Optioned Portfolios: The Trade-off between Expected and Guaranteed Returns

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Abstract

Many investors have come to realize that they have to distinguish between downside risk and upward potential whenever a trade-off is made between risk and expected return. The use of optioned portfolios provides a natural way to reflect the trade-off between downside risk and expected return in portfolio decisions. In this paper, we show to what extent expected return has to be sacrified in return for the guarantee that the worst case return will exceed a prespecified level. Using option theory, such relationships can be derived analytically. In addition, we show how these relationships can be obtained in practice by composing efficiently optioned portfolios, i.e. portfolios that provide the highest expected return, given a required guaranteed return. These portfolios and their return characteristics are based on actual market prices of the European Options Exchange.

Résumé

Beaucoup d´investisseurs se réalisent de plus en plus qu´ils doivent faire la distinction entre le risque de déficit et le potentiel en haut quand ils doivent mettre en balance "risque" et "rendement espéré". Il parait donc naturel de se servir de portefeuilles, constitués en partie d´options, lorsqu´on pèse le risque de déficit contre le rendement espéré. Dans ce papier, nous montrons jusqu´a quel point il faut sacrifier le rendement pour avoir la garantie que le rendement le plûs mauvais sera supérieur à un niveau spécifié d´avance. Une dérivation théorique des relations entre le risque de déficit et le rendement espéré, basée sur la theorie des options, est presentée. Nous montrons également qu´on peut obtenir ces relations en pratique en composant des portefeuilles d´options efficients, c.a.d. des portefeuilles qui, donné une garantie de rendement minimal, rapportent le rendement espéré le plus élevée. Nous nous servons des prix réalisés de European Options Exchange pour composer ces portefeuilles.

Keywords
Downside risk, portfolio insurance, optioned portfolios.

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1 Introduction

Many investors, private as well as institutional, are looking for investment opportunities that offer a high expected return and limited risk. This brings us to the trade-off between risk and return, which we shall analyze in this paper.

Both in practice and in the financial literature, this trade-off has often been analyzed in a mean-variance framework (e.g. Markowitz (1952)). In this approach, risk is quantified as the standard deviation or the variance of the return on the portfolio about its mean. Although standard deviation is a widely used risk measure, there have been many publications which point out the limitations of the standard deviation as a measure of risk, among them Markowitz (1959), Haigigi and Kluger (1987) and Sortino and Van Der Meer (1991). One of the main drawbacks of standard deviation as a measure of risk is that it does not distinguish between returns higher than expected and returns lower than expected.

However, even when one uses a proper risk measure, opportunities to compose investment portfolios that offer upside potential and limited downside risk are limited if one is restricted to invest in traditional asset classes such as stocks, bonds, cash and property. Using options, however, one is able to obtain a protection against (large) losses whilst maintaining sufficient exposure to profit from favourable market conditions. Securities brokers and asset managers are now offering investment products which offer participation in the upside swings of prices of risky assets, but also offer downside protection in the form of a guaranteed minimum level of return.

In the remainder of this paper, the trade-off between guaranteed minimum levels of return and expected return is investigated. Section 2 contains a theoretical analysis, which results in optimality conditions for some simple optioned portfolio structures. Using a scenario optimization model, real world optimal portfolios have been determined, based on market conditions on the European Options Exchange. In order to be able to compare the theoretical and empirical results, the scenario optimizer has been constrained to select portfolios of the structure that has also been analyzed in section 2.

2 Deriving feasible optioned portfolios: theory

In this section, we will relate the level of a required minimum guarantee on an optioned portfolio over a fixed time interval $[0, \tau]$ to the expected return on this portfolio. For reasons of simplicity, we assume to deal with an investor who invests all his initial capital in a stock index. In order to guarantee a given required return on his portfolio, he decides to insure his portfolio by buying put options on the index, expiring at a date $T \geq \tau$. No further transactions in either the index or the options take place until the horizon date τ, once the portfolio insurance is acquired. The index options used here are of the European type, so there are no early exercise opportunities. The option premium is assumed to be financed by borrowing at the riskless rate r. The value of the underlying index will be denoted by $S(t)$, $0 \leq t \leq T$, which equals value of the index at time t, where all dividends paid out during the planning period $[0, \tau]$ are excluded. The present value of dividends paid out until the investment horizon of the option is denoted by $D(\tau)$. We assume $D(\tau)$ to be known in advance. Denoting the option premium at time 0 for a put with strike price K and expiration date T by $P_0(S(0), K, T)$, and the amount of this option by $x(K, T)$, we can define the final value V_τ of the optioned portfolio:

$$V_\tau = S(\tau) + D(\tau) + x(K,T)[P_\tau(S(\tau), K, T - \tau) - e^{r\tau} P_0(1, K, T)],$$

where it is assumed that the initial capital equals one unit of the index, which has an initial value $S(0) = 1$. The minimum portfolio pay-off requirements are represented by a fraction $\theta \leq e^{r\tau}$ of the initial investment that should be guaranteed at time τ. We therefore have to select an exercise price K, a time to maturity T and a hedge ratio $x(K, T)$ for the put option such that

$$S(\tau) + D(\tau) + x(K,T)[P_\tau(S(\tau), K, T - \tau) - e^{r\tau} P_0(1, K, T)] \geq \theta,$$

for all possible realisations of the index value at the horizon $S(\tau)$.

2.1 Selecting a hedge ratio

First, let us focus on the problem of determining the hedge ratio which results in the highest guaranteed pay-off. For an option series with a given strike price and expiration date, this problem can be reduced to solving the following optimization problem:

$$\max_{x\geq 0}\left\{\min_{S\geq 0}\{S + x[P_\tau(S,K,T-\tau) - e^{r\tau}P_0(1,K,T)]\}\right\}, \tag{1}$$

where the arguments of x and S are omitted for reasons of clarity. The inner minimization problem, which determines the minimum pay-off given a hedge ratio x, becomes trivial whenever the hedge ratio x is smaller than one; in this case, the pay-off at the horizon is an nondecreasing function of the index value S, and, consequently, the minimum pay-off is attained when the index becomes worthless, i.e. when $S = 0$. In order to determine the minimum pay-off for hedge ratios larger than one, let us take the derivative of V_τ with respect to the index value at the horizon date, S:

$$\frac{dV_\tau}{dS} = 1 + x\Delta_\tau(S,K,T-\tau) = 0,$$

where Δ_τ equals the derivative of the price P_τ of the option at the horizon with respect to the value S of the index. It coincides with the ordinary Δ in options theory. Here, we restrict ourselves to the case $T > \tau$, because of the differentiability of P_τ, the case $T = \tau$ will be treated separately. Solving this equation results in the following expression for S^*, the index value corresponding to the minimum value of the total portfolio given a hedge ratio $x > 1$:

$$S^* = \Delta_\tau^{-1}\left(-\frac{1}{x}, K, T-\tau\right).$$

Using this solution to the inner minimization problem of (1), we can subsequently derive the *maximum guaranteed pay-off* hedge ratio by solving the following maximization problem:

$$\max_{x\geq 1}\left\{\Delta_\tau^{-1}\left(-\frac{1}{x}\right) + x\left[P_\tau\left(\Delta_\tau^{-1}\left(-\frac{1}{x}\right), K, T-\tau\right) - e^{r\tau}P_0(1,K,T)\right]\right\}.$$

We restrict ourselves to the interval $x \in [1, \infty)$, since the guaranteed pay-off can be shown to be a nondecreasing function of the hedge ratio x on $[0, 1]$. The first and second order conditions provide the following equation that needs to be satisfied by a maximum:

$$P_\tau \left(\Delta_\tau^{-1} \left(-\frac{1}{x} \right), K, T \right) = e^{r\tau} P_0(1, K, T).$$

Defining $\gamma \equiv e^{r\tau} P_0(1, K, T)$, this condition can be rewritten as

$$x^* = -\frac{1}{\Delta_\tau(P_\tau^{-1}(\gamma))}.$$

By differentiating the identity $P_\tau(P_\tau^{-1}(\gamma)) \equiv \gamma$, this can be simplified to

$$x^* = -\frac{\partial P^{-1}}{\partial \gamma}(\gamma).$$

It can easily be verified that $x^* > 1$, so the highest possible guaranteed return will be attained at hedge ratios larger than one. The implication is that these strategies do not result in pay-offs that increase monotonically in the index value. These strategies are inefficient, if one seeks to maximize expected return and assuming dynamic strategies as in Dybvig (1988) are admissable.

Optimal hedge ratios if the expiration date equals the horizon date

For the case $T = \tau$, i.e. when the time to maturity equals the horizon, these hedge ratios can be found explicitly by solving the following optimization problem:

$$\max_{x(K,\tau) \geq 0} \left\{ \min_{S(\tau) \geq 0} \{ S(\tau) + x(K, \tau)[\max(K - S(\tau), 0) - e^{r\tau} P_0(1, K, \tau)] \} \right\}. \tag{2}$$

It is easy to show that in this case the maximum guaranteed pay-off is attained at $x^*(K, \tau) = 1$. The resulting strategies will be monotonically increasing in the value of the index at the horizon date τ. If options with all possible exercise prices are available, every level of required pay-off θ, $0 \leq \theta < e^{r\tau}$, becomes attainable if we fix the desired hedge ratio to $x(K, \tau) = 1$. It follows directly from (2) that the expected pay-off of strategies with these unitary hedge ratios, $m(K)$, is given by

$$m(K) = e^{\mu\tau} + D(\tau) + E[\max(K - S(\tau), 0)] - e^{r\tau} P_0(1, K, \tau),$$

where μ equals the annualized expected return on the index. Under standard assumptions, for instance those that are used in the derivation of Black-Scholes option prices (Black and Scholes (1973)), and $\mu > r$, the function m can be shown to be monotonically decreasing in K, which implies that selecting higher exercise prices leads to lower expected pay-offs. Under the same assumptions, we can also show that the pay-off of the option is negative. If hedge ratios smaller than or equal to 1 are chosen, the guaranteed pay-off level increases when the exercise price is raised (this can be verified by taking the derivative with respect to K of the guaranteed pay-off level $K + D(\tau) - e^{r\tau}P_0(1, K, \tau)$). As a consequence, requiring a higher minimum pay-off of your portfolio will automatically lead to a decline of the expected return of the total portfolio. This relationship will be shown graphically in section 3.2 (based on both theory and real data), both in the case that the portfolio is constrained to contain only one option series and the case that several options may be used.

The hedge ratios derived in this section shouldn't be confused with *optimal* hedge ratios. They only indicate what guarantees are possible, once the exercise price and maturity of the option is fixed. Since the expected return on the total portfolio decreases with the fraction hedged by put options in the portfolio, hedge ratios larger than the maximum guaranteed return hedge ratios x^* are clearly inefficient in our framework: they result in portfolios with both lower guaranteed returns and lower expected returns.

2.2 Selecting optimal exercise prices

Sofar, we have analyzed the guaranteed pay-off of hedging strategies as a function of the hedge ratio, given values for the expiration dates and the exercise prices of the put option. Now, suppose we can compose portfolios of several puts options, all maturing at the same horizon date τ, but with different strike prices K_i, $K_1 < K_2 < \ldots < K_n$, where n denotes the number of options available. As our objective is to maximize the expected pay-off of the portfolio, given a minimum pay-off level of our portfolio, selecting the optimal options strategy is equivalent to solving the following semi-infinite dimensional optimization problem:

$$\begin{aligned}
\max_{\mathbf{x}} \quad & \sum_{i=1}^{n}\{E[\max(K_i - S(\tau),0)] - e^{r\tau}P_0(1,K_i,\tau)\}\ x_i \\
\text{s.t.} \quad & S(\tau) + D(\tau) + \\
& \sum_{i=1}^{n}\{\max(K_i - S(\tau),0) - e^{r\tau}P_0(1,K_i,\tau)\}\ x_i \geq \theta, \quad \forall\ S(\tau) \geq 0 \qquad (3) \\
& x_i \geq 0, \quad i = 1,2,\ldots,n
\end{aligned}$$

The following result shows that the monotonocity property, which was discussed for the single option series case, still holds in this case, i.e. that the pay-offs of any optimal portfolio is an increasing function of the value of the underlying index.

Lemma 2.1 *If* $\mathbf{x}^*$ *is an optimal solution to (3), then the pay-off function of* $\mathbf{x}^*$ *at the investment horizon* τ *is a monotonically increasing function of* $S(\tau)$.

PROOF: First, we will show that a portfolio $\mathbf{x}$ results in a monotonically increasing pay-off function in $S(\tau)$ if and only if $\sum_{i=1}^{n} x_i \leq 1$. We therefor take the derivative of the pay-off at the horizon with respect to $S(\tau)$. This derivative equals (note that the pay-off isn't differentiable in $S(\tau) = K_i, i = 1,2,\ldots,n$):

$$1 - \sum_{i:K_i > S(\tau)}^{n} x_i.$$

Requiring monotonicity implies a nonnegative derivative for all $S(\tau) \geq 0$. This results in the requirement $\sum_{i=1}^{n} x_i \leq 1$.

Now, suppose that $\mathbf{x}^*$ is an optimal solution with $\sum_{i=1}^{n} x_i^* > 1$. In this case, the derivative has to change sign at $S(\tau) = K_j$, where j is defined by

$$\begin{aligned}
\sum_{i=j}^{n} x_i^* &> 1 \qquad (4) \\
\sum_{i=j+1}^{n} x_i^* &\leq 1
\end{aligned}$$

This implies that the pay-off function attains a local minimum at $S(\tau) = K_j$. It is easy to verify that the minimum must also be the global minimum of the pay-off function. So the minimum pay-off equals

$$K_j + \sum_{i=j+1}^{n} (K_i - K_j)x_i^* - \sum_{i=1}^{n} e^{r\tau}P_0(1,K_i,\tau)x_i^*. \qquad (5)$$

Now since $x_j^* > 0$ (this follows directly from the definition of j), it is possible to decrease x_j^* by a value $\epsilon > 0$ such that the inequality (4) still holds. So the minimum pay-off is still attained at $S(\tau) = K_j$. It is easy to check feasibility by looking at (5): reducing x_j^* increases also the minimum pay-off by an amount $e^{r\tau}P_0(1, K_j, \tau)\epsilon$. Since the coefficients in the objective function of problem (3) are all negative, this new feasible solution results in a larger objective function value. Hence the solution $\mathbf{x}^*$ isn't optimal, which proves the lemma. □

Selecting optimal option series

As a consequence of lemma 2.1, we know in advance that the worst outcome possible in all portfolios of interest corresponds to the index becoming worthless, i.e. $S(\tau) = 0$. This enables us to simplify the option selection problem to:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \sum_{i=1}^{n} w_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} h_i x_i \geq \theta && (6) \\ & \sum_{i=1}^{n} x_i \leq 1 && (7) \\ & x_i \geq 0, \quad i = 1, 2, \ldots, n \end{aligned}$$

where w_i and h_i denote the expected pay-off (but with a reversed sign) and guaranteed pay-off of the unitary hedge ratio of the i-th option, respectively. So

$$\begin{aligned} w_i &= e^{r\tau}P_0(1, K_i, \tau) - E[\max(K_i - S(\tau), 0)], \quad i = 1, 2, \ldots, n \\ h_i &= K_i + D(\tau) - e^{r\tau}P_0(1, K_i, \tau), \quad i = 1, 2, \ldots, n \end{aligned}$$

Furthermore, we assume the following to hold:

$$w_i < w_{i+1}, i = 1, 2, \ldots, n-1, \tag{8}$$

$$h_i < h_{i+1}, i = 1, 2, \ldots, n-1, \tag{9}$$

$$\frac{w_i}{h_i} < \frac{w_{i+1}}{h_{i+1}}, i = 1, 2, \ldots, n-1. \tag{10}$$

Relation (8) states that the expected pay-offs of the options increase when the exercise price is raised. In (9), guaranteed pay-offs are assumed to increase with the exercise price, and, finally, (10) provides us with a relationship between the ratio of expected pay-off and guaranteed pay-off of the options: these ratios do also increase with the exercise price. Although we have not been able yet to prove that these relationships always hold, we conjecture that they do hold under the assumptions often used in option theory. Note that the values of the $w_i, i = 1, 2, \ldots, n$, depend on the expected return and volatility of the index, which can both be controlled by the user of the model. In the empirical example of the next section, where we use actual option prices, implied volatilities and estimates for expected returns based on historical averages, all three requirements (8)-(10) are satisfied.

Now, we only have to consider guaranteed pay-off levels satisfying $\theta \in [h_1, h_n]$: if the option with the lowest exercise price ($i = 1$) is sufficient to guarantee θ, i.e. $\theta \leq h_1$, then only this option will be used (because of the reduncancy of constraint (7) and assumption (10)). Levels of θ larger than h_n clearly lead to infeasibility of the minimization problem above. For values of θ within the interval $[h_1, h_n]$, we shall show the following to hold:

Lemma 2.2 *Given $\theta \in [h_1, h_n]$, constraints (6) and (7) are binding if the solution $\mathbf{x}^*(\theta)$ is optimal. Moreover, there is an optimal solution where at most two options series are used, for every $\theta \in [h_1, h_n]$.*

PROOF: It is obvious that constraint (6) is binding whenever the weigths in the objective function, w_i, are positive. Now let us assume we have an optimal solution $\mathbf{x}^*(\theta)$ (where an argument θ is added to indicate the dependence on the guaranteed pay-off level θ with (7) non-binding: $\sum_{i=1}^{n} x_i^*(\theta) < 1$. Now define $j = \max\{i | x_i^*(\theta) > 0\}$, i.e. the option series with the highest exercise price that is used in the optimal solution. The assumption $\theta \geq h_1$, implies $j > 1$. Now construct an alternative solution $\mathbf{y}$ as follows:

$$\begin{aligned} y_1(\theta) &= x_1^*(\theta) + \frac{h_j}{h_1}\epsilon, \\ y_j(\theta) &= x_j^*(\theta) - \epsilon, \\ y_i(\theta) &= x_i^*(\theta), \quad \forall i > 1, i \neq j. \end{aligned}$$

Since $\epsilon > 0$ can be chosen arbitrarily small, we can construct such a solution, whilst preserving feasibility in (7). Feasibility with respect to (6) is preserved by the construction of the solution $\mathbf{y}$. The change in the objective function, z, equals:

$$\begin{aligned}\Delta z &= \frac{h_j}{h_1} w_1 \epsilon - w_j \epsilon \\ &= \frac{\epsilon}{h_j}\left(\frac{w_1}{h_1} - \frac{w_j}{h_j}\right) < 0.\end{aligned}$$

Hence, $\mathbf{x}^*(\theta)$ is not the optimal solution.

The number of options used in the optimal solution follows directly from the number of constraints in the LP-problem and the nondegeneracy of the problem. □

Now that we have established some results concerning the structure of the optimal solution of the option selection problem, the crucial question remains: which options will be chosen, given a guaranteed pay-off level of θ? The following condition tells us whether a particular option will be used or not in optimal solutions of the problem.

Lemma 2.3 *An option j, $1 < j < n$, will not appear in optimal solutions $\mathbf{x}^*(\theta)$, $h_1 \leq \theta \leq h_n$, whenever there exist a pair of indices (i, k), $1 \leq i < j < k \leq n$ such that:*

$$(h_k - h_i)w_j > (h_k - h_j)w_i + (h_j - h_i)w_k. \tag{11}$$

PROOF: First, we will show that (11) provides a necessary condition: If x_j doesn't appear in the optimal solution $\mathbf{x}^*(\theta)$, $h_1 \leq \theta \leq h_n$, it cannot be optimal for the guaranteed pay-off level $\theta = h_j$. In this case, the solution $x_j = 1$, with objective function value w_j, is dominated by a solution (y_i, y_k), with:

$$y_i h_i + y_k h_k = h_j, \tag{12}$$

$$y_i + y_k = 1, \tag{13}$$

$$y_i w_i + y_k w_k < w_j. \tag{14}$$

Notice that conditions (12) and (13) follow directly from lemma 2.2: in optimal solutions, both constraints are binding and at most two option series are used. Substituting (12) and (13) into (14) gives us the desired result. Condition (11) is also sufficient: a solution with $x_j > \alpha$, $0 < \alpha \leq 1$, is always be dominated by $(\alpha y_i, \alpha y_k)$. Hence, solutions with $x_j > 0$ cannot be optimal. □

3 Determining optioned portfolios with real data

In the previous section, we have used option theory in order to specify theoretical relationships between guaranteed levels of return and expected returns of portfolios consisting of one unit of a stock index, a short position in cash and a long position in put options. In this section, we shall present the relationship between guaranteed return and expected return of optimal portfolios, based on market data of the European Options Exchange. The optimal portfolios have been obtained by means of a scenario optimization model. Scenario optimization techniques are suitable to solving more general option selection problems than the ones we will treat here; short positions in options and varying expiration dates can easily be incorporated in this approach, in contrast to the direct optimization approach used in section 2.2. The stock index that we consider is the Dutch AEX index. We have determined portfolios that maximize expected return, subject to the requirement that the return at the horizon may never be worse than a prespecified minimum level of return. In order to ensure that the experimental results can be compared with the theoretical results of the previous section, we have imposed constraints on the composition of the portfolios that correspond with the assumptions that have been made to facilitate theoretical analysis. These assumptions include:

- no transaction costs,
- always one unit of the stock index in the portfolio,
- no long position in cash,
- options can be traded at the average of bid and ask price,
- once the portfolio has been bought, there are no mutations until the investment horizon,

- The return distribution of the AEX index is lognormal; expected return and future dividends are based on historical averages, the standard deviation is based on the implied volatility of the options.

3.1 Selection of optimal strike price and hedge ratio

Table 1:

Market data

expected return underlying index (%)	13.1
annual return on 3 month deposit (%)	3.20
volatility underlying value (%)	15.0
expected annual dividends	10.0
starting value AEX index (10/4/96)	543
time to maturity (days)	100

Table 2:

Option characteristics

	1	2	3	4	5	6
K	500	510	520	530	540	550
p_0	2.65	4.10	6.25	9.25	14.00	19.50
w	2.67	4.13	6.30	8.76	12.25	15.13
h	497.3	505.9	513.7	520.7	525.9	530.3
$\frac{w}{h}$	.0054	.0081	.0122	.0168	.0223	.0285

The computational results, based on the data presented in tables 1 and 2, are represented in table 3. They have been obtained by maximizing the expected return on the portfolio, subject to the constraints specified earlier. In addition to these constraints, the choice of portfolios has been restricted to portfolios which hold a position in at most one option series, in order to compare the results with the theory of section 2.1, which discussed the optimal hedge ratio in combination with selection of the optimal strike price.

Table 3:

Optimal solutions: one series allowed

θ	exp. ret.	x_1	x_2	x_3	x_4	x_5	x_6
488.7	10.38	.9764					
491.3	10.37	.9818					
493.9	10.36	.9872					
496.5	10.35	.9926					
499.1	10.34	.9980					
501.7	9.64		.9859				
504.3	9.63		.9912				
506.9	9.62		.9965				
509.5	8.74			.9861			
512.1	8.72			.9913			
514.7	8.70			.9965			
517.3	7.71				.9879		
519.0	7.69				.9930		
522.5	7.67				.9982		
525.1	6.15					.9931	
527.7	6.12					.9982	
530.3	4.91						.9946
532.9	4.87						.9997

Given the constraints on portfolio composition, one would expect optimal portfolios to include a long position in put options which is precisely sufficient to offer the required guaranteed return. Moreover, in view of the maximization of expected return, one would expect that precisely those options are selected that offer the highest ratio of contribution to the expected return over 'dollars insured' (the ratio $\frac{w_i}{h_i}$ in table 2). As an illustration: in order to insure 1 dollar of the portfolio by means of the put with exercise price $K = 530$ (x_4), the investor has to sacrifice 0.0168 dollar of expected terminal portfolio value. Notice that these figures only hold under the assumption that it is possible to obtain the desired guarantee by holding the prevailing option in the first place.

This is precisely what is reflected by table 3: as can easily be verified, the value of the investment horizon of a portfolio consisting of .9765 units of the $K = 500$ option, -2.5878 units of cash and 1 unit of the index, is always greater than or equal to:

$$.9765 * 500 + 10 * \frac{100}{365} - 2.5878 * e^{.032*(\frac{100}{365})} = 488.1,$$

which is equal to the minimum required pay-off level (the second factor equals the dividend payments). When the required guarantee is increased, the amount invested in the $K = 500$ put option increases, just enough to guarantee the increased level of insurance. For instance, a minimally required pay-off level of 499.1 requires a holding of .9981 units of the cheapest option:

$$.9981 * 500 + 10 * \frac{100}{365} - 2.6451 * e^{.032*(\frac{100}{365})} = 499.7.$$

Increasing the guaranteed level to 501.7 implies that holdings in x_1 cannot suffice any more to obtain the desired insurance; the maximum guarantee, which would be obtained at a hedge ratio equal to 1, would give:

$$1 * 500 + 10 * \frac{100}{365} - 2.65 * exp^{.032*(\frac{100}{365})} = 500,$$

which coincidentally equals the exercise price of the put options that is used. So to insure a level of 501.7, we need an option with a higher strike price. A portfolio with .9861 units of the $K = 510$ put, the index and the corresponding cash position to finance the put option, is sufficient in this case.

This pattern of portfolios is repeated as the level of guarantee is increased. Every time that the required level of guarantee cannot be sustained anymore by holding the option that was optimal for the previous level of guarantee, the next best option is selected at the lowest hedge ratio that suffices to offer the desired pay-off. These practical results correspond rather well with the theoretical analysis that has been presented in section 2:

- the hedge ratios are always between 0 and 1,
- the pattern of optimal holdings, as a function of the required guarantee seems to justify the conjecture that hedge ratios would always be equal to 1, if options with the appropriate strike prices had been available.

3.2 Selection of optimal strike price and hedge ratio when several series are allowed

As compared to the setting in section 3.1, the constraint that only one series may be included in the portfolio has been relaxed. The portfolios for levels of guarantee from 488.7 to 499.1 are identical to those in table 3. For higher levels of guarantee, however, they do differ: in this setting it is not necessary to exclude holdings of an option which is in itself insufficient to safeguard the desired level of guarantee.
As an example, consider the optimal portfolio when a guarantee of 501.7 is required. Analogous to the examples in section 3.1, one can verify that the additional holding in the $K = 510$ option is precisely sufficient to guarantee the remaining portfolio value:

$$
\begin{aligned}
.8109 * 500 + 10 * \frac{100}{365} - .8109 * 2.65 * e^{.032*(\frac{100}{365})} &= 406.0 \\
.1891 * 520 - .1891 * 4.10 * e^{.032*(\frac{100}{365})} &= 95.7
\end{aligned}
$$

which results in a guaranteed level of $406.0 + 95.7 = 501.7$, precisely the required value. Notice that this solution comes with an expected return on the portfolio equal to 10.20%, whereas the solution in the table resulted in an expected return of 9.64%. The expected return is plotted as a function of guaranteed return in figure 1. It clearly shows the advantage of the multiple options case over the single options case. Maybe more interesting is the result for a guaranteed pay-off level equal to 525.1: the solution does not, as one might expect, consist of the $K = 530$

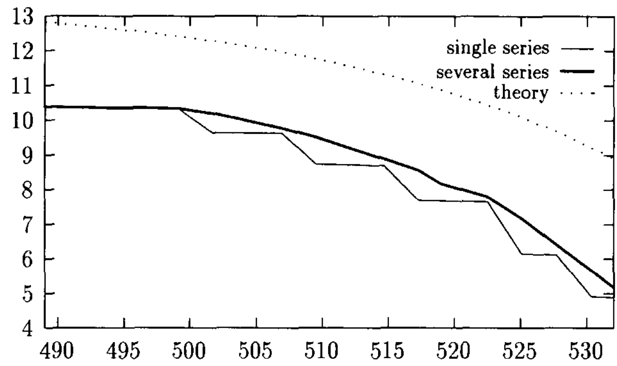

Figure 1: Trading-off guaranteed and expected returns

option in combination with the $K = 540$ one. Instead, the optimal portfolio contains $K = 530$ in combination with $K = 550$ options. When positions in several series may be held, one cannot obtain the optimal portfolio by simply selecting the option with the (next) best cost to insurance ratio. Instead, one has to consider combinations of options, as derived in section 2.2, expression (9). If we verify the inequality, which ensures exclusion of a particular option from all possible optimal portfolios, it turns out to be satisfied if we compare x_4 and x_6 with x_5.

4 Concluding Remarks

In this paper, the relationship between the level of expected return and the worst case level of return has been analyzed for portfolios with simple structures. The theoretical part of this paper states necessary and sufficient conditions for the composition of portfolios in order to be efficient in the sense that the portfolio offers the highest possible expected return given the desired level of guaranteed return.

In the second part of the paper, the theoretical results are corroborated by the results from an experiment in which optimal portfolios have been determined using real market data from the European Option Exchange. Although we have chosen to present results which, at this stage, can only be shown to hold for a relatively small class of optioned portfolios, we tend to believe that these analyses do provide useful insights in the amount to which expected return has to be sacrificed in return for reduction of downside risk.

We hope that this contribution will stimulate further research along these lines which, in our opinion, has a great potential to contribute to the theory and the practice of exploiting derivatives in asset management.

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Table 4:

Optimal solutions: several series are allowed

θ	exp. ret.	x_1	x_2	x_3	x_4	x_5	x_6
488.7	10.38	.9764					
491.3	10.37	.9818					
493.9	10.36	.9872					
496.5	10.35	.9926					
499.1	10.34	.9980					
501.7	10.20	.8108	.1891				
504.3	9.98	.5062	.4938				
506.9	9.76	.2016	.7984				
509.5	9.51		.8877	.1123			
512.1	9.20		.5556	.4444			
514.7	8.90		.2235	.7765			
517.3	8.57			.8780	.1220		
519.0	8.18			.5051	.4949		
522.5	7.80			.1321	.8679		
525.1	7.18				.8262		.1738
527.7	6.42				.5569		.4431
530.3	5.67				.2877		.7123
532.9	4.92				.0184		.9816