

## **Paradoxes Regarding the Calculation of Options The Keys to the Enigma ...?**

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### **Abstract**

The importance of derivatives in the financial world should not have to be proved any longer. But, for most managers, they still come out of a mysterious "black box" which gives them their valuations. A good perception of the valuation of these products is therefore a pledge for security. We do not need to be reminded here of the recommendations of the committee of Basle about the risk of derivatives.

We all know the limits of the classic formulas used in the calculation of options. These limits come from the hypotheses which simplify the environment. But, few operators want to recognise that the use of the formulas sometimes ends in paradoxes. Maybe the methods currently used are not satisfactory.

The author suggests following a way which was abandoned a long time ago. In doing so, he is led to examine some principles of modern theory of options such as the risk-neutral probability and the use, in certain cases, of the free-arbitrage theory. For example the author assumes that it is impossible to duplicate an option with a riskless portfolio when using the binomial process for a neutral to risk investor ...

The method generalises the classic methods and allows us to value options for which no market exists and therefore arbitrage operations are not possible. That is the case, for example, for options on interest rate or on insurance products. Furthermore, it is now easier to analyse the structure of bond rates, by introducing the notion of optional yield.

This study could appear iconoclastic, but it is nothing less than the expression of the interrogations of a manager facing the use of the current formulas in markets.

### **Keywords**

Options, Black and Scholes, Cox, Ross and Rubinstein, Arbitrage Pricing Theory, risk neutral manager, risk-neutral probability, yield-neutral probability.

### Résumé

L'importance des produits dérivés pour le monde financier, n'est plus à démontrer. Mais, pour beaucoup d'intervenants, ils sortent encore d'une mystérieuse "boite noire" qui en donnerait la valorisation. Une bonne perception de l'évaluation de ces produits est pourtant un gage de sécurité. Il n'est pas besoin de rappeler, ici, les recommandations du comité de Bâle sur les risques des produits dérivés.

Or, nous connaissons tous, les limites des formules classiques employées pour la valorisation des options. Ces limites proviennent d'hypothèses nécessairement réductrices de l'environnement. Mais, peu d'opérateurs reconnaissent volontiers, que l'utilisation de ces formules aboutit, aussi, quelque fois à des paradoxes. Il se pourrait donc que les méthodes actuelles ne soient pas satisfaisantes.

L'auteur propose, alors, de reprendre une voie, qui avait été abandonnée en son temps. Ce faisant, il est obligé de mettre en cause certains principes de la théorie moderne des options comme la probabilité de risque neutre, et l'utilisation, dans certains cas, du principe de l'Absence d'Opportunité d'Arbitrage. Par exemple, il semble impossible de dupliquer une option avec un portefeuille sans risque dans le processus binomial pour un porteur neutre au risque ...

La méthode proposée généralise les méthodes classiques et peut valoriser des options pour lesquelles il n'existe pas de marché et donc, pas d'arbitrage possible. C'est le cas, par exemple, des options sur taux et des produits d'assurance. De plus, elle permet d'aborder d'une façon simple l'analyse de la structure des taux des emprunts obligataires en introduisant la notion de taux de rendement actuariel optionnel.

Cette étude pourra paraître iconoclaste, mais elle n'est que la relation des interrogations que peut se poser un gestionnaire face à l'utilisation des formules classiques sur les marchés.

### Mots clefs

Options, Black et Scholes, Cox, Ross et Rubinstein, absence d'opportunité d'arbitrage, porteur neutre au risque, probabilité de risque neutre, probabilité de rendement neutre.

*A first version of this study was presented to the financial group of French Actuarial Association I.S.F.A. in September 1992 (Jousseau 1993), then at the International A.F.I.R. Colloquiums, in Orlando in April 1994. and in Brussels in September 1995.*

*The following version puts together the main parts of these articles, taking into considerations the remarks which have been presented to the author. It brings the last analysis of the latter in and also, perhaps, the keys to the enigma.*

### Critical analysis of the current methods

#### The Black and Scholes 's approach.

In 1973, Fisher Black and Myron Scholes proposed to solve the problem of the valuation of options by a new approach based on stochastic differential equations. The question was to study, at any point in time, the constraints which determine the price of the option, knowing the variation law of the underlying stock according to the Arbitrage Pricing Theory. They obtained a relation between the derivatives of the function of the price of the stock and of the function of the price of the option.

The process used by Black and Scholes is based on the analysis of the variations of the value of a portfolio built with stocks and one option on this stock.

The number of stocks for this option must change to counterbalance the variations of the option. This portfolio is, therefore, riskless, and must yield the riskless rate.

Consequently, the price of the call must reply to the following identity :

$$\text{Call} = S_0 \cdot N(d_1) - E \cdot e^{-(r \cdot t)} \cdot N(d_2)$$

$$d_1 = \frac{\text{Log}(S_0/E) + (r + \sigma^2/2) t}{(\sigma \cdot \sqrt{t})}$$

$$d_2 = \frac{\text{Log}(S_0/E) + (r - \sigma^2/2) t}{(\sigma \cdot \sqrt{t})}$$

$$d_2 = d_1 - \sigma \sqrt{t}$$

with :

$S_0$  : Price of the stock at calculation,  $t = 0$

$E$  : Strike price of the option

$r$  : Riskless rate

$t$  : Maturity of the option

$N(d)$  : Probability of Gauss

A characteristic of the Black and Scholes formula is the simplicity of the result :

Not many parameters:

- the price of the underlying stock.
- the strike of the option.
- the maturity of the option.
- the riskless rate.
- a parameter of the dispersion of the price of the stock - the volatility.

Very simple means of calculation:

- the logarithm.
- the Normal probability.

But we have to keep in mind that the formula depends on several hypotheses, and in particular, on the law of the underlying stock.

### The limits of the Black and Scholes formula

We all know the limits of the Black and Scholes formula. And Fisher Black himself reminds us that they are some unrealistic hypotheses in this model:

- There is no transaction fee.
- There is no margin.
- The volatility is known and constant until the exercise of the option.
- There is no jump in the variation of the price of the underlying stock.
- The riskless rate does not change during the period.
- Dividends and callables before the exercise are not allowed.
- There is no take-over bid that may change the nature of the stock and force managers call the option before the end of the period.

Nevertheless, this remarkable result has been used to enable the development of the option markets. This approach has determined the main conventions necessary to the running of the market. Some studies were done to generalise this result concerning particular problems ( such as currency options, future options, variable volatility ). But all authors, wanting to be credible, refer back to this formula again. We shall soon see the consequences of this remark.

### Paradoxes regarding the use of Black and Scholes formula

But after analysis and acceptance of the rules described by the hypotheses, six paradoxes could cast discredit on the reliability of the results obtained by this formula.

1 ) **The impact of a variation on the riskless rate** seems to be contrary to the anticipated results. In the Black and Scholes formula, a downturn of

the riskless rate generates a drop in the price of the option. We could hope, if the area did not change, that the expected profit, at exercise, did not vary. Then a downturn of the riskless rate would generate a growth of the price of the option. **The riskless rate used in the formula does not seem to be a simple actualisation rate.** The price of the option should react as a actualisation value. The latest researches confirmed this principle but did not look at all the consequences.

2 ) In the logic of the area described by Black and Scholes, **the limit of the price obtained by the formula when the volatility goes to zero must be the one of a option of a certain event.** We should have the actuarial value of the difference between the average price of the price of the underlying stock and the strike price.

According to the law of the variation of the underlying stock, the average value at exercise must be equal to :

$$E(S, t) = S_0 e^{(\mu.t)}$$

So :

$$\begin{aligned} \text{Call} &= [S_0 e^{(\mu.t)} - E] e^{-(r'.t)} & (\text{if } \sigma = 0) \\ \text{Call} &= F(\mu, r') \end{aligned}$$

This value is different from the one obtained by the Black and Scholes.

$$\begin{aligned} \text{Call (B. \& S.)} &= S_0 - E \cdot e^{-(r.t)} & (\text{if } \sigma = 0) \\ \text{Call} &= F(r, r) \end{aligned}$$

Let us add, the extra axiom which supposes that all certain events yield the riskless rate. When  $\sigma$  decreases the drift  $\mu$  of the underlying stock gets nearer the riskless rate  $r$ . In fact the price of the call option tends towards  $F(\mu, r')$  which tends itself towards  $F(r, r)$ . To tend directly towards  $F(r, r)$  does not seem to be correct from a mathematical point of view.

We could analyse in the same way the case where  $N(d1)$  and  $N(d2)$  tend towards one.

3 ) The volatility does not depend on the option itself, but on the law of the underlying stock. Now, operators change this parameter according to their mood or their anticipation. The volatility is not the same for two strike prices; or for a call option or a put option. It works like if **the Black and Scholes formula does not have enough degrees of freedom** to simulate the aleatory nature of the market. We have to utilise another parameter, the volatility, under the pretext of keeping a scientific character to the quoted prices.

4 ) In this formula the volatility is the only parameter which shows the trajectory of the stock. Consequently two stocks which have the same volatility but not the same drift, generate the same value of the option... We could think that the whole information of the drift is contained in the price of the underlying stock. But it seems that it is not. Otherwise we could say the same thing about the volatility. In fact, at any point in time, the stock price takes into account the drift and the volatility. And one could be satisfied with this unique parameter. But the price is the representation of an infinite number of couples ( drift, volatility ). Hence to calculate the price of the option we have to know the trajectory of the underlying stock and not only its value at a particular time. We have to specify in the formula the value of the drift and of the volatility of the stock.

All the processes, which exclude one of these two parameters ( the drift or the volatility), lead to contradictions. A trajectory has two degrees of freedom. This formula has only one.

In fact, it seems that, in the Black and Scholes formula, all the underlying stocks are reduced to a unique virtual model and loses their characteristics.

Therefore, some authors refuse the idea of the presence of the value of the drift in the formula of the price of the option. In fact, for them, it forbids every market. Each operator is supposed to have its own value of the drift. We may remark that this argument leads to the negation of the stock market itself.

5 ) Let us suppose that we buy a call option and sell a put option, we will receive in all cases the algebraic difference between the price of the stock at expiry and the strike price. If we repeat this operation many times with different maturities but with the same strike price, our profit (or loss) will be equal to the capitalised difference between the average prices of the stock and the common strike price. This result is not the result given by the Black and Scholes process ( parity law ). The price of options is not linked to the average price of the stock at exercise. In this way we could obtain a certain profit ( or loss ) which is contrary to the Arbitrage Pricing Theory.

6 ) We can observe that a simple change of items enables us to go from the Black and Scholes formula to the Black formula ( option on future ), that makes us to suppose that the future price of the stock is equal to the spot price capitalised with the riskless rate. If it was so, the drift of the stock would be equal to the riskless rate and not to  $\mu$ .

#### The binomial approach : The Cox, Ross and Rubinstein model

Suppose that we have a portfolio composed with  $\Delta$  stocks and a riskless asset B ( see. Cox, Ross, Rubinstein process ).

$$P_0 = \Delta S + B$$

$$P_0 \begin{cases} P_u = \Delta S_u + B r & \text{probability } p \\ P_d = \Delta S_d + B r & \text{probability } (1 - p) \end{cases}$$

Let us choose  $\Delta$  and  $B$  so that this portfolio "duplicates" the option of the stock  $S$

In that case we obtain two equations with two unknown quantities.

$$\begin{aligned} C_u &= \Delta S_u + B r \\ C_d &= \Delta S_d + B r \end{aligned}$$

so :

$$\Delta = \frac{C_u - C_d}{S(u - d)} \quad B = \frac{u C_d - d C_u}{(u - d)r}$$

then :

$$C = \Delta S + B$$

and :

$$C = [p' C_u + (1 - p') C_d] / r$$

$$\text{with } p' = \frac{r - d}{u - d}$$

The price of the option is equal to the actuarial value with the riskless rate of the future amounts of the option with the probability  $p'$ . This probability is called "the risk-neutral probability".

Step by step, beginning with the value of the option at exercise, we obtain the Cox, Ross, Rubinstein formula.

$$C_0 = \frac{1}{r^n} \left[ \sum_{k=0}^n \frac{n!}{k! (n-k)!} p'^k (1-p')^{(n-k)} \text{MAX.} [ (u^k d^{(n-k)} S_0 - E), 0 ] \right]$$

We can see that the constant  $p'$  used in the Cox, Ross and Rubinstein formula, is not the probability of the favourable event, but a number included between zero and one, that we may assimilate to a probability ( of an event we have to define ). We only have used a clever arithmetic trick.

We notice that the result of the formula is independent on the value of  $p$ .

The analogy with the Black and Scholes formula seems to be quite good, we also have :

$$p' = p \Leftrightarrow \mu = r$$

### The binomial approach by the Black and Scholes method

Suppose that we now have a portfolio composed with  $n$  stocks and the sale of call option .

$$P = n S - C$$

$$P \begin{cases} P_u = n S u - C_u & \text{probability } p \\ P_d = n S d - C_d & \text{probability } (1 - p) \end{cases}$$

Let us choose  $n$  so that the two values of this portfolio are equal:

$$P_u = n S u - C_u = P_d = n S d - C_d$$

so :

$$n = \frac{C_u - C_d}{S(u - d)}$$

with this remark and according to the Arbitrage Pricing Theory, the portfolio becomes a riskless asset, it must yield the riskless rate

$$P_u = n S u - C_u = P_d = n S d - C_d = P \cdot r$$

it follows :

$$C = [p' C_u + (1 - p') C_d] / r$$

$$\text{then } p' = \frac{r - d}{u - d}$$

The price of the option is equal to the actuarial value with the riskless rate of the future amounts of the option with the probability  $p'$ .

Step by step, beginning by the value of the option at exercise, we obtain the Cox Ross and Rubinstein formula.

### Paradoxes regarding the use of the Cox, Ross and Rubinstein formula

1) The Cox, Ross and Rubinstein formula utilises a "risk-neutral probability" as a mathematical trick. This probability is not connected with an existing event. **The price of the call is the actuarial value of a virtual event.**

2 ) What would be the law of probability of the underlying stock, this is modified in the same way. This formula, just like the Black and Scholes formula "standardises" all underlying stocks.

### Actuarial probable approach

#### Concerning the valuation of the price of an option

These observations can leave doubts about the results of the formula as it is used in the market.

Nowadays, operators favour the arbitrage as a source of the definition of the price of financial instruments. Indeed, this process seems to be free of difficulties like the perception of the risk by investors.

But we would consider that **an option is also an insurance product with a single premium and a certain maturity**. In that case, **the fair price would be equal to the sum of the probable profits, until the exercise date**, actualised at the moment of calculation. This way is not odd, it was studied by Cox and Ross, in 1976, to find the result of the Black and Scholes formula again. We shall see that actuarial process gives a generalisation to this formula, and enables to explain the above paradoxes.

A well informed reader could be opposed to this hypotheses, because of the famous **paradox of St. Petersburg** according to which a player does consider the expectation of the profits to estimate his premium. We find the **utility theory** again. We could also refer to the paradox of Allais, quoted by Clarkson ( AFIR IV - 1994 ).

In fact, in this study, we are not interested in an unique player who would play one time, but professionals who are used to the valuation of risk, and who would repeat operations a great number of times on an efficient market. That is the main difference between a single game and a **regulated market**, where the number of operators leads the **fair price to equilibrate** itself around the definition quoted above. In the opposite case, a riskless profit ( or loss ) could occur. We know that is contrary to the Arbitrage Pricing Theory.

So, we could admit that operators are neutral to the risk. Id est, they are not interested in the scattering of the risk or in the importance of the volatility but in average profit. For that kind of operators, the aim is to maximise average yield.

**In a regulated market, where a great number of dealings take place, operators can only be neutral to risk. That is to say, they are not interested in the scattering of the risk but in the average yield.**

On the other hand, the standpoint facing the quality of the risk (like failure) can be evaluated in this theory, by the **value of the rate asked by the investor**. This is the only parameter which the operator could modify.

**The Actuarial Probable Approach.**

For an option, profit is a unique amount paid at the expiry. The value is equal to the positive difference between the price of the stock and the strike price at exercise.

It follows that the price of a call option is equal to :

$$\text{Call} = \frac{1}{(1 + t_x)^t} \int_E^{+\infty} (X - E) f_x(x) dx$$

or in the case of a continuous compounding with :  $1 + t_x = e^t$

$\text{Call} = \int_E^{+\infty} (X - E) e^{-(r.t)} f_x(x) dx$
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with :

$f_x(x)$  : function of the density of the studied law  
 $E$  : strike of the option.

The integration is calculated for prices  $X$  of the stock higher than the strike  $E$  at exercise.

**The Black and Scholes environment - the actuarial method**

It will be important to analyse, further on, the hypotheses used in the Black and Scholes model and particular those connected to the law of the variation of the price of the underlying stock  $S(t)$ .

We know that :

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dZ(t)$$

where :

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma N(t) \sqrt{dt}$$

$N(t)$  follows a Normal Distribution.

then, using the Itô lemma, we could demonstrate that:

$\text{Log} ( S / S_0 )$  follows a Normal Distribution

mean  $(\mu - 1/2 \sigma^2) t$   
 standard deviation  $\sigma \sqrt{t}$

( in this hypothesis the mean  $S(t)$  is equal to :  $S_0 e^{\mu t}$  )

$$\text{Call} = \int_E^{+\infty} (X - E) e^{-(r.t)} f_x(x) dx$$

that is to say :

$$\text{Call} = \int_E^{+\infty} X e^{-(r.t)} f_X(x) dx - \int_E^{+\infty} E e^{-(r.t)} f_X(x) dx$$

$$\text{Call} = C_1 - C_2$$

$$1) \text{ Calculation of } C_1 = \int_E^{+\infty} X e^{-(r.t)} f_X(x) dx$$

so :

$$C_1 = S_0 e^{((\mu-r)t)} N[ ( \text{Log}(S_0/E) + (\mu+\sigma^2/2)t ) / ( \sigma \sqrt{t} ) ]$$

$$2) \text{ Calculation of } C_2 = \int_E^{+\infty} E e^{-(r.t)} f_X(x) dx$$

where :

$$C_2 = E e^{-(r.t)} N[ ( \text{Log}(S_0/E) + (\mu-\sigma^2/2)t ) / ( \sigma \sqrt{t} ) ]$$

from which the formula :

$$\text{Call} = S_0 e^{((\mu-r)t)} N(d_1) - E e^{-(r.t)} N(d_2)$$

$$\text{Call} = [ S_0 e^{(\mu.t)} N(d_1) - E N(d_2) ] e^{-(r.t)}$$

$$d_1 = [ \text{Log}(S_0/E) + (\mu+\sigma^2/2)t ] / ( \sigma \sqrt{t} )$$

$$d_2 = [ \text{Log}(S_0/E) + (\mu-\sigma^2/2)t ] / ( \sigma \sqrt{t} )$$

The value of a call option is equal to the difference between the average price of the stock at exercise and the strike price, these two parameters would be indexed to the probabilities  $N(d_1)$  and  $N(d_2)$ , discounted to the moment of calculation.

Whatever the law of the price of the stock  $X(t)$  :

$$\text{Call} = \int_E^{+\infty} (X - E) e^{-(r.t)} f_X(x) dx$$

$$\text{Put} = \int_{-\infty}^E (E - X) e^{-(r.t)} f_X(x) dx$$

$$\text{Call} - \text{Put} = \int_{-\infty}^{+\infty} (X - E) e^{-(r.t)} f_X(x) dx$$

$$\text{Call} - \text{Put} = (S(t) - E) e^{-r.t}$$

Call - Put = Discounted value of the average profit

$S(t)$  : average value of the underlying stock at exercise.

To calculate the price of a put option, we only need to observe that :

$$\text{Call} - \text{Put} = S_0 e^{((\mu-r).t)} - E e^{-r.t}$$

$$\text{Put} = E e^{-r.t} N(-d_2) - S_0 e^{((\mu-r).t)} N(-d_1)$$

$$\text{Put} = [ E N(-d_2) - S_0 e^{(\mu.t)} N(-d_1) ] e^{-r.t}$$

These formulas have a continuous discounting factor which can be replaced by a discrete discounting factor.

$$\text{Call} = \frac{S_0 (1 + \mu)^t N(d_1) - E N(d_2)}{(1 + r)^t}$$

$$\text{Put} = \frac{E N(-d_2) - S_0 (1 + \mu)^t N(-d_1)}{(1 + r)^t}$$

No particular hypothesis is used to calculate this value, except those which are connected to the law of the variation of the stock. We check that the fair price is dependent on the drift of the stock. That seems logical to differentiate the prices of calls of two stocks which have the same volatility but two different drifts. On the other hand, this formula solves the above paradoxes.

#### Comparison between the current and the actuarial formula

To find the Black and Scholes formula again, we have to simulate (to add ?) the equality of the drift of the stock and of the riskless rate.

$$\text{Call} = S_0 N(d_1) - E e^{-r.t} N(d_2)$$

(Black and Scholes formula)

**The drift of the stock does not seem to be absent of the Black and Scholes formula. This parameter is, in fact, equal to the riskless rate. This formula is nothing but a particular case of a general formula.**

In fact, the value of the drift may be very different from the value of the riskless rate, and so, the application of the Black and Scholes formula leads then to aberrations that we try to ease by modifications of volatility.

#### Remarks about the Cox Ross formula

Several authors have proved the Black and Scholes formula (among them Cox and Ross in 1976), by using the probable actuarial yield process. But to conclude they all needed to specify extra hypotheses:

a ) The investors must be neutral facing the risk.

That was explained by the authors as follows:

The price of the option must be equal to the actuarial value of the average profit with the riskless rate.

But, to actualise an asset with a defined rate means that we define its own yield or drift. And, in this particular case, the drift is equal to the riskless rate. That proves that this asset is a riskless asset...

b ) On the other hand, in the demonstration, when using the Itô lemma, we can see that the drift of the underlying stock is supposed to be equal to the riskless rate. This property is opposite to the hypotheses of the Black and Scholes model.

This is explained by the use of the risk-neutral probability.

It seems surprising that to demonstrate directly the Black and Scholes formula we have to add an extra hypothesis which comes from the result of the binomial process. All goes as if this "shadow" hypothesis has been added surreptitiously during the calculus of the Black and Scholes formula and becomes then necessary.

These two "remarks" show "the price to be paid" for finding the Black and Scholes formula by a direct discounting again.

In fact, we could again find the suggested result by the Cox and Ross process. For that, we have to admit the Arbitrage Pricing Theory imposes that the price of an option is equal to the average profit discounted at the moment of the calculation with a rate  $r'$  chosen by the investor. The bearers are well supposed to be neutral facing with the risk.

If  $P(t)$  is the value of the price of the underlying stock at instant  $t$  :

$$C(P(t),t) = e^{-r't} E [ C(P(0),0) ] \quad t=0 \text{ at exercise.}$$

Let say  $X(t) = \text{Log}(P(t))$  and let us use the Itô lemma :

$$dX(t) = \left[ \frac{\delta(\text{Log}(P(t)))}{\delta P} \mu P(t) + \frac{\delta^2(\text{Log}(P(t)))}{\delta P^2} \sigma^2 P^2(t) \right] dt$$

$$+ \frac{\delta(\text{Log}(P(t)))}{\delta P} \sigma P(t) dZ(t)$$

$$dX(t) = (\mu - \sigma^2/2) dt + \sigma dZ(t)$$

$X(t)$  follows a normal law with the parameters :

Mean	$X_{(t_0)} + (\mu - \sigma^2/2)$
Standard deviation	$\sigma^2 t$
$t_0$	date of the calculation

Direct integration gives the probable actuarial formula.

### Options on futures

In fact the value of the underlying stock which has to be used in the option formula is the one which is supposed to happen at exercise, and not the one at the valuation date. According to that, for an underlying stock on which is based a future market, the operators are used to using the future price in place of the spot price. This practice seems quite justified bearing in mind that the futures price is a good estimation of the value of  $S_0 e^{(\mu.t)}$ .

Then the formula becomes :

$$\text{Call} = [ F_n N(d_1) - E N(d_2) ] e^{-(r t)}$$

$$\text{Put} = [ E N(-d_2) - F_n N(-d_1) ] e^{-(r t)}$$

with  $F_n$  : value of the futures price.

( Black formula )

We can see that operators who deny the notion of a drift in the Black and Scholes formula, are using it, without knowing it, in the Black formula.

$$F_n \# S_0 e^{(\mu.t)}$$

Options on currencies

Let us take, now, this actuarial formula in the case of currency options. If the profit at exercise  $t$  will be considered as equal to :

$$\text{Profit} = X e^{(t_i.t)} - E$$

with:

$X$  price of the asset in currency  
 $e^{(t_i.t)}$  discounting factor with the rate  $t_i$  of the currency  
 $E$  strike price of the asset in currency of the contract

then :

$$\text{Option} = \int_E^{+\infty} (X e^{(t_i.t)} - E) e^{-(r.t)} f_X(x) dx$$

so :

$$\text{Call} = S_0 e^{((t_i+\mu-r).t)} N(d_1) - E e^{-(r.t)} N(d_2)$$

$$\text{Put} = E e^{-(r.t)} N(-d_2) - S_0 e^{((t_i+\mu-r).t)} N(-d_1)$$

With a discrete discounting the result becomes :

$$\text{Call} = \frac{S_0 (1 + t_i)^t (1 + \mu)^t N(d_1) - E N(d_2)}{(1 + r)^t}$$

$$\text{Put} = \frac{E N(-d_2) - S_0 (1 + t_i)^t (1 + \mu)^t N(-d_1)}{(1 + r)^t}$$

in the case of  $\mu$  is equal to  $r$  :

$$\text{Call} = S_0 e^{(t_i.t)} N(d_1) - E e^{-(r.t)} N(d_2)$$

( Garman-Kohlhagen formula )

If, furthermore,  $t_i$  is equal to zero, which means that the amount of the underlying asset can't be invested on a financial market, we find the result of the Black and Scholes formula again.

**The binomial process - the actuarial approach**

Let us study the suggested method in the case of an asset which follows a binomial process. That is to say, that the asset is supposed to take, at time  $t_{(k+1)}$  only two values :  $S_k u$  and  $S_k d$

In particular for  $k = 0$ , we have :

$$S_0 \begin{cases} S_1(1) = S_0 u & \text{probability } p \\ S_1(0) = S_0 d & \text{probability } (1 - p) \end{cases}$$

We know that in a governed market :

$$S_0 \mu = E(S_1) = p S_0 u + (1-p) S_0 d$$

(  $\mu$  drift of the stock or actuarial yield of the stock.)

That gives the **fundamental relation** which describes the system :

$$p = \frac{\mu - d}{u - d}$$

Three of these four parameters are enough to determine the system.

So for step  $n$ , state  $k$  :

$$S_n(k) = S_0 u^k d^{(n-k)} \quad \text{with } k=0,n$$

$$\text{Prob. } (X_n = S_n(k)) = \frac{n!}{k! (n-k)!} p^k (1-p)^{(n-k)}$$

probability that the price of the stock  $X_n$  at the step  $n$  is equal to  $S_n(k)$ .

If a call option on that asset exists, supposing that the strike price is equal to  $E$ , the valuation of this option must be equal to the probable profit:

$$C_0 = \frac{\sum_{k=a}^{k=n} \text{Prob. } (X_n = S_n(k)) \text{ MAX. } [ ( S_n(k) - E ) , 0 ]}{\mu^n}$$

$$C_0 = \frac{1}{\mu^n} \sum_{k=0}^n \frac{n!}{k!(n-k)!} p^k (1-p)^{(n-k)} \text{MAX.} [ (u^k d^{(n-k)} S_0 - E), 0 ]$$

$\mu'$  = drift of the option or actualisation factor of the option.

The law of the variation of the price of the underlying stock is sufficient to determine the valuation of the option.

### Comparison between the two methods

#### Comparative analysis between the current and the actuarial method

In conclusion, we can observe that :

- 1 ) The use of the Arbitrage Pricing Theory is not necessary to determine a price of option.
- 2 ) The current formulas used in the market:
  - lead to the value of a price which is an actuarial value of the future profit. That justifies the proposed process.
  - suppose ( or simulate ) the equality of the drift of the underlying stock and of the riskless rate.
- 3 ) The current formulas and the actuarial formula assumes the bearers to be risk-neutral.

We know that we can link the Black and Scholes formula to the Cox, Ross and Rubinstein formula.

It would be interesting to study these two current methods and try to find the hypotheses which limit their use facing the actuarial method.

Then by comparison with the actuarial process and with the current models, these last formulas used two added theories :

- the Arbitrage Pricing Theory
- the risk-neutral probability

#### The special case of the Black and Scholes formula

As we have seen, the variation of the price of the underlying stock is divided into two components; the drift of the stock and the stochastic scattering of the stock around this drift. The aim of the process proposed by Black and Scholes is to free from this scattering by using a portfolio. It seems logical to think that in this case, the portfolio will yield "on average" as the drift of the underlying and not as a riskless portfolio. The Arbitrage Pricing Theory leads to impose the yield of the riskless rate.

Let us take the portfolio described by Black and Scholes and the proper number of stocks needed to "cover" aleatory effects.

$$P = C(S, t) + n S(\mu, \sigma, t)$$

$$\text{with : } n = - \frac{\partial C}{\partial S}$$

Let us suppose that the yield of this portfolio is constant ( that is not obligatory ) and equal to  $\lambda$ .

We have then :

$$dP = P \lambda dt$$

$$dP = \left[ \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right] dt = \lambda \left[ C - S \frac{\partial C}{\partial S} \right] dt$$

We can follow the Black and Scholes process without using the Arbitrage Pricing Theory.

After finding the general solution to the equation, we have to determine the solution linked to the limit conditions.

Price of the call equal to  $\text{MAX}(S - K, 0)$  ( at exercise )

Price of the call equal to  $[ S_0 e^{(\mu \cdot t)} - E ] e^{-(r \cdot t)}$  ( if  $\sigma = 0$  )

The analysis of these conditions leads to pose  $\lambda = \mu$

That forbids the use of the Arbitrage Pricing Theory.

Otherwise, the number of stocks needed to balance the stochastic variations is itself an stochastic item. This difficulty can only be erased in a two state process. In the general case ( more than two states or with continued items ), we can assume that it is not possible to determine this number. **That can cast doubt over obtaining a riskless portfolio.** We have to remember that only this property enables us to use the Arbitrage Pricing Theory in the Black and Scholes formula.

### The Arbitrage Pricing Theory

Before studying the binomial process we have to recall the Arbitrage Pricing Theory.

In accordance with this theory :

" A certain asset can only yield the riskless rate of its own maturity. "

" An operator cannot hope to earn more than the riskless rate without taking risk. "

A consequence of this theory is that the price of a stochastic asset must be equal to the actuarial value of the average profit. In the other case the use of the Law of the Large Numbers leads to a riskless profit (or loss) which is forbidden by the Arbitrage Pricing Theory.

... used in the binomial process

Let us suppose we have two assets which follow the binomial process but with different values.

We can construct a portfolio composed of these two assets. The yield of this portfolio is equal to the average yield of these two assets indexed by the importance of each of these assets in the portfolio. This yield only depends on the yield of each of these assets and on the composition of the portfolio.

Otherwise we know that we can compose a portfolio so that this one will always have the same value.

Using the Arbitrage Pricing Theory we conclude that this portfolio yields the riskless rate.

The portfolio has then two yields, the yield linked to the yields of the two assets and the riskless rate. That is impossible.

Let us have  $S_1$  and  $S_2$  following the same binomial process.

$$S_1 \begin{cases} S_1 u_1 \\ S_1 d_1 \end{cases} \quad \text{and} \quad S_2 \begin{cases} S_2 u_2 & \text{probability } p \\ S_2 d_2 & \text{probability } (1-p) \end{cases}$$

we can define two items  $\mu_1$  and  $\mu_2$ .

$$\begin{cases} S_1 \mu_1 = p S_1 u_1 + (1-p) S_1 d_1 \\ S_2 \mu_2 = p S_2 u_2 + (1-p) S_2 d_2 \end{cases}$$

then :

$$p = \frac{\mu_1 - d_1}{u_1 - d_1} = \frac{\mu_2 - d_2}{u_2 - d_2}$$

Let us take a portfolio composed of  $n_1$  stocks  $S_1$  and of  $n_2$  stocks  $S_2$ .

$$n_1 S_1 + n_2 S_2 = F$$

$$\begin{cases} n_1 S_1 u_1 + n_2 S_2 u_2 = F u & \text{probability } p \\ n_1 S_1 d_1 + n_2 S_2 d_2 = F d & \text{probability } (1-p) \end{cases}$$

With :

$$F \mu = p F u + (1-p) F d$$

$$\begin{aligned} F \mu &= p [n_1 S_1 u_1 + n_2 S_2 u_2] + (1-p) [n_1 S_1 d_1 + n_2 S_2 d_2] \\ F \mu &= n_1 S_1 \mu_1 + n_2 S_2 \mu_2 \end{aligned}$$

$$\mu = \frac{n_1 S_1}{F} \mu_1 + \frac{n_2 S_2}{F} \mu_2$$

The yield  $\mu$  of the portfolio F is well defined .

We have also a remarkable relation

$$\frac{n_1 S_1}{n_2 S_2} = \frac{\mu - \mu_2}{\mu - \mu_1}$$

If we have more :  $F u = F d$

$$\begin{aligned} n_1 S_1 u_1 + n_2 S_2 u_2 &= n_1 S_1 d_1 + n_2 S_2 d_2 \\ \frac{n_1 S_1 (u_1 - d_1)}{n_2 S_2 (u_2 - d_2)} &= \frac{n_1 S_1 d_1 + n_2 S_2 d_2 - n_1 S_1 d_1 - n_2 S_2 d_2}{n_2 S_2 (u_2 - d_2)} \end{aligned}$$

$$\frac{n_1}{n_2} = \frac{S_2 (u_2 - d_2)}{S_1 (u_1 - d_1)} = \frac{\Delta S_2}{\Delta S_1}$$

The portfolio F is riskless, that means that according to the Arbitrage Pricing Theory its own yield  $\mu$  is also equal to the riskless rate  $r$ ...

In fact the binomial process because it allows only two possible states is a bad system to analyse the market when using the Arbitrage Pricing Theory. The stochastic nature is too easily erasable by a proper choice of the composition of the portfolio. That is too far from the financial market's realities.

**The use of the Arbitrage Pricing Theory in a binomial process adds a logical but conflicting relation to the equation system. That leads to a contradiction.**

In case the two drifts should be equal ( $\mu = r$ ), and the two values of the portfolio should be equal, we have the following relation.

$$\frac{n_1 S_1}{n_2 S_2} = \frac{r - \mu_2}{r - \mu_1} = \frac{(u_2 - d_2)}{(u_1 - d_1)} = k$$

with  $k > 0$

$$\text{So : } r \cdot (1 - k) = \mu_2 - k \cdot \mu_1$$

Then we have:

$$\begin{cases} n_1 S_1 u_1 + n_2 S_2 u_2 = F r & \text{probability } p \\ n_1 S_1 d_1 + n_2 S_2 d_2 = F r & \text{probability } (1 - p) \end{cases}$$

with the solution :

$$\frac{n_1}{n_2} = - \frac{S_2 (u_2 - d_2)}{S_1 (u_1 - d_1)} \quad F = \frac{n_2 S_2 (u_1 d_2 - d_1 u_2)}{(u_1 - d_1) r}$$

Hence the values of  $S_1$  and  $S_2$ .

$$S_1 = \frac{1}{r} (p' S_1 u_1 + (1 - p') S_1 d_1) \quad (A)$$

$$\text{with : } p' = \frac{(r - d_2)}{(u_2 - d_2)}$$

and

$$S_2 = \frac{1}{r} (p'' S_2 u_2 + (1 - p'') S_2 d_2) \quad (B)$$

$$\text{with : } p'' = \frac{(r - d_1)}{(u_1 - d_1)}$$

But if we divide the equation (A) by  $S_1$  we obtain:

$$p' = \frac{(r - d_2)}{(u_2 - d_2)} = \frac{(r - d_1)}{(u_1 - d_1)} = p''$$

So the remarkable results:

$$S_1 = \frac{1}{r} (p' S_1 u_1 + (1 - p') S_1 d_1)$$

$$\text{with : } p' = \frac{(r - d_1)}{(u_1 - d_1)}$$

and

$$S_2 = \frac{1}{r} (p'' S_2 u_2 + (1 - p'') S_2 d_2)$$

$$\text{with : } p'' = \frac{(r - d_2)}{(u_2 - d_2)}$$

Each of the assets is equal to the discounting value of the future amounts with the probability  $p'$  ( or  $p''$  ) linked to the riskless rate and to the parameters of the distribution, and discounted with this riskless rate.

We can note that:

**The value of the asset  $S_1$  is independent of the asset  $S_2$  and in particular of its drift  $\mu_2$**

**The value of the asset  $S_2$  is independent of the asset  $S_1$  and in particular of its drift  $\mu_1$**

The use of this portfolio enables us to define a new process to discount a stochastic asset. This actualisation is done in a riskless environment and in a yieldfree environment. The result is independent of the properties of the assets ( stock, option ) but this process adds to their properties the neutrality facing with the risk and the neutrality facing with the yield of the associate asset.

Knowing the value of  $S_1$ , by  $S$ ,  $u_1$ ,  $d_1$ , and the value of the riskless rate  $r$ , we can determine  $S_2$  according to the values of  $S_{2u}$  and  $S_{2d}$

On the other hand knowing  $S_2$ , we could wonder how many assets  $S_1$  could satisfy to the previously indicated rules. In fact, each of the assets  $S_2$  is linked to an infinite number of assets  $S_1$ .  $S_1$  and  $S_2$  belong to the same set of assets defined by the following relation:

$$r = p' u + (1 - p') d$$

The factor  $p'$  is not linked to the probability  $p$  of the binomial process. These assets are interchangeable.

Therefore a remarkable case can occurs:  $p = p'$ .

We have then  $\mu_1 = \mu_2 = \mu$

Let us take the system proposed by Cox, Ross and Rubinstein.

$$P_0 = \Delta S + B$$

$$P_0 \begin{cases} P_u = \Delta S.u + B r & \text{probability } p \\ P_d = \Delta S.d + B r & \text{probability } (1 - p) \end{cases}$$

Let us choose  $\Delta$  and  $B$  so that this portfolio "duplicates" the option on the stock  $S$

We can obtain two equations with two unknown quantities.

$$\begin{cases} C_u = \Delta S.u + B r \\ C_d = \Delta S.d + B r \end{cases}$$

in that case :

$$\Delta = \frac{C_u - C_d}{S(u - d)} \quad B = \frac{u C_d - d C_u}{(u - d) r}$$

We know that :

$$C + r = [p' C_u + (1 - p') C_d] \quad (1)$$

with  $p' = \frac{r - d}{u - d}$

We can write the equation :

$$- \Delta S + C = B$$

We can observe that:

- C and S follow a binomial process.
- The final portfolio is constant and independent of the probability p.

This process uses, without saying it, the Arbitrage Pricing Theory. This theory wants that a portfolio composed of two stochastic items is itself non stochastic. This can only happen in case of important constraints on the price of the stochastic assets. We know that this determine a relation between the drifts of the assets and the one of the constructed portfolio.

Hence we have :

$$C + \mu_2 = [p C_u + (1 - p) C_d] \quad (2)$$

with:  $\mu_2 = k \mu + r(1 - k)$   
 $\mu_2$  real drift of the option

These relations describe a martingale which defines the price of the option. This martingale is the result of the constraints wanted by the Arbitrage Pricing Theory when using the binomial process. It owns in itself, like each other martingale, all the characteristics of the strategy of the investor, and in particular his attitude facing with the risk.

Formula ( 2 ) shows that the investor should be **risk-neutral**. The price is defined as an discounted value of the probable profit calculated with the real probabilities.

Formula ( 1 ) shows that the price is also independent on the value of the drift of the stock ( S is independent on S et  $\mu$  ). We could consider that the investor must be **yield-neutral** facing with the yield of the stock, and that he only perceives one unique yield, the riskless rate (in fact the yield of the constructed portfolio)

The process determined in the Cox Ross and Rubinstein formula is linked to a very particular case.

The risk-neutral probability must be called the **yield-neutral probability**.

**We cannot "duplicate" an option in the binomial process with a riskless portfolio for a risk-neutral investor...**

In conclusion, we have two processes to calculate the price of an option.

- For risk-neutral investors, we can use the Actuarial Probable Approach.

- For risk-neutral and yield-neutral facing the yield of the stock investors, the formulas to be used are the current formulas.

We can remark that if we add the yield-neutral theory when using the Actuarial Probable Approach, the results are equal to the ones of the current formulas.

This remark enables us to explain the paradoxes and the existence of the "phantom" hypotheses needed to resolve the price of the option when using direct calculation.

### **Remarks concerning the Arbitrage Pricing Theory**

It seems difficult to deny this principle, but in the process of the valuation of an option by arbitrage we have to be careful of :

- The martingale must not alter the law of the future prices of the underlying stock.

- The arbitrage principle must take advantage of the necessary simplicity of the model to obtain results which cannot be transposed on the financial market.

In all cases the price determined by arbitrage must also be the discounted value of future amounts.

### **The risk-neutral probability principle**

In the method of **duplication of the option** we try to determine a portfolio which reproduces the price of the option. **It is not certain we can succeed. In doing so, we have to verify that the obtain result is not conflicting with the hypotheses.** We know now that it is when using the binomial process.

In the binomial process this method leads to the existence of the risk-neutral probability.

Some authors justify the use of the risk-neutral probability by the fact that the operators must be neutral facing the risk.

**But, the neutrality facing the risk does not mean the use of the risk-neutral probability.**

In the first case the bearer is insensible to the scattering of the risk. He only considers the average probable profit. That is the reason why he does not modify his perception of the distribution of the probability of events.

In the second case they are an other distribution table and so a perception of an other average profit.

Other authors describe a risk-neutral universe.

But in this universe we know that all the investors must be risk-neutral and yield-neutral.

In fact the risk-neutral probability needs nothing but to find the result of the Black and Scholes formula again when using the binomial process. In doing so we forget that this result comes from a wrong application of the Arbitrage Pricing Theory. Even so the use of the yield-neutral principle is necessary to find the result of the Black and Scholes formula again by a direct integration.

On the other hand we can remark that the duplication strategy comes back to the analysis of a Black and Scholes portfolio.

We then have only one alternative:

- To develop a virtual environment around the Black and Scholes formula with the use of the risk-neutral probability. It is the aim of the most current studies.
- To abandon this formula or at least its current version.

Therefore we have just demonstrated that the risk-neutral probability principle is contrary to the initial hypotheses of the system.

The aim of this study is to make the financial world aware of the fact that if many studies have generalised the result of the current methods, very few have dared to analyse the results itself. What would have happened if Cox Ross and Rubinstein had found their formula before Black and Scholes?

**Conclusion - The stochastic nature in the field of option pricing - The neutral delta principle - The arbitrage notion**

The value of an option concerns two types of people:

- The investors
- The brokers

The former hope to earn a profit connected to the value of the asset.

Currently the latter use to offer price defined by arbitrages linked to their portfolios. Their aim is to earn a profit on each contract.

Therefore an option is linked to stochastic events like an insurance product. Logically a portfolio composed of two stochastic assets cannot be reduced to an riskless asset except if these two assets have the same distribution of probability which is not the case for an option and for the underlying stock.

We can perceive the importance of the choice of the arbitrage models and their appropriateness to the market. Is the investor really conscious of the hypotheses needed for the calculation of the price of the option? In particular those concerning the variation of the underlying assets. What can we say about the current models in the case of very strong variations of the riskless rate, if we know that this rate is also supposed to be the drift of the underlying stock?

Experience shows that the use of the neutral delta method is perverse. It works for little change of the underlying stock but becomes false in the case of substantial changes just when we need it because of the large amount of possible losses. We can ask ourselves if a blind application of this process is not the cause of the problem for many financial companies.

The modern financial theory generalises arbitrage for pricing an asset.

First of all this process possesses its own limit: Arbitrage wants a market so it is not possible to price an asset in all cases. This method is not a general method.

Further on the arbitrage principle leads to a price which is not connected to the real value of the asset. We know that the use of the Law of Large Numbers leads to riskless profit or loss. This is contrary to the Arbitrage Pricing Theory.

At last the arbitrage process contains in itself an attitude facing the risk, even if it is not clearly described. This attitude depends on the equations of the arbitrage process.

The calculus of future SWAPS, gathers all the contradictions of the definition of a price by arbitrage. For this product we generally use the forward-forward rates. But forward-forward rates are generally very bad estimation of the price at exercise. The price of SWAP is not connected to the real value of the financial assets.

Forward SWAPS can be considered as the most perfect example of the inadequacy of the arbitrage principle for pricing an asset.

*Derivatives which have all the characteristics of insurance products are sold by agents who dislike risk. In doing so many imbalances can occur in the definition of prices. These imbalances are dangerous as the current formulas are not beyond criticisms.*

### Generalisation

The proposed method disregards the intermediate variations of the financial items but we need to know the value of the underlying stock at exercise. In doing so this process can resolve many particular problems.

Principally we have to define three items .

- The mean variation law of  $X(t)$  :  $M(t)$  or  $M(t,x)$
- The standard deviation law of  $X(t)$  :  $V(t)$  or  $V(t,x)$
- The density function of  $X(t)$ .

For example  $M(t,x)$  can be connected to the yield curve.

An other interesting example is the method described by Simon Rosenblatt and Ould Amar Yahya [AFIR 1994]

The Actuarial Probable Approach enables us to be free of too restricting hypotheses and even to evaluate the option according to the own appreciation of the risk of each operator. We shall soon see that the adaptability of this process makes the evaluation of long term options and options with a varying yield curve possible.

#### Long term options - study of the price of the call-warrant

As we have seen before the actuarial probable formula uses the law of the underlying asset at exercise. The price at the valuation time is the one which represents the future events at exercise. So a generalisation is possible for long term options. We only have to deduct from the quoted price the actuarial values of the intermediate events.

For the study of call-warrants on bonds, we prefer to take as underlying asset the yield of bonds in place of the price which can only change by the modification of the maturity. This trick enable us to exclude the impact of the intermediate revenues. Experience shows that results are very near quoted prices.

It is important to indicate that specially in the case of long term options we have a particular result. It occurs that the "time value" is negative. We are used to dividing the option price in two items, intrinsic

value and "time value". But as far as the option is an actuarial value, it can occur that this value for high rate and important maturity is lower than the intrinsic value. This only happens for European options.

The simulation of price of option can be used with more complex laws like Levy law or in fractal environment (Walter 1989).

### Convertible bonds

Convertible bonds can be analysed as a bond and an option. The knowledge of the price of the bond can be calculated by discounting. The value of the option is therefore the difference between the price and the actuarial value. If we consider that the volatility must be the historic volatility, the value of the drift used by the market can be obtained. We have therefore to compare this value of the drift with our anticipations.

### The special cases of SWAP CAP and FLOOR

This method can be used to price financial instruments such as CAPS, FLOORS or SWAPS.

A CAP is nothing but many call options on rate.

A FLOOR is nothing but many put options on rate.

A SWAP is the purchase of a CAP and the sell of a FLOOR.

It is easy to prove that for a SWAP we have the following relation

$$0 = \frac{M}{(1 + t_x)^d} \int_{-\infty}^{+\infty} (T - T_0) f_t(t) dt$$

with :

M	Capital
$f_t(t)$	density function of the variation law of the rate.
$t_x$	discounting rate
d	maturity

then :

$$0 = \bar{T} - T_0$$

$\bar{T}$  average value of the rate at exercise

The SWAP rate corresponds to the equation :  $T_0 = \bar{T}$

For a CAP with a rate  $T_1$ , we should have :

$$\text{CAP} = \frac{M}{(1 + t_x)^d} \int_{T_1}^{+\infty} (T - T_1) f_t(t) dt$$

For a FLOOR with a rate  $T_2$ , we should have :

$$\text{FLOOR} = \frac{-M}{(1 + t_x)^d} \int_{-\infty}^{T_2} (T - T_2) f_t(t) dt$$

And for a tunnel with the two rates  $T_2$   $T_1$ , we then have the following equation.

$$\int_{T_1}^{+\infty} (T - T_1) f_t(t) dt = - \int_{-\infty}^{T_2} (T - T_2) f_t(t) dt$$

### Analysis of the yield structure

In the past investors managed their portfolios very passively. They only subscribed and waited until exercise. Today they sell and buy more often. So it is very important to anticipate the future variations of the price during the life of bonds .

It seems logical to determine two parts in future according to the anticipations of investors.

- A short term ( for example one or two years ) during which investors can make hypotheses quite realistic.
- A long term during which it seems difficult to predict anything.

A purchase comes down to the following question; What would be the price of the bonds at the end of the first period? A probable approach leads to the equation.

Price	=	actuarial value of intermediate revenues	+	Call warrant strike price = 0 exercise 1 year
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This method is very general and takes into consideration:

- The term structure of rate for one year
- The volatility of the rate.
- The concavity of the yield curve.

Further more it becomes easy to compare stocks and bonds. This process determines the risk premium between these markets.

We can explain by this equation the main type of yield curve such as inverse structure or very sloping structure.

### Dynamic analysis of the price of the options

#### Dynamic analysis of the price of the options

The principles of management of options are connected to the use of derivabilities of the price of the underlying stock. Therefore the stochastic aspect of the latter seems to forbid the existence of this derivability. In particular the equality between the right and the left derivabilities is perhaps not verified for continuous items. Nevertheless for little change of the price of the underlying stock some parameters such as the coefficient delta and coefficient gamma, give good results, but for important variations the obtained results are not valid.

According to the hypotheses used by Black and Scholes we know the law of the studied stock. In particular the average price follows the following law:

$$S_{\text{average}}(t) = S_0 e^{\mu t}$$

So we can determine two important probabilities:

- Probability  $P_E$  that the price of the underlying stock will be higher than the strike price  $E$  :

$$P_E = N [ \text{Log}(S_0/E) + (\mu - \sigma^2/2) t ] / ( \sigma \sqrt{t} )$$

- Probability  $P_{E'}$  that the price of the underlying stock will be higher than the strike price  $E$  increased plus the premium :

$$P_{E'} = N [ \text{Log}(S_0/(E+P)) + (\mu - \sigma^2/2) t ] / ( \sigma \sqrt{t} )$$

For example an option with six months maturity on the French CAC 40 gives the following results :

$S_0 = 1950$	$\sigma = 20 \%$	$r = 5,5 \%$	$\mu = 7,5 \%$
$E = 1800$	call = 245	$P_E = 77,5 \%$	$P_{E'} = 42,8 \%$
$E = 1900$	call = 176	$P_E = 64,4 \%$	$P_{E'} = 39,7 \%$
$E = 2000$	call = 120	$P_E = 50,1 \%$	$P_{E'} = 34,0 \%$
$E = 2100$	call = 78	$P_E = 36,5 \%$	$P_{E'} = 27,3 \%$
$E = 2200$	call = 48	$P_E = 24,9 \%$	$P_{E'} = 20,3 \%$

In general the value of  $P_{E'}$  is around the value of 30 %.

But the Black and Scholes formula supposes that the average stock follows a law indexed on the riskless rate :

$$S_{\text{average B.S.}(t)} = S_0 e^{rt}$$

According to this remark we can observe that price of a call option on assets like stock is under evaluated in comparison with its financial value.

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