Linear Approach for Solving Large-Scale Portfolio Optimization Problems in a Lognormal Market

Faris Hamza and Jacques Janssen

Abstract

This paper presents a linear approach for solving large scale portfolio optimization problems using asymmetric risk functions. This approach is based on separable convex programming techniques. In addition, we discuss our model in the particular case where the investment is lognormally distributed. It is shown in this case that the selection of optimal portfolios is independent of the investor´s choice of the parameters in the asymmetric risk function. Moreover, it will be shown that the non linear programming model obtained can be well approximated by a piecewise linear approach in order to be solved in a practical amount of time.

Résumé

Dans ce papier, nous présentons une approximation linéaire pour la résolution des problèmes d´optimisation de portefeuille de grande taille, suivant le critère moyenne-risque asymétrique. Cet approche est basé sur les techniques de la programmation convexe séparable. De plus nous discutons notre modèle dans le cas particulier où le rendement global de portefeuille est distribué suivant une log-normale. Il est démontré dans ce cas que la sélection des portefeuilles optimaux est indépendante du choix des paramètres de la fonction de risque. Le modèle non linéaire obtenu peut être efficacement approximé par une approche linéaire afin d´être résolu en un temps raisonnable.

Keywords

Portfolio optimization, asymmetric risk function, linear programming, mean-variance model, lognormal market.

Mots cléfs

Optimisation de portefeuille, risque asymétrique, programmation linéaire, modèle moyenne-variance, marché log-normal.

Université Libre de Bruxelles, Ecole de Commerce Solvay - CADEPS, Avenue F.D. Roosevelt, 50 - P.B. 194/7, B - 1050 Brussels (Belgium); Tel: + 32-2-650 38 83, Fax: + 32-2-650 27 85, E-mail: Janssen@ulb.ac.be

1. Introduction

The basic portfolio optimization methodology started with the famous seminal work of Markowitz (1952). This latter, formulated the portfolio optimization problem as a quadratic programming problem in which the risk function measured by variance of return out of the portfolio is minimized subject to the constraint on the average return. This formulation which is considered as the basis of financial modern portfolio theory, has not been extensively used in practice. First, this mean-variance frame work is known to be valid if the rates of return are random variables normally distributed or if the investor's utility function is quadratic. Second, empirical evidence with attempts to solve a large scale mean-variance portfolio problem with existing software for quadratic programming indicate that the problem is very hard, if not impossible. We refer to Zenios (1993) for different difficulties in solving the large scale quadratic model. More recently, with growing importance of determining optimal portfolio allocations when there are a large number of assets, several alternative models were developed as an alternative to the classical Markowit'z model. These alternatives do not require any assumption on the distribution of the rates of return. Konno and Yamazaki (1991) have been proposed the absolute deviation, while Speranza (1993) has been proposed the semi-absolute deviation as measures of risk instead of variance. King and Jensen (1992), King (1993), Markowitz (1991) and Markowitz et all (1993) adopted semi-variance as the risk measure. Hamza and Janssen (1995) introduced a new risk function measured by a convex combination of the two semi-variances of the portfolio rate of return from the mean value. The model generalizes both of the mean-variance and semi-variance models according to a suitable choice of the coefficients in the risk function.

In this paper, a linearization of the quadratic objective function, based on separable convex programming techniques have been made. The resulting optimization model generalizes the mean-absolute model. Moreover, the mean-variance model developed by Konno and Suzuki (1992) appears as a particular case of this our linear approximation. In addition, several empirical tests have been found that actual security price data are lognormally distributed and a discussion of the model under this assumption seems very interesting.

The paper is organized as follows. In section 2 we recall the Hamza and Janssen's model. In section 3 a linear approach of our model based on separable convex programming is presented. In section 4, we analyse optimal portfolios in a lognormal market. It will be shown, in this case that the selection of efficient portfolios is independent of the investor's choice of the parameters in our risk function. Finally, some concluding remarks will be given.

2. The Hamza and Janssen's model

Let be n stocks denoted by A_j ($j = 1, \ldots\ldots, n$) and let R_j the random variable representing the rate of return per period of the asset A_j. We denote by x_j the proportion of the capital invested in security j ($j = 1, \ldots\ldots, n$). The rate of return of the portfolio is given by:

$$R(x) = \sum_{j=1}^{n} R_j x_j$$

Hamza and Janssen (1995) propose consideration of the convex combination of the two semi-variances as measure of risk:

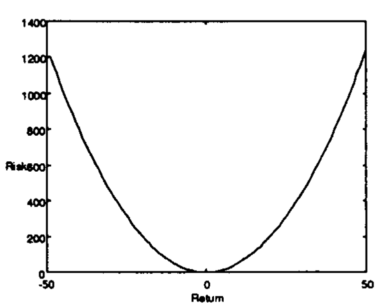
$$N_{(\alpha,\beta)}(x) = \alpha E\{min(0, R(x) - ER(x))\}^2 + \beta E\{max(0, R(x) - ER(x))\}^2$$

where the notation $ER(x)$ represents the expected value of the random variable $R(x)$ and α and β are two positive parameters representing the degree of risk aversion of the investor.

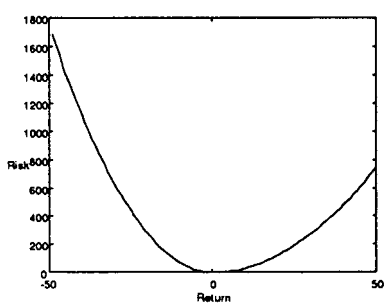
This model which does not require any assumption on the distribution of the rates of returns, was validated by a result which showd that, if the rates of returns of assets are multivariate normally distributed, our risk function is equivalent to the traditional measure of risk, the variance of returns (see Hamza and Janssen -1995).

This asymmetric risk function has also an interesting interpretation in terms of preferences of an investor (a detailed utility analysis is out of the scope of this paper).

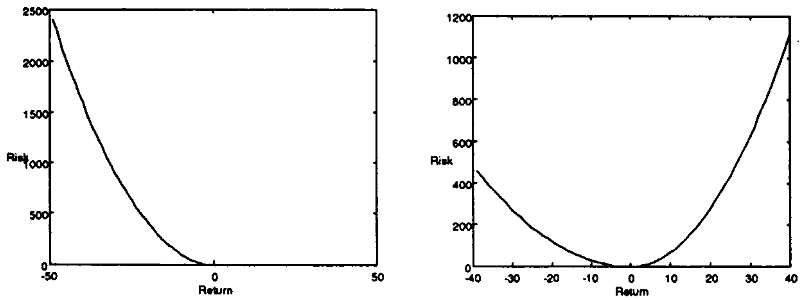
Graphically, It can be seen from Fig.1 that $\alpha=\beta$ (a) corresponds to the variance risk. Also $\beta=0$ (b) corresponds to an investor who uses an asymmetric measure of risk that focuses on squared return deviations below the mean of the distribution, called the semi-variance risk. But $\alpha>\beta$ (c) represents an investor whose risk associated with "below the average" return is compensated to some extent by "above the average" return. Moreover $\alpha<\beta$ (d) can be associated with an investor who is not averse to risk.



(a) Variance ($\alpha=\beta=0.5$)



(c) Asymmetric risk ($\alpha=0.7$ and $\beta=0.3$)



(b) Semi-variance (α=1 and β=0)

(d) Asymmetric risk (α=0.3 and β=0.7)

Fig. 1. Penalty Functions For Risk measured by the convex combination of the two semi-variances below and above the mean (ρ=0).

Numerical example:

Let us consider two following investments A and B:

A		B	
Probability	Outcome (%)	Probability	Outcome (%)
0.2	0	0.3	-1
0.1	1	0.4	4
0.4	2	0.1	5
0.3	7	0.2	6
0.0	otherwise	0.0	otherwise

What can be said about the desirability of the investments A and B using our risk function ?

Simple arithmetic shows that:

mean (A) = mean (B) = 3.

a) A and B are identical from the viewpoint of $E\text{-}N(0.5, 0.5)$ model, (risk (A) = risk (B) = 3.7);

b) A is preferred to B in $E\text{-}N(1,0)$ model, (risk (A) = 2.6 and risk (B) = 4.8);

c) A is preferred to B in $E\text{-}N(0.7,0.3)$ model, (risk (A) = 3.26 and risk (B) = 4.14);

d) B is preferred to A in $E\text{-}N(0.3,0.7)$ model, (risk (A) = 4.14 and risk (B) = 3.26).

3. Piecewise linear approximation to the risk function and portfolio optimization

It was shown by Hamza and Janssen (1995) that the portfolio optimization problem using this class of alternative risk function can be selected by solving the following program

$$Min \frac{1}{T-1} \sum_{t=1}^{T} \left\{ \alpha u_t^2 + \beta v_t^2 \right\}$$

$$\text{subject to} \begin{cases} \sum_{j=1}^{n} \hat{r}_j x_j \geq \rho \\ \sum_{j=1}^{n} x_j = 1 \\ \sum_{j=1}^{n} \left(r_{jt} - \hat{r}_j \right) x_j = v_t - u_t & t = 1, \ldots\ldots, T \\ x_j \geq 0 & j = 1, \ldots\ldots, n \\ u_t \geq 0, \quad v_t \geq 0 & t = 1, \ldots\ldots, T. \end{cases} \tag{1}$$

where

$$\hat{r}_j = \frac{1}{T} \sum_{t=1}^{T} r_{jt} \qquad for \ \ j = 1, \ldots, n$$

and $(r_{1t}, \ldots\ldots, r_{nt})$ where $t = 1, \ldots\ldots T$; are T independent samples of the random variable $(R_1, \ldots\ldots, R_n)$ which is assumed to be available through the historical data.

In order to solve the basic mean-asymmetric risk optimization program, we can also solve the program (2) instead the program (1) formulated as follow

$$Min \frac{1}{T-1}\sum_{t=1}^{T} z_t^2$$

$$\text{subject to}\begin{cases}\sum_{j=1}^{n}\hat{r}_j x_j \ge \rho & \\ \sum_{j=1}^{n} x_j = 1 & \\ \sum_{j=1}^{n}\left(r_{jt}-\hat{r}_j\right)x_j = v_t - u_t & t=1,\dots\dots,T \\ z_t = \sqrt{\beta}v_t - \sqrt{\alpha}u_t & t=1,\dots\dots,T \\ x_j \ge 0 & j=1,\dots\dots,n \\ u_t \ge 0, \quad v_t \ge 0 & t=1,\dots\dots,T.\end{cases} \tag{2}$$

This formulation called the « Compact factorization scheme » was first introduced by Konno and Suzuki (1992) for solving the mean-variance problem, obtained in our model by the particular case where α=β.

Observing that the objective function can be written in the form

$$\Phi(z_1,\dots..,z_T)=\sum_{t=1}^{T} z_t^2 = \sum_{t=1}^{T}\delta_t(z_t)$$

where

$$\delta_t(z_t) = z_t^2 \qquad \text{for all } t=1,\dots\dots,T$$

are T convex functions. Then the program (2) belongs to a class of convex separable programming problems.
Therefore each function $\delta_t(z_t)$ can be approximated by a broken line function in the plan (z_t,δ_t), and then we can use the simplex algorithm.
To do this, we select K breakpoints A_k $(k=1,\dots.,K)$ on the curve represented by their abscissas a_k $(k=1,\dots.,K)$ with

$$a_1=\sqrt{\beta}c^{+}-\sqrt{\alpha}c^{-} \quad and \quad a_K=\sqrt{\beta}d^{+}-\sqrt{\alpha}d^{-}$$

where

$$c = min\left\{r_{jt}-\hat{r}_j \,/\, j=1,\dots.,n \ \ and\ \ t=1,\dots.,T\right\}$$

and

$$d = max\left\{r_{jt}-\hat{r}_j \,/\, j=1,\dots.,n \ \ and\ \ t=1,\dots.,T\right\}.$$

Recall that:

$$|\xi|_+ = \max(0,\xi) = \begin{cases} \xi & \text{if } \xi \geq 0 \\ 0 & \text{if not} \end{cases} \quad \text{and} \quad |\xi|_- = -\min(0,\xi) = \begin{cases} 0 & \text{if } \xi \geq 0 \\ -\xi & \text{if not} \end{cases}$$

If the risk is measured by the traditional measure of risk, the variance of returns, which is obtained when the two parameters α and β take the same value in our risk function, the extreme points are given by the boundaries c and d .

Obviously, the approximation is more better if we increase the number of breakpoints A_k $(k = 1,\ldots,K)$.

Putting

$$\delta_t = \delta(a_t).$$

for all z_t $(t = 1,\ldots,T)$ such that $a_k \leq z_t \leq a_{k+1}$ $(t = 1,\ldots,T \text{ and } k = 1,\ldots,K)$

i.e. $z_t = \lambda_{tk} a_k + \lambda_{tk+1} a_{k+1}$

with

$$\lambda_{tk} + \lambda_{tk+1} = 1 \text{ and } \lambda_{tk} , \lambda_{tk+1} \geq 0. \qquad (i)$$

then $\delta_t(z_t)$ can be approximated by, the straight line which has the following equation

$$\delta_t'(z_t) = \delta_t(a_k) + \frac{\delta_t(a_{k+1}) - \delta_t(a_k)}{a_{k+1} - a_k}(z_t - a_k). \qquad (ii)$$

Using *(i)*, The expression *(ii)* yields

$$\delta_t'(z_t) = \delta_t(a_k) + \lambda_{tk+1}(\delta_t(a_{k+1}) - \delta_t(a_k)).$$

Then, we easily replaced $\delta_t(z_t)$ by $\delta_t'(z_t)$ such that

$$\lambda_{tk} + \lambda_{tk+1} = 1$$

$$\lambda_{tk}, \lambda_{tk+1} \geq 0$$

$$z_t = \lambda_{tk} a_k + \lambda_{tk+1} a_{k+1}$$

and

$$\delta_t'(z_t) = \delta_t(a_k)\lambda_{tk} + \delta_t(a_{k+1})\lambda_{tk+1}..$$

For all z_t such that $a_1 \leq z_t \leq a_k$ $(t = 1,....,T \text{ and } k = 1,....,K)$ we can write, more generally

$$\sum_{k=1}^{K} \lambda_{tk} = 1$$

$$\sum_{k=1}^{K} \lambda_{tk} a_k - z_t = 0 \quad \forall t = 1,....,T$$

$$\lambda_{tk} \geq 0 \qquad \forall t = 1,....,T.$$

Substituting $\delta_t(z_t)$ by it approximation

$$\sum_{k=1}^{K} \delta_t(a_k)\lambda_{tk} \quad \textit{where } \delta_t(z_t) = z_t^2 .$$

we get

$$z_t^2 \approx \sum_{k=1}^{K} a_k^2 \lambda_{tk} .$$

Then we need to solve the following linear program

$$
Min \frac{1}{T-1} \sum_{t=1}^{T} \sum_{k=1}^{K} a_k^2 \lambda_{tk}
$$

$$
\text{subject to} \begin{cases}
\sum_{j=1}^{n} \hat{r}_j x_j \geq \rho & \\
\sum_{j=1}^{n} x_j = 1 & \\
\sum_{j=1}^{n} \left(r_{jt} - \hat{r}_j \right) x_j = v_t - u_t & t = 1, \ldots\ldots, T \\
\sum_{k=1}^{K} a_k \lambda_{tk} = \sqrt{\beta} v_t - \sqrt{\alpha} u_t & t = 1, \ldots\ldots, T \\
\sum_{k=1}^{K} \lambda_{tk} = 1 & t = 1, \ldots\ldots, T \\
x_j \geq 0 & j = 1, \ldots\ldots, n \\
\lambda_{tk} \geq 0 & t = 1, \ldots\ldots, T \text{ and } k = 1, \ldots.., K \\
u_t \geq 0,\ v_t \geq 0 & t = 1, \ldots\ldots, T
\end{cases} \qquad (3)
$$

This program consists of $n+T(k+2)$ variables and $3T+2$ constraints.

To illustrate this approximation, the figure 2 shows the converting of the convex risk function measured by $N_{(0.7,0.3)}$ (see Fig.1 sheme (c)) into a piecewise linear format.

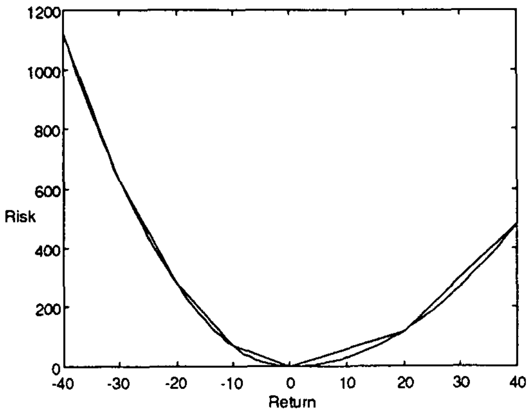


Fig-2. Converting a convex risk function $N(0.7,0.3)$ into a piecewise linear format

Observing that, the mean-absolute deviation model developed by Konno and Yamazaki (1991), and Speranza (1993), can be viewed as a particular case of this piecewise linear approximation, obtained with two linear pieces (K=2) instead of K-1 pieces (see Figure 3).

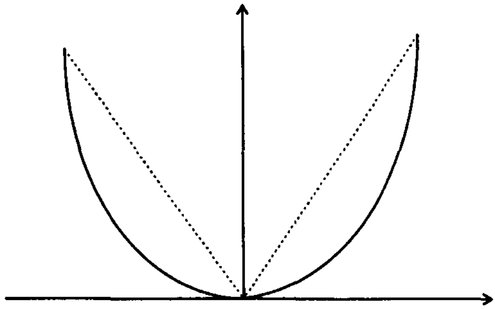


Fig. 3

4. Mean-asymmetric risk analysis in a lognormal market

The assumption of normally distributed returns has been questioned by several authors because return distributions must be bounded from below and because empirical evidence suggests that returns are not normally distributed. Lintner (1980) has analyzed security-price data and concluded that lognormal distribution is significant. Several authors have confirmed the empirical Lintner's framework and they found that actual security price data are well be approximated by the lognormal distribution. For a special references to the lognormal distribution applications in finance, the reader can refer to Kunio and Edwin (1988).

4.A. Global return of the portfolio is lognormally distributed

First, we propose to reformulate portfolio theory under the assumption that global investment $R(x)=\sum_{i=1}^{n} x_i R_i$ is lognormally distributed:

$$R(x) \sim LN\left(\mu(x), \sigma^2(x)\right)$$

or equivalently

$$LnR(x) \sim N\left(\mu(x), \sigma^2(x)\right).$$

It is known that the mean value for the lognormal random variable *R(x)* is given by

$$ER(x) = \exp\left(\mu(x) + \frac{1}{2}\sigma^2(x)\right)$$

which represents the expected rate of return of portfolio, with

$$\mu(x) = E\left(\sum_{i=1}^{n} x_i \ln R_i\right) = \sum_{i=1}^{n} x_i E(\ln R_i)$$

and

$$\sigma^2(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \operatorname{cov}(\ln R_i, \ln R_j).$$

Next theorem, shows that the selection of optimal portfolios is independent of the investor's choice of the parameters in the asymmetric risk function $N_{(\alpha,\beta)}(x)$.

Theorem 4.1

If the global return of the portfolio of an investor is lognormally distributed, then the risk measured by $N_{(\alpha,\beta)}(x)$ is independent of the choice of the parameters α and β. Moreover, the optimum portfolio will be the one which has the minimum possible variance, for a given average rate of return.

Proof:

The risk function used by the investor is given by

$$N_{(\alpha,\beta)}(x) = \alpha E\{\min(0, R(x) - ER(x))\}^2 + \beta E\{\max(0, R(x) - ER(x))\}^2.$$

Both random variables

$$E\{\min(0, R(x) - ER(x))\}^2$$

and

$$E\{\max(0, R(x) - ER(x))\}^2$$

have the same density of probabilities function given by

$$g(u) = \frac{f(\sqrt{u})}{2\sqrt{u}}$$

where f is the density function of lognormal distribution $LN(0, \sigma^2(x))$ given by:

$$f(u)=\begin{cases}\dfrac{1}{\sqrt{2\pi}\sigma(x)u}exp\left[\dfrac{-(lnu)^2}{2\sigma^2(x)}\right] & if\ u>0\\ 0 & if\ u\leq 0\end{cases}$$

therefore

$$g(u)=\begin{cases}\dfrac{1}{2\sqrt{2\pi}\sigma(x)u}exp\left[\dfrac{-(lnu)^2}{8\sigma^2(x)}\right] & if\ u>0\\ 0 & if\ u\leq 0\end{cases}$$

It follows that

$$N_{(\alpha,\beta)}(x)=(\alpha+\beta)\int_0^{+\infty}ug(u)du$$
$$=\frac{(\alpha+\beta)}{2\sqrt{2\pi}\sigma(x)}\int_0^{+\infty}\exp\left(\frac{-(\ln u)^2}{8\sigma^2(x)}\right)du.$$

Putting

$$t=\ln u$$

we get

$$N_{(\alpha,\beta)}(x)=\frac{(\alpha+\beta)}{2\sqrt{2\pi}\sigma(x)}\int_{-\infty}^{+\infty}\exp\left(\frac{-t^2}{8\sigma^2(x)}+t\right)dt.$$

Since the remarkable equality

$$\frac{-t^2}{8\sigma^2(x)}+t=-\frac{1}{2}\left[\left(\frac{t}{2\sigma(x)}-2\sigma(x)\right)^2-\left(4\sigma^2(x)\right)\right]$$

and putting again

$$v=\frac{t}{2\sigma(x)}-2\sigma(x):$$

we get

$$N_{(\alpha,\beta)}(x)=\frac{(\alpha+\beta)}{2\sqrt{2\pi}\sigma(x)}2\sigma(x)exp\left(2\sigma^2(x)\right)\int_{-\infty}^{+\infty}exp\left(\frac{-v^2}{2}\right)dv$$
$$=\frac{(\alpha+\beta)}{2\sqrt{2\pi}\sigma(x)}2\sigma(x)exp\left(2\sigma^2(x)\right)\sqrt{2\pi}$$
$$=(\alpha+\beta)exp\left(2\sigma^2(x)\right).$$

Recall that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{v^2}{2}} dv = 1 = \Phi(+\infty)$$

where Φ is the standard normal distribution $N(0,1)$.

Then the resulting $E\text{-}N_{(\alpha,\beta)}(x)$ portfolio optimization program takes the following form

$$Min \;\; (\alpha+\beta)\, exp\left(2\sigma^2(x)\right)$$

$$subject\ to \begin{cases} E(R(x)) = exp\left[\mu(x) + \frac{1}{2}\sigma^2(x)\right] \geq \rho \\ \sum\limits_{j=1}^{n} x_j = 1 \\ x_j \geq 0 \quad j = 1,\ldots,n \end{cases} \tag{1}$$

Because both of exponential and logarithmic functions are non decreasing, the program (1) is equivalent to the following program (2) given by

$$Min\ \sigma^2(x) = x'\Sigma x$$

$$subject\ to \begin{cases} \mu(x) + \frac{1}{2}\sigma^2(x) \geq ln\rho \\ \sum\limits_{j=1}^{n} x_j = 1 \\ x_j \geq 0 \quad j = 1,\ldots,n \end{cases} \tag{2}$$

where Σ is the variance-covariance matrix .

Finally, the investor chooses its optimum portfolios using the mean-variance approach. □

Under the assumptions of theorem 4.1, we have the following result.

Theorem 4.2

If the variance-covariance matrix Σ is positive definite, then the investor will have a unique optimal composition for his portfolio.

Proof:

The optimization portfolio problem is given by the following program:

$$Min\ \sigma^2(x) = x'\Sigma x$$

$$subject\ to \begin{cases} \mu(x) + \frac{1}{2}\sigma^2(x) \geq ln\rho \\ \sum_{j=1}^{n} x_j = 1 \\ x_j \geq 0 \quad j = 1,\dots,n \end{cases}$$

Putting

$$K_1 = \left\{ x \in R^n / \sum_{j=1}^{n} x_j = 1, x_j \geq 0,\ j = 1,\dots,n \right\}$$

K_1 is a closed and bounded set, then it is compact.

Putting also

$$K_2 = \left\{ x \in R^n / \mu(x) + \frac{1}{2}\sigma^2(x) \geq \ln\rho \right\}$$

K_2 is also a closed set . In order to see this, we consider the following function

$$f: R^n \to R$$
$$x \mapsto \mu(x) + \frac{1}{2}\sigma^2(x)$$

f is a quadratic and continuous function , and consequently

$K_2 = f^{-1}([\ln\rho, +\infty[)$ is a closed set of R .

Therefore $D = K_1 \cap K_2$ is a closed set of K_1 which is compact, and then D is compact.

Now we can write the portfolio optimization program (P) as:

$$\underset{x\in D}{Min}\ \sigma^2(x) = x'\Sigma x$$

Since D is compact, proving that the program *(P)* has a unique optimal solution, it suffices to show that the objective function is finite , strictly convex on D and continuous.

(i) $x'\Sigma x$ is finite, because of

$$\begin{aligned}\underset{x\in D}{Min}\ \sigma^2(x) = x'\Sigma x &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j \\ &\le \left(\sum_{i=1}^{n} x_i\sigma_i\right)^2 \\ &\le \left(\max_i \sigma_i^2 \sum_{i=1}^{n} x_i\right)^2 \\ &= \max_i \sigma_i^2 \\ &< \infty\end{aligned}$$

(ii) The objective function is strictly convex because the variance-covariance matrix Σ is positive definite. To see This, we observe that the Hessian matrix of the variance $\sigma^2(x)$ given by $H(x) = \left(\dfrac{\partial^2\sigma^2(x)}{\partial x_i \partial x_j}\right)_{i,j=1,\dots,n}$ is exactly the variance-covariance matrix Σ which is assumed positive definite and then

$$x'\Sigma x > 0 \quad \forall x \in R^n .$$

(iii) The continuity of the function f on D is obvious.

Consequently, the investor has a unique optimal composition for his portfolio. □

4.B. Returns of securities are lognormally distributed

This case is more realistic than the previous case, but much more complex. If the rate of returns of securities are multivariate lognormally distributed, then the rate of return of portfolio can not be lognormally distributed. We recall that a linear combination of random variables which are lognormally distributed is not necessarily lognormal distributed. Thus, we may have distributions which are approximately but not precisely lognormal. In financial literature, we often utilize the approximation that sums of lognormal variables are also lognormal; that is, assumed that every portfolio has lognormal distribution, through it cannot be justified theoretically. For more details, the reader can refer to Elton and Gruber (1974), Ohlson and Ziemba (1976), and Levy and Samuelson (1992).

In the particular case, when returns of assets are mutually independents lognormally distributed, the following theorem holds.

Theorem 4.3

If the n assets which form the portfolio of the investor are mutually independents lognormally distributed, and if the investor decides to invest in the n assets, then the risk incurred by the investor will be constant, and his optimum expected return will be obtained for the uniform composition between the n assets.

Proof:

Denoting by $R_1,\dots,R_n$ n random variables representing the rate of return per period of the n assets which form the portfolio. We assume that $R_1,\dots,R_n$ are mutually independents with

$$R_i \sim LN(\mu_i, \sigma_i^2).$$

The assumption that The investor decides to invest in the n assets can be formulated as

$$x_i > 0 \quad \forall i = 1,\dots,n$$

Then, (see Johnson and Kotz-1970- p.119) the rate of return of portfolio $R(x) = \sum_{i=1}^{n} x_i R_i$ is lognormally distributed

$$LN\left(\mu(x) = \sum_{i=1}^{n} (\mu_i + \ln x_i), \sigma^2 = \sum_{i=1}^{n} \sigma_i^2\right).$$

Using theorem 4.1, the asymmetric risk function yields:

$$N_{(\alpha,\beta)}(x) = (\alpha+\beta) exp\left[2\sum_{i=1}^{n} \sigma_i^2\right].$$

Therefore the risk function is a constant independent of x.

The investor is then driven by the desire to get maximum expected return from the portfolio which consists of n assets, each purchased in quantity of proportion $x_i > 0 \quad \forall i = 1,\dots,n$; and he needs to solve the following optimization program:

$$(P_0) \quad Max\, E(R(x))$$

$$subject\ to \begin{cases} \sum_{j=1}^{n} x_j = 1 \\ x_j > 0 \qquad j = 1,\dots,n \end{cases}$$

$$\begin{cases} x_2 x_3 \ldots . x_n + \lambda = 0 \\ x_1 x_3 \ldots . x_n + \lambda = 0 \\ . \\ . \\ \ldots\ldots\ldots\ldots\ldots \\ x_1 x_2 \ldots . x_{n-1} + \lambda = 0 \\ x_1 + x_2 + \ldots . + x_n = 1 \end{cases}$$

It can be easily shown that the solution of this system is given by the equal investment in the n assets $\hat{x} = \left(\frac{1}{n}, \ldots\ldots, \frac{1}{n}\right)$ which represents the optimal portfolio of the investor.

□

4.C. Piecewise linear approximation for lognormal portfolio selection

Taking into account theorem 4.1, the investor selects optimal portfolios by solving the following program

$$Min\ \sigma^2(x) = x'\Sigma x$$

$$subject\ to \begin{cases} \mu(x) + \frac{1}{2}\sigma^2(x) \geq ln\rho \\ \sum_{j=1}^{n} x_j = 1 \\ x_j \geq 0 \quad j = 1, \ldots, n \end{cases} \qquad (1)$$

Using the following unbiased estimates of $\mu(x)$ and $\sigma^2(x) = x'\Sigma x$ given respectively by

$$\sum_{j=1}^{n} \hat{r}_j x_j \quad \text{and} \quad \frac{1}{T-1}\sum_{t=1}^{T}\left\{\sum_{j=1}^{n}\left(r_{jt} - \hat{r}_j\right)x_j\right\}^2 \quad \text{where} \quad \hat{r}_j = \frac{1}{T}\sum_{t=1}^{T} r_{jt}.$$

Then the program (1) yields:

$$Min\ \frac{1}{T-1}\sum_{t=1}^{T}\left\{\sum_{j=1}^{n}\left(r_{jt}-\hat{r}_j\right)x_j\right\}^2$$

$$subject\ to\begin{cases}\sum_{j=1}^{n}\hat{r}_j x_j+\frac{1}{2(T-1)}\sum_{t=1}^{T}\left\{\sum_{j=1}^{n}\left(r_{jt}-\hat{r}_j\right)x_j\right\}^2\geq ln\rho\\ \sum_{j=1}^{n}x_j=1\\ x_j\geq 0 \qquad j=1,\dots,n\end{cases}$$

Let us introduce T auxiliary variables

$$z_t=\sum_{j=1}^{n}\left(r_{jt}-\hat{r}_j\right)x_j \quad t=1,\dots,T.$$

We obtain then an alternative representation of the program (1) given by:

$$Min\ \frac{1}{T-1}\sum_{t=1}^{T}z_t^2$$

$$subject\ to\begin{cases}\sum_{j=1}^{n}\hat{r}_j x_j+\frac{1}{2(T-1)}\sum_{t=1}^{T}z_t^2\geq ln\rho\\ z_t-\sum_{j=1}^{n}\left(r_{jt}-\hat{r}_j\right)x_j=0 \qquad t=1,\dots,T\\ \sum_{j=1}^{n}x_j=1\\ x_j\geq 0 \qquad j=1,\dots,n\end{cases}$$

Therefore we can apply standard piecewise linear approximation of a separable convex function, and a separable convex constraint in order to solve a linear version of a non linear convex program given by :

$$Min \frac{1}{T-1} \sum_{t=1}^{T} \sum_{k=1}^{K} \lambda_{tk} a_k^2$$

$$\text{subject to} \begin{cases} \sum_{j=1}^{n} \hat{r}_j x_j + \frac{1}{2(T-1)} \sum_{t=1}^{T} \sum_{k=1}^{K} \lambda_{tk} a_k^2 \geq \ln \rho \\ \sum_{k=1}^{K} \lambda_{tk} a_k - \sum_{j=1}^{n} \left(r_{jt} - \hat{r}_j \right) x_j = 0 & t = 1,\dots\dots,T \\ \sum_{j=1}^{n} x_j = 1 \\ \sum_{k=1}^{K} \lambda_{tk} = 1 & t = 1,\dots\dots,T \\ x_j \geq 0 & j = 1,\dots\dots,n \\ \lambda_{tk} \geq 0 & t = 1,\dots\dots,T \ \ and\ \ k = 1,\dots\dots,K \end{cases}$$

This is a linear program which consists on $n+TK$ variables and $2T+2$ constraints. Obviously, it suffices to choose as extremities of abscissas of the K breakpoints introduced the previous expressions given by c and d defined in the second section, this is because the investor selects its optimum portfolios using the mean-variance criterion.

5. Concluding remarks

The purpose of this paper has been to show how the separable convex programming techniques can be used as a practical tool to solve a large scale portfolio optimization problems using asymmetric risk functions. The problem we need to solve is a linear program instead a quadratic program, so that we can handle a large scale portfolio consisting of a large number of assets, in a practical amount of time. In addition, we discuss our model in the particular case where the expected rate of return of portfolio is lognormally distributed. It was shown in this case, that the selection of optimal portfolios is independent of the investor's choice of the parameters in the asymmetric risk function.

In order to show the usefulness of our approach, we are planning an experiment based on the historical data of 158 stocks of the Belgium Stock Market, using the « OSL » (Optimization Subroutine Library) package installed on IBM RS/ 6000 system. The results of these experiments will be reported subsequently.

References

Albrecht. P. (1993), *Normal and Lognormal Shortfall risk*, 3rd AFIR, Rome, Italy.

Chvàtal. V. (1983), *Linear Programming*, Freeman and Co.

Elton. E.J. and Gruber. M.J. (1974), *Portfolio theory when Investment relatives are Lognormally distributed*, J. Finance, 29, p.1265-1273.

Elton. E.J. and Gruber. M.J. (1991), *Modern Portfolio Theory and Investment Analysis.* John Wiley and Sons, NewYork

Hamza. F. and Janssen. J. (1995), *Portfolio Optimization Model Using Asymmetric risk Functions.* 5th AFIR, Brussels, Belgium.

King A.J. and Jensen D.L. (1992), *Linear-Quadratic Efficient Frontiers for Portfolio Optimization Model.* Applied Stochastic Models and Data Analysis, 8, p. 195-207.

King A.J. (1993), *Asymmetric Risk Measures and Tracking Models for Portfolio Optimization Under Uncertainty.* Annals of Operations research, 45, p. 165-178.

Konno. H. and Yamazaki. H. (1991), *A Mean Absolute Deviation Portfolio Optimization Model and its Applications to Tokyo Stock Market.* Management Science, 37, p. 519-531.

Konno. H. and Suzuki. K. (1992), *A fast Algorithm for Large Scale Mean-Variance Models by Compact Factorization of Covariance Matrices.* J. Op.Res.Soc.Jap.,35, p. 93-104.

Konno. H. Pliska. S.R. and Suzuki K.I. (1993), *Optimal portfolios with asymptotic criteria.* Annals of Operations research.,45, p. 187-204.

Kunio. S. and Edwin L.C. (1988), *Lognormal Distributions: Theory and Applications.* Marcel Dekker (eds), INC, NewYork and Basel.

Johnson. N.L. and Kotz. S. (1970), *Distributions in Statistics: Continuous univariate distributions -1-.* Houghton Mifflin Company, Boston.

Levy. H. and Samuelson. P.A. (1992), *The Capital Asset Pricing Model with Diverse Holding Periods.* Management Science, 38, p. 1592-1542.

Lintner. J. (1980), *The Lognormality of Security Returns. Portfolio Selection and Market Equilibrium.* Working Paper, Graduate School of Business Administration, Harvard University.

Luenberger. D.G. (1984), *Linear and Nonlinear Programming.* Second edition, Addison- wesley.

Markowitz. H. (1991), *Foundations of Portfolio Optimizations.* J. Finance., 2, p. 469-477.

Markowitz. H., Todd. P., Xu. G., and Yamane. Y. (1993), *Computation of mean-semi-variance efficient sets by the Critical Line Algorithm.* Ann. Op. Res., 45, p. 307-318.

Merton. R.C. and Samuelson. P.A. (1974), *Fallacy of Lognormal Approximation to Optimal Decision Making Over Many Periods.* J. Financial Economics, 1, p. 67-95.

Ohlsen. J.A. and Ziemba. W.T. (1976), *Portfolio Selections in a Lognormal Market when Investor has a Power Utility Function.* J. Financial and Quant. Anal., 7, p. 57-71.

Speranza M.G. (1993), *Linear Programming Models for Portfolio Optimization.* Finance, 14, p. 107-123.

Zenois. S.A. and Ziemba. W.T. (1992), *Introduction to the Focused issue on Financial Modeling.* Management Science. Vol 38, N°11, p. 1525-1528.

Zenois. S.A. (1993), *Financial Optimization.* Cambridge University Press.