

Mean-Variance Portfolio Selection under Portfolio Insurance

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Abstract

The problem of selecting a portfolio of risky assets such that a given possibly negative return may be guaranteed is considered. For example this may be done by buying portfolio insurance. Variance reduction through diversification in an option environment is justified on the basis of a statistical and an actuarial argument, which leads to the notion of total option variance. A maximum variance reduction using options as diversification instruments is achieved by minimizing the total option variance. It is shown that the average option minimizes the total option variance provided the random value of the risky investment fund is normally distributed. In fact this choice is a preferred one for portfolio insurance as is also justified rationally. The guaranteed rate of return offered by the preferred portfolio insurance strategy depends on the expected return and the variance of the portfolio of risky assets, whose random return should be protected. The portfolio selection problem under portfolio insurance is formulated as optimal selection of a collection of risky assets such that a given return should be guaranteed under minimizing hedging costs. It leads to an extended mean-variance portfolio analysis in the sense that the classical approach by Markowitz can be included in the analysis. An analytical geometric method, which is applied to the extended mean-variance portfolio problem, is developed.

Keywords

Portfolio selection, portfolio insurance, total option variance, average option.

1. Variance reduction and diversification.

The main aim of diversifying investment opportunities is to reduce risk by equal expected return. Measuring investment risk with the variance has led Markowitz(1952/59) to lay down the foundations of Modern Portfolio Theory. How can variance reduction through diversification be justified in an economy with derivative instruments ? Two simple explanations are presented.

1.1. A statistical argument.

Let X be a random variable with finite mean and variance. Consider the set of all bivariate random splitting pairs $BD(X) := \{ (Y, Z) : X = Y + Z \}$. For the analysis of variance reduction (or increase) one has to compare the variance $\text{Var}[X]$ with the *total splitting risk* (as measured by variance) $R_x[Y, Z] := \text{Var}[Y] + \text{Var}[Z]$, where $(Y, Z) \in BD(X)$. The Cauchy-Schwarz inequality implies the best lower bound

$$R_x[Y, Z] \geq \frac{1}{2} \text{Var}[X], \text{ for all } (Y, Z) \in BD(X),$$

which is attained exactly when $Y = \frac{1}{2}(X - E[X]) + E[Y]$, $Z = \frac{1}{2}(X - E[X]) + E[Z]$.

Suppose one is only interested in variance reduction, which is the case here. Since $R_x[Y, Z] = \text{Var}[X] - 2\text{Cov}[Y, Z]$ one needs the constraint $\text{Cov}[Y, Z] \geq 0$. The obtained feasible subset of bivariate random variables is $BD^+(X) := \{ (Y, Z) \in BD(X) : \text{Cov}[Y, Z] \geq 0 \}$. If Y or Z is a constant, then $\text{Cov}[Y, Z] = 0$. Thus this set is non-void. A sufficient condition for $(Y, Z) \in BD^+(X)$ is the *comonotonic* property. By definition $(Y, Z) \in \text{Com}(X)$ if and only if there exists u, v continuous increasing functions such that $u(x) + v(x) = x$ and $Y = u(X)$, $Z = v(X)$. Since the pair (X, X) is positive quadrant dependent, one has $\text{Cov}[Y, Z] \geq 0$ for all $(Y, Z) \in \text{Com}(X)$, hence $\text{Com}(X)$ is a subset of $BD^+(X)$. One has the best bounds

$$(1.1) \quad \frac{1}{2} \text{Var}[X] \leq R_x[Y, Z] \leq \text{Var}[X], \text{ for all } (Y, Z) \in \text{Com}(X).$$

The upper bound is attained if Y or Z is constant, the lower bound if $Y=Z=\frac{1}{2}X$.

In reinsurance and option markets, the comonotonic property can be fulfilled in a natural way. For example in an option environment, the random variable $Z=(X-d)_+$ models the outcome at maturity of a call-option with exercise price d . Setting $Y=X-Z=d-(d-X)_+=\min\{d,X\}$, one has $(Y,Z)\in\text{Com}(X)$ and the total splitting risk is

$$(1.2) \quad R_X[Y,Z]=\text{Var}[(X-d)_+] + \text{Var}[(d-X)_+]=\text{Var}[X] - 2E[(X-d)_+]\cdot E[(d-X)_+].$$

The first representation expresses the total splitting risk as the sum of the call- and put-option variance. For this reason it is called *total option variance*. The second equality quantifies the variance reduction.

1.2. An actuarial argument.

Let $A=\{A(t)\}$, $t\geq 0$, $L=\{L(t)\}$, $t\geq 0$, be stochastic processes representing accumulated values of assets and liabilities at a future time $t\geq 0$. Then the stochastic process $G=\{G(t)\}$, $t\geq 0$, defined by $G(t)=A(t)-L(t)$, represents the *financial gain*, while $V=\{V(t)\}$, $t\geq 0$, defined by $V(t)=L(t)-A(t)$, is the *financial loss*. For simplicity indices will be omitted, but it will be understood that corresponding affirmations can be made whatever the time parameter is. Since $G=G^+-V^+$ the financial gain can be represented as difference between (positive) absolute *gain* and (positive) *absolute loss*. The *gain identity* $G+V^+=G^+$, in words

$$\text{financial gain} + \text{absolute loss} = \text{absolute gain},$$

is a main object of interest, whose general study has been named "AFIR problem" by Bühlmann(1995), and which is also the basic idea underlying Portfolio Insurance created by Leland (cf. Luskin(1988)).

Consider an individual financial contract and its associated gain identity. In a general ALM context (ALM=Asset and Liability Management), suppose a risk manager looks for an "optimal" hedging strategy, which is able to cover a possible loss

taking into account a possible gain according to the ALM relationship

$$(1.3) \quad A + (L - A)_+ = L + (A - L)_+,$$

which holds with probability one. Optimality of a pair (A, L) is considered with respect to the following two properties :

(P1) Acceptable for the risk manager are only mean self-financing hedging strategies

(P2) The (financial) risk premium needed to cover the liability should be a minimum

A time dependent mean self-financing strategy can be formulated as follows. At the beginning of a period, the risk manager puts aside the present value of the asset position A and pays the hedging costs for the option $(L-A)_+$ to exchange A for L . At the end of the considered period, the outcome $(L-A)_+$ together with the asset value A suffices to meet the liability L , and there remains as possible gain the outcome $(A-L)_+$ of an option to exchange L for A , which can be reinvested to finance the asset position plus the hedging costs of the next period, and so on. To be *mean self-financing*, the expected absolute gain must at least be equal to the expected absolute loss, that is

$$(1.4) \quad E[G] = E[G_+] - E[V_+] \geq 0.$$

In a pure Finance environment the exchange option can be priced according to Margrabe(1978) in general, or to Black-Scholes(1973) if A or L is deterministic. However in a dual random ALM environment of Finance and Insurance, these derivative instruments must be priced differently. As simple practical alternative, suppose to fix ideas that prices are set according to the variance premium principle $H[\cdot] = E[\cdot] + \theta \cdot \text{Var}[\cdot]$. Then, given a possible absolute gain of amount G_+ , the needed periodic random price to meet the liability L equals

$$(1.5) \quad H[L | G_+] = E[A + V_+] + \theta \cdot \text{Var}[A + V_+] - G_+.$$

Its variance premium equals

$$\begin{aligned}
 (1.6) \quad H[L] &:= H[H[L \mid G_+]] = E[H[L \mid G_+]] + \theta \cdot \text{Var}[H[L \mid G_+]] \\
 &= E[A + V_+ - G_+] + \theta \cdot (\text{Var}[A + V_+] + \text{Var}[G_+]) \\
 &= E[L] + \theta \cdot R_L[A + V_+, -G_+],
 \end{aligned}$$

which is the sum of the expected liability payment plus the total splitting risk loading. By the property (P2) the latter quantity should be minimized.

To illustrate look at a particular case. If $d = P^N = A$ is interpreted as a deterministic accumulated value of insurance premiums, net of the hedging costs for the option payment $(X-d)_+$, $L = X$ the insurance claims, the above variance premium to meet X equals

$$\begin{aligned}
 (1.7) \quad H[X] &= E[X] + \theta \cdot (\text{Var}[(X-d)_+] + \text{Var}[(d-X)_+]) \\
 &= E[X] + \theta \cdot R_X[\min\{d, X\}, (X-d)_+],
 \end{aligned}$$

which is the sum of the expected insurance claims plus the total option variance loading, subject to minimization. The latter representation in (1.7) shows that the part of the risk stemming from the dependence between the retained risk of the insurer and the option risk has been eliminated by the option contract (variance reduction through diversification). In this situation the mean self-financing constraint (1.4) says that $d = P^N \geq \mu$, the mean insurance claims.

2. Minimum of the total option variance.

It has been argued in Section 1 that to achieve a maximum variance reduction using options as diversification instruments, the total option variance has to be minimized.

Consider a random variable X with finite mean μ and variance σ^2 . For a real number d , let $V = X-d$ the financial loss, $G = d-X$ the financial gain, $V_+ = (X-d)_+$ the absolute loss as call-option, $G_+ = (d-X)_+$ the absolute gain as put-option. Future

expected values of the options are denoted by $C(d)=E[V_+]$, $P(d)=E[G_+]$, and their variances are $VC(d)=\text{Var}[V_+]$, $VP(d)=\text{Var}[G_+]$. Suppose that $F(x)$ is the distribution function of X , which is assumed to have for simplicity a density $f(x)=F'(x)$. The survival function is $S(x)=1-F(x)$. Setting $Y=X-(X-d)_+=d-(d-X)_+$, $Z=(X-d)_+$, one has $(Y,Z) \in \text{Com}(X)$, and the total option variance is given by, say

$$(2.1) \quad R(d) = R_x[Y,Z] = VC(d) + VP(d) = \sigma^2 - 2C(d)P(d).$$

Our optimization problem reads

$$(2.2) \quad R(d) = \min. \quad \text{under the constraint } d \geq \mu.$$

The first-order condition $R'(d)=0$ is necessary for a minimum :

$$(2.3) \quad R'(d) = 2 \cdot (S(d)P(d) - F(d)C(d)) = 0, \quad d \geq \mu,$$

or equivalently the conditional expected equation

$$(2.4) \quad E[X - d \mid X > d] = E[d - X \mid X \leq d], \quad d \geq \mu.$$

In order that an exercise price d minimizes the total option variance, it is necessary that the conditional call-option payment given that a payment occurs equals the conditional put-option payment given that a payment occurs.

A stationary solution $d \geq \mu$ to (2.3) or (2.4) will be a guaranteed local minimum provided the following second-order condition holds :

$$(2.5) \quad R''(d) = 4F(d)S(d) - 2f(d) \cdot (C(d) + P(d)) > 0.$$

Example 2.1. If $F(x)$ is normally distributed with mean μ and variance σ^2 , one has $C(\mu)=P(\mu)=\sigma/(2\pi)^{1/2}$, $F(\mu)=S(\mu)=1/2$, $R'(\mu)=0$, $R''(\mu)=1-2\sigma/\pi$. Thus, the "average option" with exercise price $d=\mu$ is optimal in a neighbourhood of μ

provided $\sigma < \frac{1}{2}\pi$, which is almost always fulfilled in Finance. This is an additional argument in favour of the preferred choice made in Section 3 for portfolio selection under portfolio insurance in the practically important special case of a normal distribution.

Example 2.2. Let $F(x)$ be the "option extremal" distribution defined by

$$(2.6) \quad F(x) = \frac{1}{2} \{1 + (x - \mu) / (\sigma^2 + (x - \mu)^2)\}, \quad x \in (-\infty, \infty),$$

such that the call-option future prices

$$(2.7) \quad C(d) = \frac{1}{2} \{(\sigma^2 + (d - \mu)^2)^{-1/2} - (d - \mu)\}$$

are maximum given known values of the mean and variance (inequality of Bowers(1969)). One checks that $S(d)P(d) = F(d)C(d)$ uniformly for all d , and that $R(d) = \sigma^2 - 2C(d)P(d) = \frac{1}{2}\sigma^2$. In this "extreme" dangerous situation, the minimum total option variance is attained for every exercise price.

3. The guaranteed rate of return for Portfolio Insurance.

Let $\{S(t)\}$, $0 \leq t \leq T$, be a stochastic process, whose sample paths describe the price evolution of a risky investment fund, and let L be the guaranteed limit the price should not fall below at time T . Portfolio Insurance is based on the identity

$$(3.1) \quad S(T) + (L - S(T))_+ = L + (S(T) - L)_+,$$

which states the economic equivalence between the "hold fund-buy put" strategy and the capital protection option strategy called "hold cash-buy call".

Let $P(t)$ be the price to be paid at time t for Portfolio Insurance, and let $P(T, L)$, resp. $C(T, L)$, denote the value at time $t=0$ of a European put-, resp. call-option, on $S(T)$ with maturity date T and exercise price L . The *guaranteed compounded rate of return* of the portfolio insurance strategy is defined by the relation

$$(3.2) \quad L = P(0) \cdot \exp(gT),$$

where $g < 0$ is allowed. Discounting (3.1) with the compounded rate δ , the initial price of portfolio insurance satisfies

$$(3.3) \quad P(0) = S(0) + P(T,L) = L \cdot \exp(-\delta T) + C(T,L).$$

In particular one has the inequality

$$(3.4) \quad P(0) \geq L \cdot \exp(-\delta T) = P(0) \cdot \exp\{(g-\delta)T\},$$

which shows that $g \leq \delta$. For mathematical convenience set the guaranteed limit proportional to the expected future price of the risky fund

$$(3.5) \quad L = \beta \cdot E[S(T)] = \beta \cdot S(0) \cdot \exp(\delta T) \text{ for some } \beta > 0,$$

where use has been made of the natural condition of no-arbitrage. Combining (3.2) and (3.5) one gets

$$(3.6) \quad P(0)/S(0) = \beta \cdot \exp\{(\delta-g)T\}.$$

Inserting in (3.3) the following relationships are obtained :

$$(3.7) \quad \beta \cdot \exp\{(\delta-g)T\} = 1 + P(T,L)/S(0) = \beta + C(T,L)/S(0).$$

Example 3.1. Let the relative price $S(T)/S(0)$ be lognormally distributed $\ln N(\delta - \frac{1}{2}v^2, v)$, where v is the volatility. This Gaussian specification of the compounded rate of return includes in particular the geometric Brownian motion (model of Black and Scholes(1973)) and the Ornstein-Uhlenbeck process (model of Vasicek(1977)). A calculation shows that the guaranteed compounded rate of return satisfies the equation

$$(3.8) \quad \beta \cdot \exp\{(\delta-g)T\} = N(\frac{1}{2}\sqrt{T} \cdot \ln\{\beta\}/\sqrt{T}) + N(\frac{1}{2}\sqrt{T} + \ln\{\beta\}/\sqrt{T}),$$

where $N(x)$ is the standard normal distribution. If $\beta=1$ one obtains the relation

$$(3.9) \quad \exp\{(\delta-g)T\} = 2 \cdot N(1/2\sigma\sqrt{T}).$$

It is important to observe that the simple choice $\beta=1$ can be rationally justified as follows. Let $H[S(T)]$ be the functional, which represent the expected future price of portfolio insurance at maturity date T such that

$$(3.10) \quad H[S(T)] = P(T) = E[S(T)] + E[(\beta E[S(T)] - S(T))_+].$$

If decision makers have non-decreasing utility and are risk-averse, the functional $H[\cdot]$ preserves stochastic dominance and stop-loss order, as should be. Moreover under the existence of risk-free investments it is plausible to require the translation-invariant property $H[S(T)+R(T)]=H[S(T)]+R(T)$, where $R(T)$ is the accumulated value of a risk-free investment. Under this additional condition, the unique choice of β in (3.7) is indeed $\beta=1$. In the following only this special choice is retained.

Example 3.2. If the relative price $S(T)/S(0)$ is normally distributed $N(\exp(\delta T), \sigma\sqrt{T})$ and $g \leq \delta$ is used as discount rate, one gets

$$P(T, S(0)\exp(\delta T)) = C(T, S(0)\exp(\delta T)) = S(0) \cdot \exp(-gT) \cdot \sigma \cdot (T/2\pi)^{1/2}.$$

Thus (3.7) yields the relation

$$(3.11) \quad \exp\{(\delta-g)T\} = 1 + \exp(-gT) \cdot \sigma \cdot (T/2\pi)^{1/2}.$$

Example 3.3. Assume the guaranteed mean $S(0)\exp(gT)$ and the variance $(S(0) \cdot \sigma)^2 \cdot T$ of the price $S(T)$ is known. Then a distribution-free arbitrage-free binomial option pricing model can be constructed such that

$$P(T, S(0)\exp(\delta T)) = C(T, S(0)\exp(\delta T)) = 1/2 S(0) \cdot \exp(-gT) \cdot \sigma\sqrt{T}.$$

Inserting in (3.7) one obtains

$$(3.12) \quad \exp\{(\delta-g)T\} = 1 + \exp(-gT) \cdot \frac{1}{2} \sigma \sqrt{T}.$$

Now set $T=1$ (say one-year time horizon) and consider the accumulated rates of return $r = \exp(\delta)$, $r_g = \exp(g)$. Then the relations (3.9), (3.11), (3.12) can be rewritten as

$$(3.13) \quad \begin{aligned} r &= r_g + \{2N(\frac{1}{2}v) - 1\} \cdot r_g \quad (\text{lognormal case}) \\ r &= r_g + \sigma \sqrt{2\pi} \quad (\text{normal case}) \\ r &= r_g + \frac{1}{2} \sigma \quad (\text{distribution-free case}) \end{aligned}$$

Under a lognormal assumption one can set approximately $\sigma = r_g \cdot (\exp(v^2) - 1)$. Suppose the volatility v is sufficiently small such that the following approximations hold :

$$(3.14) \quad \sigma = r_g \cdot (\exp(v^2) - 1) \approx r_g \cdot v, \quad 2N(\frac{1}{2}v) - 1 \approx v \sqrt{2\pi}.$$

Then the first relation in (3.13) equals approximately $r \approx r_g + \sigma \sqrt{2\pi}$, the relation in the normal case.

4. Portfolio Selection under Portfolio Insurance.

In Finance practice the goal of Portfolio Selection is to minimize the variance by given mean return. Thus v and σ will often be sufficiently small. Therefore it can reasonably be assumed that the following relations hold

$$(4.1) \quad \begin{aligned} r &= r_g + \sigma \sqrt{2\pi} \quad (\text{realistic case}) \\ r &= r_g + \frac{1}{2} \sigma \quad (\text{distribution-free case}) \end{aligned}$$

In both situations the price of the put- or call-option to guarantee the return r_g is proportional to the standard deviation of return.

Suppose a risk manager wants to select a portfolio of risky assets such that the

return r_g is guaranteed and such that his hedging costs are as small as possible. Then he has to minimize the standard deviation under the condition that the return of the portfolio satisfies a relation $r=r_g + c \cdot \sigma$ with $c=1/\sqrt{2\pi}$ or $c=1/2$ as in (3.15).

To formulate precisely this *Portfolio Selection problem under Portfolio Insurance*, let us introduce the required (partly changed) notations :

r_f : risk-free accumulated rate of return

E_g : guaranteed accumulated rate of return

$\mathbf{R}=(R_1, \dots, R_{n+1})$: vector of accumulated random returns on $(n+1)$ risky assets

$\boldsymbol{\mu}=(\mu_1, \dots, \mu_{n+1})$: vector of expected accumulated returns

$\mathbf{X}=(X_1, \dots, X_{n+1})$: vector of fractions of the portfolio held in each risky asset

$\mathbf{I}=(1, \dots, 1)$: unit vector

$R=\mathbf{X}^T \cdot \mathbf{R}$: accumulated random return of the portfolio

$E=E[R]=\mathbf{X}^T \cdot \boldsymbol{\mu}$: accumulated expected return of the portfolio

$\mathbf{C}=(\sigma_{ij})$, $\sigma_{ij}=\text{Cov}[R_i, R_j]$: matrix of covariances between the random returns

$V=\text{Var}[R]=\mathbf{X}^T \cdot \mathbf{C} \cdot \mathbf{X}$: variance of return on the portfolio

As for the classical mean-variance portfolio selection problem by Markowitz(1952/59), several types of assumptions may be considered. The main distinction is as follows (e.g. Markowitz(1989), part I, or Elton and Gruber(1991), chap. 4) :

- (P1) Short sales allowed with riskless lending and borrowing
- (P2) Short sales allowed but no riskless lending and borrowing
- (P3) Riskless lending and borrowing with short sales disallowed
- (P4) Riskless lending, borrowing, and short sales all disallowed
- (P5) Additional linear inequality constraints on the portfolio weights, for example
 - minimum dividend yield
 - upper limits on the fractions held in each risky asset
 - consideration of transaction costs

In a first analysis we restrict our attention to the simplest cases (P1), (P2), for which the following optimization problems must be solved (cf. Hürlimann(1991)) :

$$(P1) \quad \min\{ \frac{1}{2} \mathbf{X}^T \cdot \mathbf{C} \cdot \mathbf{X} \} \text{ under the constraint} \\ \mathbf{X}^T \cdot (\boldsymbol{\mu} - \mathbf{I}r_f) + r_f = E_g + c \cdot (\mathbf{X}^T \cdot \mathbf{C} \cdot \mathbf{X})^{1/2}, \text{ with} \\ c = 1/\sqrt{2\pi} \text{ (realistic case), } c = 1/2 \text{ (distribution-free case).}$$

$$(P2) \quad \min\{ \frac{1}{2} \mathbf{X}^T \cdot \mathbf{C} \cdot \mathbf{X} \} \text{ under the constraints} \\ \mathbf{X}^T \cdot \mathbf{I} = 1, \quad \mathbf{X}^T \cdot \boldsymbol{\mu} = E_g + c \cdot (\mathbf{X}^T \cdot \mathbf{C} \cdot \mathbf{X})^{1/2}, \text{ with } c = 1/\sqrt{2\pi}, \text{ or } c = 1/2.$$

Remarks 4.1.

(a) Portfolio analysis under portfolio insurance includes as special case $c=0$ the classical approach by Markowitz, which does not worry about hedging concerns. In this situation E_g is simply reinterpreted as the (unhedged) return E of the portfolio, which varies between the return on the minimum variance portfolio and the return on the maximum return portfolio, as traced out by the efficient set of portfolio combinations. For this reason one may speak of an *extended portfolio analysis*.

(b) As in the classical case, the assumption of a normal distribution leads to exact (or rigorous) results. Moreover the distribution-free case, for which the "safe" portfolio hedging costs of amount $\frac{1}{2}(\mathbf{X}^T \cdot \mathbf{C} \cdot \mathbf{X})^{1/2}$ are attained by a diatomic distribution (inequality of Bowers(1969)), may also be viewed as a rigorous theory.

(c) Problem (P1) has been solved in Hürlimann(1991), Section 4. Its solution is very similar to the classical one (see Huang and Litzenberger(1988), Section 3.18, p.76-80). Problem (P2) has been tackled by the author(1991), Section 5, but not solved in a satisfactory way. An improved discussion of its solution is presented in Section 6 of the present paper. It is based on the idea of a "canonical" form for portfolio analysis suggested by Markowitz(1987), chap. 11. A more advanced treatment of this mathematical tool is the subject of the next Section 5.

5. Canonical orthogonal form of the standard portfolio selection model.

The "canonical" method consists to reduce using non-singular affine transformations the equations for mean-variance analysis to simple algebraic expressions. A canonical form permits powerful *graphical analysis* and clear *analytical discussions*.

Recall that the random return on the portfolio is

$$(5.1) \quad R = \sum_{i=1}^{n+1} X_i R_i$$

with an expected return of

$$(5.2) \quad E = \sum_{i=1}^{n+1} X_i \mu_i,$$

and a variance of return on the portfolio of

$$(5.3) \quad V = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} X_i X_j \sigma_{ij}.$$

As usual the variances σ_{ii} are written σ_i^2 . The fractions X_1, \dots, X_{n+1} are subject to the constraints

$$(5.4) \quad \sum_{i=1}^{n+1} X_i = 1, \quad X_i \geq 0, \quad i=1, \dots, n+1.$$

Through elimination of the variable $X_{n+1} = 1 - \sum_{i=1}^n X_i$ in (5.4) one sees that the feasible set for the standard $(n+1)$ -security analysis is the inner region of the n -dimensional standard simplex in \mathbf{R}^n with coordinate vector $\mathbf{X}^T = (X_1, \dots, X_n)$. The vertices of the simplex are $\mathbf{e}_{n+1} = 0$, and the \mathbf{e}_i 's, which are vectors with a one in its i -th place and zeros elsewhere, $i=1, \dots, n$. The above substitution made in the return space leads to a non-singular transformation with matrix $\mathbf{S} = (s_{ij})$ such that

$$(5.5) \quad \begin{aligned} s_{ij} &= 0, \quad j=1, \dots, n, \quad s_{in+1} = 1, \\ s_{ij} &= \delta_{i-1,j}, \quad j=1, \dots, n, \\ s_{in+1} &= -1, \quad i=2, \dots, n+1. \end{aligned}$$

In the new return space with random return $\mathbf{U}^T = (U_0, \dots, U_n)$ the expected return and the covariance matrix of the random return transformed vector $\mathbf{U} = \mathbf{S} \cdot \mathbf{R}$ are given by

$$(5.6) \quad \mathbf{v} = (v_i) = \mathbf{S} \cdot \boldsymbol{\mu}, \quad \boldsymbol{\tau} = (\tau_{ij}) = \mathbf{S} \cdot \mathbf{C} \cdot \mathbf{S}^T.$$

Explicitly one has

$$(5.7) \quad v_0 = \mu_{n+1}, \quad v_i = \mu_i - \mu_{n+1}, \quad i=1, \dots, n,$$

$$\tau_{00} = (\sigma_{n+1})^2, \quad \tau_{0i} = \tau_{i0} = \sigma_{in+1} - (\sigma_{n+1})^2, \quad i=1, \dots, n,$$

$$(5.8) \quad \tau_{ii} = (\sigma_i)^2 - 2\sigma_{in+1} + (\sigma_{n+1})^2, \quad i=1, \dots, n,$$

$$\tau_{ij} = \tau_{ji} = \sigma_{ij} - \sigma_{in+1} - \sigma_{jn+1}, \quad 1 \leq i < j \leq n.$$

In this representation the expected return and the variance of return on the portfolio are with $\mathbf{X}_* = (1, X_1, \dots, X_n)^T$:

$$(5.9) \quad \mathbf{E} = (\mathbf{X}_*)^T \cdot \mathbf{v} = v_0 + \mathbf{X}^T \cdot \mathbf{v}^*,$$

$$\mathbf{V} = (\mathbf{X}_*)^T \cdot \boldsymbol{\tau} \cdot \mathbf{X}_* = \tau_{00} + \mathbf{X}^T \cdot \boldsymbol{\tau}_0 + \mathbf{X}^T \cdot \boldsymbol{\tau}^* \cdot \mathbf{X},$$

where one sets $\mathbf{v}^* = (v_1, \dots, v_n)^T$, $\boldsymbol{\tau}_0 = (\tau_{01}, \dots, \tau_{0n})^T$, $\boldsymbol{\tau}^* = (\tau_{ij})$, $1 \leq i, j \leq n$. Our purpose is to derive the following *canonical orthogonal form* for a n -dimensional mean-variance portfolio geometric analysis. The displayed canonical form is effective, that is numerically computable. It generalizes theorem 11.7 in Markowitz(1987), which is the special case $n=2$, $k=1$ of our result.

Theorem 5.1. Let \mathbf{E} and \mathbf{V} be given by (5.9) such that $\mathbf{v}^* \neq 0$. Assume that $\text{rank} \boldsymbol{\tau} = k+1$ for some $1 \leq k \leq n$, and that (τ_{ij}) , $1 \leq i, j \leq k$, is of maximal rank. Then there exists a uniquely defined non-singular affine transformation (in fact a n -dimensional space motion)

$$\mathbf{Y} = \mathbf{b} + \mathbf{H} \cdot \mathbf{X}$$

such that $\mathbf{b} = (b_1, \dots, b_k, 0, \dots, 0)^T$, \mathbf{H} is orthogonal, and

$$(5.10) \quad \mathbf{V} = V_{\min} + \mathbf{Y}^T \cdot \mathbf{D} \cdot \mathbf{Y}, \quad \mathbf{E} = a_0 + \mathbf{a}^T \cdot \mathbf{Y},$$

with $\mathbf{D} = \text{diag}\{d_1, \dots, d_n\}$, $\mathbf{a} = (a_1, \dots, a_n)$. More precisely, one has

$$(5.11) \quad d_1 \neq 0, \dots, d_k \neq 0, d_{k+1} = \dots = d_n = 0 \text{ are the eigenvalues of } \boldsymbol{\tau}^*$$

$$(5.12) \quad \mathbf{H} = (\mathbf{h}^{<1>}, \dots, \mathbf{h}^{<n>}) = (\mathbf{h}_{<1>}^T, \dots, \mathbf{h}_{<n>}^T), \quad \mathbf{h}_{<i>} \text{ the eigenvector to the eigenvalue } d_i \text{ with euclidean norm } \|\mathbf{h}_{<i>}\| = 1, \text{ and } (\mathbf{h}^{<i>})_j = (\mathbf{h}_{<j>})_i.$$

(5.13) The translation vector \mathbf{b} is parallel to the vector $\mathbf{H} \cdot \boldsymbol{\tau}_0$, with non-zero components $b_i = (\mathbf{H} \cdot \boldsymbol{\tau}_0)_i / d_i$, $i=1, \dots, k$.

$$(5.14) \quad \mathbf{a} = \mathbf{H} \cdot \mathbf{v}^*$$

$$(5.15) \quad \mathbf{a}_0 = v_0 - \mathbf{a}^T \cdot \mathbf{b}$$

$$(5.16) \quad V_{\min} = \tau_{00} - \mathbf{b}^T \cdot \mathbf{D} \cdot \mathbf{b}$$

Proof. For convenience set $\mathbf{h}^{<n+1>} = 0$. Given an affine transformation $\mathbf{Y} = \mathbf{b} + \mathbf{H} \cdot \mathbf{X}$ the vertices \mathbf{e}_i of the standard simplex map to the vertices $\mathbf{b} + \mathbf{h}^{<i>}$, $i=1, \dots, n+1$. By linearity, to satisfy (5.10), it suffices to solve (if possible) the system of equations

$$(5.17) \quad \mu_i = \mathbf{a}_0 + \mathbf{a}^T \cdot (\mathbf{b} + \mathbf{h}^{<i>}), \quad i=1, \dots, n+1,$$

$$(5.18) \quad \sigma_{ij} = V_{\min} + (\mathbf{b} + \mathbf{h}^{<i>})^T \cdot \mathbf{D} \cdot (\mathbf{b} + \mathbf{h}^{<j>}), \quad 1 \leq i, j \leq n+1.$$

Rewrite this system in terms of the random return $\mathbf{U} = \mathbf{S} \cdot \mathbf{R}$. Using (5.7) the first set of equations shows that

$$(5.19) \quad \mathbf{a}_0 = v_0 - \mathbf{a}^T \cdot \mathbf{b}$$

$$(5.20) \quad \mathbf{v}^* = \mathbf{H}^T \cdot \mathbf{a}$$

In particular (5.15) is fulfilled. From the second set of equations one sees using (5.8) that

$$(5.21) \quad \tau_{00} = V_{\min} + \mathbf{b}^T \cdot \mathbf{D} \cdot \mathbf{b}$$

$$(5.22) \quad \boldsymbol{\tau}_0 = \mathbf{H}^T \cdot \mathbf{D} \cdot \mathbf{b}$$

$$(5.23) \quad \boldsymbol{\tau}^* = \mathbf{H}^T \cdot \mathbf{D} \cdot \mathbf{H}$$

In particular (5.16) is fulfilled. Now $\boldsymbol{\tau}^*$ is a symmetric matrix with real coefficients. From Linear Algebra one knows that any symmetric real matrix possesses a diagonal form with an orthogonal transformation matrix. Therefore there exists an orthogonal matrix $\mathbf{H} = (\mathbf{h}^{<1>}, \dots, \mathbf{h}^{<n>})$ such that $\mathbf{H} \cdot \boldsymbol{\tau}^* \cdot \mathbf{H}^T = \mathbf{D}$. The constructive steps of the diagonalization show that the conditions (5.11) and (5.12) can be fulfilled since $\boldsymbol{\tau}^*$ is

chosen such that $(\tau_{ij}), 1 \leq i, j \leq k$, has maximal rank. Since $H^T = H^{-1}$ the condition (5.14) follows from (5.20) above. It remains to show the validity of condition (5.13). Equivalently one has to show that (5.22) can be solved. After multiplication with H one must have $H \cdot \tau_0 = D \cdot b$. Lemma 5.1 below does the job and Theorem 5.1 is proved.

Lemma 5.1. With the assumptions of Theorem 5.1 one has $(H \cdot \tau_0)_i = 0, i = k+1, \dots, n$.

Proof. Consider the transformation $X^0 = H \cdot X$. In the new coordinates the matrix

$$\tau = \begin{pmatrix} \tau_{00} & \tau_0^T \\ \tau_0 & \tau^* \end{pmatrix}$$

is

$$\tau^0 = \begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} \tau_{00} & \tau_0^T \\ \tau_0 & \tau^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & H^T \end{pmatrix} = \begin{pmatrix} \tau_{00} & (H \cdot \tau_0)^T \\ H \cdot \tau_0 & D \end{pmatrix}$$

Set $c_i = (H \cdot \tau_0)_i, i = 1, \dots, n$, for the i -th component of $H \cdot \tau_0$. For $i \in \{k+1, \dots, n\}$ consider the $(k+2) \times (k+2)$ -submatrix $\tau^0(i)$ of τ^0 given by

$$\tau^0(i) = \begin{pmatrix} \tau_{00} & c_1 & \dots & c_k & c_i \\ c_1 & d_1 & & 0 & \\ \cdot & \cdot & & 0 & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & 0 & & \cdot & \cdot \\ c_k & & & d_k & 0 \\ c_i & 0 & & 0 & 0 \end{pmatrix}$$

Since $\text{rank } \tau^0 = \text{rank } \tau = k+1$ one has necessarily $\det \tau^0(i) = 0$. But one has

$$\det \tau^0(i) = (-1)^{k+1} \cdot c_i^2 \prod_1^k d_j.$$

This shows that $c_i = 0, i = k+1, \dots, n$, as asserted.

Example 5.1 : three-security portfolio analysis

Let τ_i , $i=1,2$, be the standard deviation of U_i and let ρ be the correlation coefficient between U_1 and U_2 . Then one has

$$\tau^* = \begin{pmatrix} \tau_1^2 & \rho\tau_1\tau_2 \\ \rho\tau_1\tau_2 & \tau_2^2 \end{pmatrix}$$

The eigenvalues of τ^* are

$$(5.24) \quad d_{1,2} = \frac{1}{2}\{\tau_1^2 + \tau_2^2 \pm \sqrt{(\tau_1^2 - \tau_2^2)^2 + 4(\rho\tau_1\tau_2)^2}\}$$

The construction of the canonical orthogonal form depends upon $\text{rank } \tau^* = 1$ or 2 .

Case 1 : $\text{rank } \tau^* = 2$, that is $-1 < \rho < 1$.

The normed eigenvectors $\mathbf{h}_{\langle i \rangle}$ to d_i , $i=1,2$, yield the orthogonal matrix

$$(5.25) \quad \mathbf{H} = \begin{pmatrix} \rho\tau_1\tau_2/\sqrt{(\rho\tau_1\tau_2)^2 + (d_1 - \tau_1^2)^2} & (d_1 - \tau_1^2)/\sqrt{(\rho\tau_1\tau_2)^2 + (d_1 - \tau_1^2)^2} \\ (d_2 - \tau_2^2)/\sqrt{(\rho\tau_1\tau_2)^2 + (d_2 - \tau_1^2)^2} & \rho\tau_1\tau_2/\sqrt{(\rho\tau_1\tau_2)^2 + (d_2 - \tau_1^2)^2} \end{pmatrix}$$

The required formulas are obtained from Theorem 5.1 :

$$(5.26) \quad \mathbf{b} = \mathbf{D}^{-1} \cdot \mathbf{H} \cdot \boldsymbol{\tau}_0, \quad \mathbf{a} = \mathbf{H} \cdot \mathbf{v}^*, \quad a_0 = v_0 - \mathbf{a}^T \cdot \mathbf{b}, \quad V_{\min} = \tau_{00} - \mathbf{b}^T \cdot \mathbf{D} \cdot \mathbf{b}.$$

In the (Y_1, Y_2) -plane the expected return and the variance of return on the portfolio are

$$(5.27) \quad \begin{aligned} V &= V_{\min} + d_1 Y_1^2 + d_2 Y_2^2, \\ E &= a_0 + a_1 Y_1 + a_2 Y_2. \end{aligned}$$

The standard three-security portfolio selection model is solved by finding the efficient set for the triangle with vertices \mathbf{b} , $\mathbf{b} + \mathbf{h}^{<1>}$, $\mathbf{b} + \mathbf{h}^{<2>}$, using analytical geometrical tools. The discussion is similar to Markowitz(1987), chap. 11. In this special case one can use the alternative simpler canonical (in general non-orthogonal) form

$$(5.28) \quad \begin{aligned} \mathbf{V} &= \mathbf{V}_{\min} + Y_1^2 + Y_2^2, \\ \mathbf{E} &= \mathbf{a}_0 + \mathbf{a}_1 Y_1. \end{aligned}$$

A derivation is found in Markowitz(1987), theorem 11.1, and appendix to chap. 11, as well as in Section 7 to the present paper. The geometrical construction of efficient portfolios for non-standard constraint sets is also possible. Results in the (X_1, X_2) -plane and U -coordinate return space are obtained using the inverse plane motion

$$(5.29) \quad \mathbf{X} = \mathbf{H}^T \cdot (\mathbf{Y} - \mathbf{b}).$$

Case 2 : rank $\tau^* = 1$

(i) If $\tau_2 = 0$ one finds that

$$(5.30) \quad \mathbf{D} = \text{diag}\{\tau_1^2, 0\}, \quad \mathbf{H} = \text{diag}\{1, 1\}, \quad \tau_{02} = 0 \quad (\text{condition (5.13)}).$$

It follows that

$$(5.31) \quad \mathbf{b} = (\tau_{01}/\tau_1^2, 0)^T, \quad \mathbf{a} = (v_1, v_2)^T, \quad \mathbf{a}_0 = v_0 - v_1 \tau_{01}/\tau_1^2, \quad \mathbf{V}_{\min} = \tau_{00} - (\tau_{01}/\tau_1)^2.$$

(ii) If $\tau_1 = 0$ then exchange U_1 and U_2 to get the preceding situation (i).

(iii) In all other possible cases one has either $\rho = -1$ or $+1$. Set $\tau = \sqrt{\tau_1^2 + \tau_2^2}$, then one has

$$(5.32) \quad \mathbf{D} = \text{diag}\{\tau^2, 0\}, \quad \mathbf{H} = \begin{pmatrix} \tau_1/\tau & \rho\tau_2/\tau \\ -\rho\tau_2/\tau & \tau_1/\tau \end{pmatrix}$$

and from (5.13) the necessary condition

$$(5.33) \quad \tau_1 \tau_{02} = \rho \tau_2 \tau_{01}.$$

Furthermore one has

$$(5.34) \quad \mathbf{b} = (\tau_{01}/(\tau_1 \tau), 0)^T, \quad \mathbf{a} = ([\tau_1 v_1 + \rho \tau_2 v_2]/\tau, [\tau_1 v_2 - \rho \tau_2 v_1]/\tau),$$

$$(5.35) \quad a_0 = v_0 - (\tau_1 v_1 + \rho \tau_2 v_2) \tau_{01}/(\tau_1 \tau^2), \quad V_{\min} = \tau_{00} - (\tau_{01}/\tau_1)^2.$$

6. Affine portfolio selection model under portfolio insurance without riskless asset.

In the notations of Section 4, the optimization problem (P2) of extended mean-variance portfolio analysis reads

$$(6.1) \quad \begin{aligned} & \min \{ \frac{1}{2} \mathbf{X}^T \cdot \mathbf{C} \cdot \mathbf{X} \} \text{ under the constraints} \\ & \mathbf{X}^T \cdot \mathbf{I} = 1, \\ & \mathbf{X}^T \cdot \boldsymbol{\mu} = E_g + c \sqrt{\mathbf{X}^T \cdot \mathbf{C} \cdot \mathbf{X}}, \end{aligned}$$

Remark 6.1. The "affine" problem (6.1) allows unlimited short selling of securities. For a *standard portfolio analysis* one requires additionally the constraints

$$(6.2) \quad X_i \geq 0, \quad i=1, \dots, n+1.$$

The *general portfolio selection model* under portfolio insurance (corresponding to form 2 in Markowitz(1987), chap. 2) requires further linear constraints, say in matrix notation

$$(6.3) \quad \mathbf{A} \cdot \mathbf{X} = \boldsymbol{\xi}.$$

In the classical case $c=0$, one knows that the "affine" solution is a component of the solution to the standard and the general portfolio selection model (see Markowitz(1987)). In the extended portfolio analysis this problem has not yet been studied. As a first step, only the affine case (6.1) is considered.

Using the canonical orthogonal form associated to a portfolio selection model, it is possible to simplify the discussion given in Hürlimann(1991), Section 5, and to

answer some questions left open there.

Making the substitution $X_{n+1} = 1 - \sum_1^n X_i$ and applying Theorem 5.1, there exists an explicitly given n -dimensional space motion $\mathbf{Y} = \mathbf{b} + \mathbf{H} \cdot \mathbf{X}$, with \mathbf{H} an orthogonal $(n \times n)$ -matrix and now $\mathbf{X}^T = (X_1, \dots, X_n)$ denotes another vector of dimension n only, such that (6.1) transforms to the optimization problem

$$(6.4) \quad \min \frac{1}{2} \{ V_{\min} + \sum_1^k d_i Y_i^2 \} \text{ under the constraint} \\ a_0 + \sum_1^k a_i Y_i = E_g + c \sqrt{V_{\min} + \sum_1^k d_i Y_i^2}$$

The classical Lagrange function

$$(6.5) \quad L = \frac{1}{2}(V_{\min} + \sum_1^k d_i Y_i^2) + \lambda(E_g + c \sqrt{V_{\min} + \sum_1^k d_i Y_i^2} - a_0 - \sum_1^k a_i Y_i)$$

yields the following necessary and sufficient conditions for a solution :

$$(6.6) \quad \partial L / \partial Y_i = d_i Y_i + \lambda(c d_i Y_i / \sqrt{V_{\min} + \sum_1^k d_i Y_i^2} - a_i) = 0, \quad i=1, \dots, k,$$

$$(6.7) \quad \partial L / \partial \lambda = E_g + c \sqrt{V_{\min} + \sum_1^k d_i Y_i^2} - a_0 - \sum_1^k a_i Y_i = 0.$$

Solving for λ in the first equations one sees that

$$(6.8) \quad \lambda = d_i Y_i \sqrt{V_{\min} + \sum_1^k d_i Y_i^2} / (a_i \sqrt{V_{\min} + \sum_1^k d_i Y_i^2} - c d_i Y_i), \quad i=1, \dots, k.$$

Equating all these expressions one gets

$$(6.9) \quad a_i d_i Y_i = a_j d_j Y_j, \quad i=2, \dots, k.$$

Introduced in (6.4) one obtains the condition

$$(6.10) \quad c \sqrt{V_{\min} + \beta Y_1^2} = (a_0 - E_g) + \alpha Y_1, \text{ with} \\ \alpha = \sum_{i=1}^k a_i^2 \prod_{j \neq i} d_j / (a_1 \prod_{i=2}^k d_i), \\ \beta = d_1 \alpha / a_1.$$

Eliminating the square root one has to solve the quadratic equation

$$(6.11) \quad (\beta c^2 - \alpha^2) Y_1^2 - 2\alpha(a_0 - E_g) Y_1 + c^2 V_{\min} - (a_0 - E_g)^2 = 0.$$

For a real solution the discriminant of this equation must be positive :

$$(6.12) \quad \Delta_c = c^2 V_{\min} \{ \alpha^2 + \beta \gamma^2 - \beta c^2 \} \geq 0,$$

where to simplify the notation one sets

$$(6.13) \quad \gamma^2 = (a_0 - E_g)^2 / V_{\min}.$$

It follows that necessarily $c \in [0, c_{\max}]$ with

$$(6.14) \quad (c_{\max})^2 = \alpha^2 / \beta + \gamma^2 = \sum_{i=1}^k a_i^2 \prod_{j \neq i} d_j / \prod_{i=1}^k d_i + (a_0 - E_g)^2 / V_{\min}.$$

Definition 6.1. For each $c \in [0, c_{\max}]$ the solution to the optimization problem (6.4) yields an "efficient" extended portfolio with expected return E_c and variance of return V_c . By construction one has

$$(6.15) \quad E_c = a_0 + \alpha Y(c) = E_g + c \sqrt{V_c},$$

$$(6.16) \quad V_c = V_{\min} + \beta Y(c)^2,$$

where the function $Y(c)$ denotes an appropriate solution Y_1 of (6.11). In the following use is made of the following facts.

Lemma 6.1. The model parameters necessarily satisfy the following properties :

$$\operatorname{sgn}(\alpha) = \operatorname{sgn}(a_1), \quad \beta > 0, \quad \gamma = (a_0 - E_g) / \sqrt{V_{\min}} > 0.$$

Proof. The relation $\operatorname{sgn}(\alpha) = \operatorname{sgn}(a_1)$ follows from the definition of α and from the

fact that the eigenvalues d_i are positive. It follows that $\beta = d_1 \alpha / a_1 > 0$. Now let $c = c_{\max} = \sqrt{\alpha^2 / \beta + \gamma^2}$. Then one has $\Delta_c = 0$, and thus $Y(c) = (\alpha / \beta) V_{\min} / (a_0 - E_g)$. Since

$$E_c = a_0 + (\alpha^2 / \beta) V_{\min} / (a_0 - E_g) > E_g,$$

one must have

$$(\alpha^2 / \beta) V_{\min} / (a_0 - E_g) > E_g - a_0,$$

which is only possible if $E_g < a_0$. Clearly γ is then positive.

From (6.11) one derives the following possible values for $Y(c)$:

$$(6.17) \quad Y(c) = \begin{cases} \sqrt{V_{\min}} (\alpha \gamma + \epsilon c \sqrt{\alpha^2 + \beta \gamma^2 - \beta c^2}) / (\beta c^2 - \alpha^2), & \epsilon = \pm 1, c \in [0, c_{\max}] \setminus \{\sqrt{\alpha^2 / \beta}\}, \\ \sqrt{V_{\min}} (\alpha^2 - \beta \gamma^2) / 2\alpha \beta \gamma, & c = \sqrt{\alpha^2 / \beta}. \end{cases}$$

The determination of the appropriate sign is done in Theorem 6.2 below. First of all let us analyze the behaviour of the function $Y(c)$ in the neighbourhood of $c = \sqrt{\alpha^2 / \beta}$. If $\epsilon = -\text{sgn}(\alpha)$, it is continuous, otherwise it is unbounded.

Lemma 6.2. The function $Y(c)$ defined by (6.17) satisfies the following property :

$$(6.18) \quad \lim_{c \rightarrow \sqrt{\alpha^2 / \beta}} Y(c) = \begin{cases} Y(\sqrt{\alpha^2 / \beta}), & \text{if } \epsilon = -\text{sgn}(\alpha), \\ \text{sgn}(\alpha) \cdot \infty, & \text{if } \epsilon = \text{sgn}(\alpha). \end{cases}$$

Proof. The substitution $t = \beta c^2 - \alpha^2$ shows that

$$\lim_{c \rightarrow \sqrt{\alpha^2 / \beta}} Y(c) = \sqrt{V_{\min}} / \beta \cdot \lim_{t \rightarrow 0} (\alpha \gamma \sqrt{\beta} + \epsilon \sqrt{\alpha^2 \beta \gamma^2 + (\beta \gamma^2 - \alpha^2) t - t^2}) / t = \begin{cases} 0/0, & \text{if } \epsilon = -\text{sgn}(\alpha), \\ 2\alpha \gamma / 0, & \text{if } \epsilon = \text{sgn}(\alpha). \end{cases}$$

In case $\epsilon = -\text{sgn}(\alpha)$ De L'Hospital's rule removes the indetermination :

$$\begin{aligned} \lim_{c \rightarrow \sqrt{\alpha^2/\beta}} Y(c) &= \sqrt{V_{\min}/\beta} \cdot \lim_{t \rightarrow 0} \{-\operatorname{sgn}(\alpha) \cdot (-2t + \beta\gamma^2 - \alpha^2) / 2\sqrt{\alpha^2\beta\gamma^2 + (\beta\gamma^2 - \alpha^2)t - t^2}\} \\ &= \sqrt{V_{\min}} \cdot (\alpha^2 - \beta\gamma^2) / 2\alpha\beta\gamma = Y(\sqrt{\alpha^2/\beta}). \end{aligned}$$

As a main application of our approach, let us answer question (a) of Hürlimann(1991), Section 5.

Theorem 6.2. Let be given the canonical orthogonal form (6.4) of the portfolio selection model (6.1). Assume that

$$(6.19) \quad E_g \leq a_0 - \sqrt{(\alpha^2/\beta)V_{\min}}, \text{ or } \beta\gamma^2 \geq \alpha^2,$$

and suppose the coordinate functions of the optimization problem (6.4) are continuous functions of c . Then this optimization problem is solved by the coordinate functions

$$Y_i = (a_i d_i / a_i d_i) Y(c), \quad i = 1, 2, \dots, k,$$

where $Y(c)$ is the uniquely defined *continuous* function given by

$$(6.20) \quad Y(c) = \begin{cases} \operatorname{sgn}(\alpha) \cdot \operatorname{sgn}(c - \gamma) \cdot \sqrt{V_{\min}} \cdot |(\gamma | \alpha | - c\sqrt{\alpha^2 + \beta\gamma^2 - \beta c^2}) / (\beta c^2 - \alpha^2)|, & c \in [0, c_{\max}] \setminus \{\sqrt{\alpha^2/\beta}\} \\ \sqrt{V_{\min}} \cdot (\alpha^2 - \beta\gamma^2) / 2\alpha\beta\gamma, & c^2 = \alpha^2/\beta. \end{cases}$$

For $c, c' \in [0, c_{\max}]$ the efficient extended portfolios have the following properties :

- (i) $E_c > E_{c'}$ whenever $c > c'$
(the expected return is monotone increasing in c)
- (ii) $V_{c_{\max}} \leq V_0$ with equality whenever $\beta\gamma^2 = \alpha^2$
- (iii) If $c, c' \in [0, \gamma]$ then $V_c < V_{c'}$ whenever $c > c'$

(the variance of return is monotone decreasing on $[0, \gamma]$)

If $c, c' \in [\gamma, c_{\max}]$ then $V_c > V_{c'}$ whenever $c > c'$

(the variance of return is monotone increasing on $[\gamma, c_{\max}]$)

Of special interest are the following cases :

(iv) If $c = \gamma$ one has

$$E_\gamma = a_0 \geq E_g + \sqrt{(\alpha^2/\beta)V_{\min}}, \text{ with equality whenever } \beta\gamma^2 = \alpha^2$$

$$V_\gamma = V_{\min} = \min \{ V_c \}$$

$$c \in [0, c_{\max}]$$

(v) One has the inequality

$$E_{c_{\max}} = a_0 + (\alpha^2/\beta\gamma)\sqrt{V_{\min}} \leq a_0 + \sqrt{(\alpha^2/\beta)V_{\min}},$$

where the upper bound is attained if $\beta\gamma^2 = \alpha^2$. In this case $(c_{\max})^2 = 2\alpha^2/\beta$ and

$$E_{c_{\max}} = E_g + 2\gamma\sqrt{V_{\min}}, \quad V_{c_{\max}} = 2 \cdot V_{\min}.$$

Proof. It is straightforward to verify that the proposed solution can be rewritten as

$$(6.21) \quad Y(c) = \begin{cases} \operatorname{sgn}(\alpha)\sqrt{V_{\min}} \cdot (\gamma |\alpha| - c\sqrt{\alpha^2 + \beta\gamma^2 - \beta c^2}) / (\beta c^2 - \alpha^2), & c^2 \neq \alpha^2/\beta, \\ \sqrt{V_{\min}} \cdot (\alpha^2 - \beta\gamma^2) / 2\alpha\beta\gamma, & c^2 = \alpha^2/\beta. \end{cases}$$

A comparison with (6.17) shows that this is the solution obtained setting $\epsilon = -\operatorname{sgn}(\alpha)$. From Lemma 6.2 one knows that the alternative choice $\epsilon = \operatorname{sgn}(\alpha)$ does not lead to a continuous solution in the interval $[0, c_{\max}]$. Let us check the properties (i) to (v).

(i) Starting with the representation (6.17) one obtains after some calculation the following value for the derivative with respect to c (assume $c^2 \neq \alpha^2/\beta$) :

$$(6.22) \quad E_c' = \alpha Y'(c) = -\alpha\sqrt{V_{\min}} \cdot [\epsilon \{ \beta(\beta\gamma^2 - \alpha^2)c^2 + \alpha^2(\alpha^2 + \beta\gamma^2) \} + 2\alpha\beta\gamma c\sqrt{\alpha^2 + \beta\gamma^2 - \beta c^2}] / [(\beta c^2 - \alpha^2)^2 \sqrt{\alpha^2 + \beta\gamma^2 - \beta c^2}]$$

One shows that

$$(6.23) \quad \begin{aligned} E_c' &> 0 \text{ if } \epsilon = -\text{sgn}(\alpha), \\ E_c' &< 0 \text{ if } \epsilon = \text{sgn}(\alpha). \end{aligned}$$

For this it suffices to show that

$$(6.24) \quad \beta(\beta\gamma^2 - \alpha^2)c^2 + \alpha^2(\alpha^2 + \beta\gamma^2) > 2|\alpha| \beta\gamma c \sqrt{\alpha^2 + \beta\gamma^2 - \beta c^2}.$$

Taking squares and doing some algebra this condition is equivalent to the inequality

$$(6.25) \quad \beta^2(\alpha^2 + \beta\gamma^2)^2(\beta c^2 - \alpha^2)^2 > 0,$$

which is always fulfilled by assumption.

(ii) One has

$$(6.26) \quad Y(0) = -(\gamma/\alpha)\sqrt{V_{\min}}, \quad Y(c_{\max}) = (\alpha/\beta\gamma)\sqrt{V_{\min}}.$$

Using the assumption $\beta\gamma^2 \geq \alpha^2$ it follows that

$$V_0 - V_{c_{\max}} = \beta(Y(0)^2 - Y(c_{\max})^2) = \beta V_{\min} \cdot (\gamma^2/\alpha^2 - \alpha^2/\beta^2\gamma^2) = V_{\min} \cdot (\beta^2\gamma^4 - \alpha^4)/\alpha^2\beta\gamma^2 \geq 0.$$

(iii) From the expressions (6.20) and (6.22) for $Y(c)$, $Y'(c)$, one obtains that

$$(6.27) \quad \text{sgn}(V_c') = \text{sgn}(2\beta Y(c)Y'(c)) = \text{sgn}(c - \gamma),$$

which shows the desired property.

(iv) If $c = \gamma$ one has $Y(\gamma) = 0$ and the property follows from the formulas (6.15), (6.16) and the assumption.

(v) This is similar to (iv) using that $Y(c_{\max}) = (\alpha/\beta\gamma)\sqrt{V_{\min}}$.

7. An alternative canonical form for a standard three-security analysis.

In the notations of Section 5 suppose that $v^* \neq 0$, and let τ^* be a non-singular 2×2 -matrix. The constructive proof of our Theorem 5.1 yields an alternative derivation of the canonical form proposed by Markowitz(1987) (see theorem 11.1 and appendix to chap. 11). Setting $n=k=2$, $d_1=d_2=1$, $a_2=0$, $H=(h_{ij})$, in equations (5.19) to (5.23), one has to solve the system of equations

$$(7.1) \quad a_0 = v_0 - a_1 b_1$$

$$(7.2) \quad v_1 = h_{11} a_1$$

$$v_2 = h_{12} a_1$$

$$(7.3) \quad \tau_{00} = V_{\min} + b_1^2 + b_2^2$$

$$(7.4) \quad \tau_{01} = h_{11} b_1 + h_{21} b_2$$

$$\tau_{02} = h_{12} b_1 + h_{22} b_2$$

$$\tau_{11} = (h_{11})^2 + (h_{21})^2$$

$$(7.5) \quad \tau_{12} = h_{11} h_{12} + h_{21} h_{22}$$

$$\tau_{22} = (h_{12})^2 + (h_{22})^2$$

Multiply equations (7.5) with a_1^2 and use (7.2) to get the system of equations

$$a_1^2 \tau_{11} = v_1^2 + (a_1 h_{21})^2$$

$$(7.6) \quad a_1^2 \tau_{12} = v_1 v_2 + a_1^2 h_{21} h_{22}$$

$$a_1^2 \tau_{22} = v_2^2 + (a_1 h_{22})^2$$

Multiply the first equation of (7.6) with $(a_1 h_{22})^2$ and use the others to get

$$(7.7) \quad (a_1^2 \tau_{11} - v_1^2) \cdot (a_1^2 \tau_{22} - v_2^2) = (a_1^2 \tau_{12} - v_1 v_2)^2.$$

Simplifying this expression one can solve for a_1 :

$$(7.8) \quad a_1^2 (\det \tau^*) = \tau_{22} v_1^2 - 2 \tau_{12} v_1 v_2 + \tau_{11} v_2^2.$$

Multiply this equation with τ_{11} , resp. τ_{22} , and use the previous set (7.6) of equations to solve for h_{21} , resp. h_{22} :

$$(7.9) \quad (a_1 h_{21})^2 (\det \tau^*) = (\tau_{12} v_1 - \tau_{11} v_2)^2$$

$$(7.10) \quad (a_1 h_{22})^2 (\det \tau^*) = (\tau_{12} v_2 - \tau_{22} v_1)^2$$

On the other side h_{11} , h_{12} are given by (7.2). Now it is possible to evaluate $\mathbf{b} = (\mathbf{H}^T)^{-1} \cdot \tau_0$. After simplification one obtains

$$(7.11) \quad b_1 = a_1 \cdot (h_{22} \tau_{01} - h_{21} \tau_{02}) / (h_{22} v_1 - h_{21} v_2)$$

$$(7.12) \quad b_2 = a_1 \cdot (h_{11} \tau_{02} - h_{12} \tau_{01}) / (h_{22} v_1 - h_{21} v_2)$$

The remaining parameters follow from (7.1) and (7.3) :

$$(7.13) \quad a_0 = v_0 - a_1 b_1$$

$$(7.14) \quad V_{\min} = \tau_{00} - b_1^2 - b_2^2$$

We have shown that the non-singular affine transformation $\mathbf{Y} = \mathbf{b} + \mathbf{H} \cdot \mathbf{X}$ yields the canonical form of portfolio analysis

$$(7.15) \quad V = V_{\min} + Y_1^2 + Y_2^2, \quad E = a_0 + a_1 Y_1.$$

Bibliographical notes. The statistical and actuarial arguments about variance reduction in an option environment are based on previous ideas expressed by the author(1994a/94b). The minimum problem about the total option variance has been considered also in Hürlimann(1994b). The extremal distribution (2.6), with respect to the second-order stochastic dominance relation or stop-loss order relation, has been introduced in Hürlimann(1993). Similar (minimal and maximal) extremal distributions can be defined in case the range of a random variable is two-side bounded or one-side bounded (e.g author(1995c)). The determination of the guaranteed rate of return for portfolio insurance is based on Hürlimann(1995a). Example 3.3 is due to the author(1995b). The material in Sections 5, 6 and 7 has been presented first in Hürlimann(1992).

References.

- Black, F., Scholes, M.* (1973). The pricing of options and corporate liabilities. *Journal of Political Economy* 81, 637-59. Reprinted in Luskin(1988).
- Bowers, N.L.* (1969). An upper bound for the net stop-loss premium. *Transactions of the Society of Actuaries* XIX, 211-216.
- Bühlmann, H.* (1995). Life insurance with stochastic interest rates. In Ottaviani, G. (Editor) : *Financial Risk in Insurance*. Springer-Verlag.
- Elton, M.J., Gruber, M.J.* (1991). *Modern Portfolio Theory and Investment Analysis* (fourth edition). John Wiley.
- Huang, C., Litzenberger, R.H.* (1988). *Foundations for Financial Economics*. Elsevier Science Publishing Co.
- Hürlimann, W.* (1991). Absicherung des Anlagerisikos, Diskontierung der Passiven und Portfoliotheorie. *Bulletin Swiss Association of Actuaries*, 217-250.
- Hürlimann, W.* (1992). A geometric approach to extended mean-variance portfolio analysis. Annual Meeting "Deutsche Mathematische Vereinigung", Berlin, September 1992.
- Hürlimann, W.* (1993). Solvabilité et Réassurance. *Bulletin Swiss Association of Actuaries*, 229-49.
- Hürlimann, W.* (1994a). Splitting risk and premium calculation. *Bulletin Swiss Association of Actuaries*, 167-97.
- Hürlimann, W.* (1994b). From the inequalities of Bowers, Kremer and Schmitter to the total stop-loss risk. *Proceedings XXV. ASTIN Colloquium, Cannes*.
- Hürlimann, W.* (1995a). Links between premium principles and reinsurance. *Proc. XXV. Int. Congress of Actuaries, Brussels, vol. 2, 141-67*.
- Hürlimann, W.* (1995b). A distribution-free all-finance binomial valuation model. *Proc. XXV. Int. Congress of Actuaries, Brussels, vol. 3, 365-382*.
- Hürlimann, W.* (1995c). A stop-loss ordered extremal distribution and some of its applications. *XXVI. ASTIN Colloquium, Leuven, September 1995*.
- Luskin, D.L.* (1988) (Editor). *Portfolio Insurance : a guide to Dynamic Hedging*. John Wiley, New York.
- Margrabe, W.* (1978). The value of an option to exchange one asset for another. *Journal of Finance* 33, 177-186.
- Markowitz, H.* (1952). Portfolio Selection. *The Journal of Finance*, 77-91.
- Markowitz, H.* (1959). *Portfolio Selection - Efficient Diversification of Investments*. John Wiley. Second Edition (1991). Basil Blackwell.
- Markowitz, H.* (1987). *Mean-variance analysis in portfolio choice and capital markets*. Basil Blackwell.
- Vasicek, O.* (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics* 5, 177-188.