(t) and t or p(t) and μ_S).
[21] For the target return the intervall [-5% ; 8%] and for the expected return the intervall [12% ; 30%] has been used. This is, because the expected return of the Minimum-Standard-Deviation-Portfolio is 10,32% in our example, which means that the intervalls for the target return and the expected return must lie below and above, respectively.
[22] Ceteris paribus means to keep the expected return constantly here.
[23] According to KATAOKA (1964) an investor should first, in contrast to the approach of ROY (1952), specify an acceptable shortfall probability, with which the target return, which is to maximise subsequentlly, can be failed to meet. The portfolio solving this optimization problem is the so-called Kataoka-Portfolio.

[24] This line runs parallel to the x-axis in three-dimensional space.
[25] This is the case for the analytical as well as graphical relationships derived so far.
[26] „No information" refers only to the *shape* of the probability distributions of the returns. Of course, also the CHEBYSHEV-estimations require at least the knowledge of the distribution parameters mean and standard deviation of all assets [see Equation (10)]. Also, one has to know the correlations between all asset returns.
[27] In this context the article of KALIN/ZAGST (1995) should be mentioned. They investigate under which generalized distributions Mean-Variance portfolio optimization and shortfall-risk-based portfolio optimization coincide. They prove the connection for the wide class of two-parameter-distributions. This class contains besides the normal-distribution, other in the context of Portfolio Theory commonly used distributions, like the two-point-distribution, the triangular-distribution and the exponential-distribution.

Appendix

In order to generate the alternative representations of efficient frontiers displayed in this article, a concrete scenario has been used. To focus attention on the main ideas behind the diagrams, it has been refrained from plotting the scale of axes in most of the diagrams. Nevertheless, in the following the underlying data of the example will be given.

Imagine an investor with two investment opportunities with stochastic returns. Mean and standard deviation of those assets are given by the following table. All data refer to a holding period of one year and are chosen to have practical relevance:

Asset	A	B
μ	9%	12%
σ	17%	18%

Table 1: Annual expected returns and standard deviations of the available risky assets.

In addition, the investor knows the covariance matrix between these two assets:

	A	B
A	289	162
B	162	324

Table 2: Covariance matrix of the risky assets.

The investor can invest any amount of his capital in both assets and is even allowed to sell them short, if he wants to. Therefore, we have an optimization problem according to BLACK, as it was stated earlier several times. With the data given above, the classical Efficient Frontier of the μ-σ-World according to equation (3) can be generated:

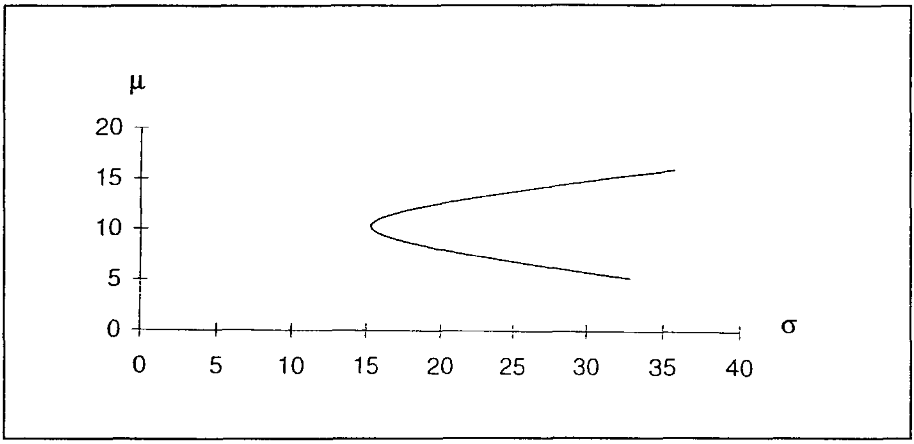

Figure 5: The Efficient Frontier in the μ-σ-World.

With the data given, the Minimum-Standard-Deviation-Portfolio has the coordinates (15.27 , 10.31). To calculate the shortfall probabilities additional information about the shape of the probability distributions are required. As assumed for the analytical analysis, we use the normal-distribution for our diagrams. Further, a target return has to be pre-specified, corresponding to which the shortfall probabilities are calculated. Here, a target return of 0% has been employed, expressing the idea of nominal capital preservation. Of course, alternative target returns are possible.

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