# Optimal Portfolio and Optimal Trading in a Dynamic Continuous Time Framework

Pierre Brugière[1]

Abstract

One of the problems encountered by traders and fund managers is to control their Profit & Loss smoothness and volatility. In fact, traders who mark to market their positions, or fund managers who want to reassure their clients are interested in exhibiting a very smooth ascending (preferably) P&L, without any large swings. Mathematically, the regularity or smoothness of the P&L, within a given period of time, can be measured by its total quadratic varation. The quadratic variation also measures the part of randomness in the trading strategy results, and is a measure of risk. We solve in this paper the problem of maximising, for a given time horizon, the expected return of a trading strategy, with a constraint on the quadratic variation of the P&L. We introduce in our demonstration the notions of "instantaneous mean-variance optimal portfolios" and "instantaneous Sharpe Ratios", "optimal mean-quadratic variation trading strategies" and "average Sharpe Ratios".

Keywords

Diffusion, quadratic-variation optimisation, Markowitz, stochastic control.

University of Paris 9 Dauphine, Place du Maréchal de Lattre de Tassigny, 75116 Paris & Compagnie Parisienne de Réescompte, 30 rue Saint Georges, 75312 Paris Cedex 09 (France); Tel: + 33-44-05 46 70 & + 33-45-96 27 91, E-mail: brugiere@ceremade.dauphine.fr.

[1] The author wishes to thank Compagnie Parisienne de Réescompte, who supported a part of this work, and particularly J. Bocquet and E. Marcombes, from the Equities and Fixed Income desks, for stimulating discussions.

## 1. Introduction

In the Markowitz discrete time single period model, the return on investment is optimised, with a mean-variance criteria. In this model, the Sharpe ratio (expected return on standard deviation) enables us to compare the performances of different portfolios. This ratio is maximal for the optimal risky portfolio. The problem of minimising the variance of the final wealth, in continuous time trading, with an appropriate continuous time strategy, has been studied by Föllmer-Schweizer (1991), Duffie Richardson (1991), Föllmer (1992), in an incomplete market framework. The problem of minimising the variance of a portfolio at a given time horizon and for a given expected return, has been studied by Bayeux-Portait (1993). Nevertheless, the mean-variance criteria, in continuous time trading, can sometimes appear to be too simple, and then utility functions at maturity can be introduced, as the probability distributions of the achievable returns could be of very different natures, and could even be orthogonals. Our point of view here is different, as we are interested in the regularity of the evolution of the value of the portfolio and as we choose the quadratic variation as the natural measure of the P&L smoothness and risk. We then maximise the expected earnings of a trading strategy, for a finite time horizon, with a constraint on the P&L total quadratic variation.

To be able to compare two different strategies, in a dynamic framework, even for two different periods of time, the quadratic variation can be divided by the time elapsed during the actual trading period and then square rooted. With this renormalisation this quantity can be interpreted as an average standard deviation, per unit time, over the trading period, of the trading earnings. The mathematical problem of optimisation, developed in this article, consists in maximising for a certain time horizon, the expected earnings of a trading strategy, with a constraint of boundedness on the quadratic variation of the P&L. The assets considered for these trading strategies are futures, to avoid any useless consideration of financing. These futures are expressed in one single currency, but the problem could also be handled with spot assets and different currencies. The futures are modelled by diffusion processes and their instantaneous covariance matrices are supposed to be non degenerate over time, so that none of them is redundant at any moment, and no arbitrage opportunity occurs. Note that in the model, the drift and variance coefficients of the diffusion processes are not supposed to be constant, and can even be non deterministic. When specifying more precisely, for example deterministically, the drifts and variances of the futures, explicit calculations can be made for the optimal strategy and the associated expected earnings. With mild conditions on the processes drift and covariance matrice, it can be demonstrated that the quadratic variation constraint saturates.

For different constraints on the P&L total quadratic variation, the optimal strategies differ from one to another only by a constant multiplicative factor. More precisely, if the quadratic variation constraint is modified in the optimisation program by the multiplication with a positive factor $\beta$, the positions on the futures in the new trading strategy are multiplied by a factor $\sqrt{\beta}$ for the whole period. Therefore, any optimal trading strategy with the futures is proportional to a reference strategy, which serves as a "Benchmark", the coefficient of proportionality depending on the total quadratic variation accepted. The method for optimal trading is then similar to that obtained in the discrete time Markowitz model: a "Benchmark Strategy" is determined as the optimal risky investment strategy, and the amount invested in this strategy depends on the accepted risk.

It is also possible to calculate in this optimal trading problem what I would call an "average Sharpe Ratio", which consists in the non dimensional ratio: expected earning per unit time divided by the square rooted quadratic variation per unit time. This ratio can be viewed as the continuous version of the discrete time Sharpe Ratio.

Note that, in the model developed here, holding positions on futures consume quadratic variation, and the philosophy of the optimal trading strategy appears quite intuitively: quadratic variation must be consumed intensively when the ratio instantaneous return on risk is high and moderately when the ratio is low. The problem of "optimal trading" is a problem of controling the quadratic variation consumption over time and with the best combination of instruments. More precisely, at every instant the quadratic variation must be consumed with the best achievable portfolio of futures, as in a classic two periods model written from time $t$ to time $t+dt$. Note also that, even for a single future, the problem of optimisation is not trivial and that the general problem can be viewed as a problem of optimaly trading a Markowitz instantaneous optimal portfolio.

## 2. The model

Our model includes several futures contracts and $F_s^i$ represents the price of the future contract $i$ at a time $s$.

The futures are modelled by diffusion processes, and their equations can be written as,

$$dF_s^i = \mu_s^i F_s^i ds + \sigma_s^i F_s^i dW_s^i \quad i \in [1,n]$$

where $W_s^i$ are brownian motions under a given probability $P$.

We note $\Gamma_s = \left[\rho_s^{i,j}\right]_{i,j\in[1,n]}$ the matrix of instantaneous correlations between the $W_s^i$.

We suppose $\Gamma_s$ to be strictly positive definite for all $s$.

We note $<,>_s$ the scalar product on $R^n$ relative to $\Gamma_s$, and defined by $\langle u,v\rangle_s = \sum_{i,j\in[1,n]} u_i v_j \rho_s^{i,j}$.

${}^t x$ denotes the transpose of vector $x$.

At every instant $s$, the number of future contracts $F_s^i$ in the portfolio equals $X_s^i$ and can be either positive,

negative or null. $(X_s)_{s\geq 0}$ stands for the $n$-dimensional process $(X_s^1, X_s^2, \cdots, X_s^n)_{s\geq 0}$ and represents a trading strategy.

We suppose that there is a constant instantaneous riskless interest rate $r$ and that during the trading period the gains or losses at financed at the interest rate $r$.

**2. The Optimisation Problem**

Trading starts at time 0 and $T$ is the time horizon, for the optimisation problem.
$G_t$ represents the earnings, cumulated at time $t$ and actualised at time 0, of the trading strategy $(X_s)_{s\geq 0}$.
$G_t$ can be viewed as the actualisation at date 0 of the trader's wealth at time $t$, (the trader starts with a null initial wealth).

We have,
$G_t = \sum_{i=1}^{n} \int_0^t e^{-rs} X_s^i dF_s^i$.
The expectation at time 0, of the actualised value of the portfolio at time $T$ equals, $\sum_{i=1}^{n} E\left[\int_0^T e^{-rs} X_s^i \mu_s^i F_s^i ds\right]$.
The total quadratic variation of the actualised value of the portfolio equals $\sum_{i,j} \int_0^T e^{-2rs} \sigma_s^i X_s^i F_s^i \sigma_s^j X_s^j F_s^j \rho_{i,j} ds$.
The problem of optimisation consists in maximising the expected earnings, with a constraint of boundedness on the total quadratic variation. This constraint is expressed under the form $\gamma^2 T$ for more convenience.

Mathematically we solve,

$$\begin{cases} \max_X \sum_{i=1}^{n} E\left[\int_0^T e^{-rs} X_s^i \mu_s^i F_s^i ds\right] \\ \sum_{i,j} \int_0^T e^{-2rs} \sigma_s^i X_s^i F_s^i \sigma_s^j X_s^j F_s^j \rho_s^{i,j} ds \leq \gamma^2 T \end{cases} \quad (2.1)$$

for more convenience we define the new variables,

$$\begin{pmatrix} Y_s^1 \\ \vdots \\ Y_s^n \end{pmatrix} = \begin{pmatrix} \frac{\mu_s^1}{\sigma_s^j} \\ \vdots \\ \frac{\mu_s^n}{\sigma_s^n} \end{pmatrix} \text{ and } Z_s = \begin{pmatrix} e^{-rs} \sigma_s^1 X_s^1 F_s^1 \\ \vdots \\ e^{-rs} \sigma_s^n X_s^n F_s^n \end{pmatrix}$$

the optimisation problem becomes,

$$\begin{cases} \max_Z \sum_{i=1}^{n} E\left[\int_0^T Y_s^i Z_s^i ds\right] \\ \sum_{i,j} \int_0^T Z_s^i Z_s^j \rho_s^{i,j} ds \leq \gamma^2 T \end{cases} \quad (2.2)$$

and the following theorem gives the solution of this problem,

**Theorem 2.1**

*The solution of problem* $(2.2)$ *is given by,*

$$Z_s = \sqrt{\frac{\gamma^2 T}{T-s}\exp\left(-\int_0^s \frac{f(u)}{T-u}du\right)}\frac{\Gamma_s^{-1}Y_s}{\sqrt{\frac{1}{T-s}E\left[\int_s^T {}^tY_u\Gamma_u^{-1}Y_u du \Big/ \Im_s\right]}}$$

*with,* $f(s) = \dfrac{{}^tY_s\Gamma_s^{-1}Y_s}{\frac{1}{T-s}E\left[\int_s^T {}^tY_u\Gamma_u^{-1}Y_u du \Big/ \Im_s\right]}.$

**Corollary 2.1**

*The optimal trading solution is given by,*

$$\begin{pmatrix} e^{-rs}\sigma_s^1 X_s^1 F_s^1 \\ \vdots \\ e^{-rs}\sigma_s^n X_s^n F_s^n \end{pmatrix} = \sqrt{\frac{\gamma^2 T}{T-s}\exp\left(-\int_0^s \frac{f(u)}{T-u}du\right)}\frac{\Gamma_s^{-1}Y_s}{\sqrt{\frac{1}{T-s}E\left[\int_s^T {}^tY_u\Gamma_u^{-1}Y_u du \Big/ \Im_s\right]}}$$

*and the instantaneous quadratic variation of the optimal portfolio equals,*

$$\left[\frac{\gamma^2 T}{T-s}\exp\left(-\int_0^s \frac{f(u)}{T-u}du\right)\right] f(s)$$

*Proof of corollary 2.1*

It's just a change of variables.

*Proof of theorem 2.1*

To solve $(2.2)$ we can solve, using a little revisited Bellman principle with constraints, the following problem

$$\begin{cases} \underset{Z}{Max}\sum_{i=1}^n E\left[\int_t^T Y_s^i Z_s^i ds \Big/ \Im_t\right] \\ \sum_{i,j} E\left[\int_0^T Z_s^i Z_s^j \rho_s^{i,j} ds \Big/ \Im_t\right] \le \gamma^2 T \end{cases}$$

and to solve this problem let us introduce the Lagrange multipliers $\lambda_t$.

The maximisation of,

$$\sum_{i=1}^{n} E\left[\int_t^T Y_s^i Z_s^i ds \Big/ \Im_t\right] - \lambda_t \sum_{i,j} E\left[\int_0^T Z_s^i Z_s^j \rho_s^{i,j} ds \Big/ \Im_t\right]$$

gives us, $\frac{Y_t^j}{2\lambda_t} = \sum_{i=1}^{n} Z_t^i \rho_t^{i,j}$ and matricially we obtain, $\frac{1}{2\lambda_t}{}^tY_t = {}^tZ_t\Gamma_t$

so, $$Z_t = \frac{1}{2\lambda_t}\Gamma_t^{-1}Y_t \tag{2.3}$$

Now we are going to saturate the constraints to eliminate $\lambda_t$.

$$\langle Z_t, Z_t\rangle_t = \frac{1}{4\lambda_t^2}{}^tY_t\Gamma_t^{-1}Y_t \tag{2.4}$$

so, $$\frac{1}{4\lambda_t^2}E\left[\int_t^T {}^tY_s\Gamma_s^{-1}Y_s ds \Big/ \Im_t\right] + \int_0^t \langle Z_s, Z_s\rangle_s ds = \gamma^2 T$$

and, $$\frac{1}{4\lambda_t^2} = \frac{\gamma^2 T - \int_0^t \langle Z_s, Z_s\rangle_s ds}{E\left[\int_t^T {}^tY_s\Gamma_s^{-1}Y_s ds \Big/ \Im_t\right]}$$

and we obtain, $$\langle Z_t, Z_t\rangle_t = \frac{{}^tY_t\Gamma_t^{-1}Y_t}{\frac{1}{T-t}E\left[\int_t^T {}^tY_s\Gamma_s^{-1}Y_s ds \Big/ \Im_t\right]} \frac{\left(\gamma^2 T - \int_0^t \langle Z_s, Z_s\rangle_s ds\right)}{T-t} \tag{2.5}$$

Let us define, $\varphi(t) = \gamma^2 T - \int_0^t \langle Z_s, Z_s\rangle_s ds$ and $f(t) = \frac{{}^tY_t\Gamma_t^{-1}Y_t}{\frac{1}{T-t}E\left[\int_t^T {}^tY_s\Gamma_s^{-1}Y_s ds \Big/ \Im_t\right]}$

(2.5) is an ordinary differential equation for $\varphi(t)$, we have $-\varphi'(t) = f(t)\varphi(t)$

so, $\varphi(t) = \gamma^2 T \exp\left(-\int_0^t \frac{f(s)}{T-s}ds\right)$ and $\varphi'(t) = -\frac{\gamma^2 T}{T-t}f(t)\exp\left(-\int_0^t \frac{f(s)}{T-s}ds\right)$

which gives, $\langle Z_t, Z_t\rangle_t = \left[\frac{\gamma^2 T}{T-t}\exp\left(-\int_0^T \frac{f(u)}{T-u}du\right)\right]f(t)$.

When replacing in (2.3) and (2.4) we obtain,

$$Z_t = \sqrt{\frac{\gamma^2 T \exp\left(-\int_0^t f(s)ds\right)}{T-t}} \frac{\Gamma_t^{-1}Y_t}{\sqrt{\frac{1}{T-t}E\left[\int_t^T {}^tY_u\Gamma_u^{-1}Y_u du \Big/ \Im_t\right]}}$$

which finishes the proof of theorem 2.1.

## 3. Interpretation of the results and remarks

First of all, note that the term $\sqrt{{}^tY_u\Gamma_u^{-1}Y_u}$ is the square of the instantaneous Sharpe Ratio for the optimal instantaneous mean-variance portfolio of futures constructed at time $u$ for the period $[u, u+du]$. The instantaneous Sharpe Ratio is the ratio, expected gain divided by standard deviation for the Markowitz optimal instantaneous portfolio.

The quantity, $\sqrt{\frac{1}{T-s}E\left[\int_s^T {}^tY_u\Gamma_u^{-1}Y_u du \Big/ \Im_s\right]}$ represents the quadratic average of the expected successive instantaneous Sharpe Ratio achievable between time $s$ and $T$.

The quantity, $\frac{\sqrt{{}^tY_s\Gamma_s^{-1}Y_s}}{\sqrt{\frac{1}{T-s}E\left[\int_s^T {}^tY_u\Gamma_u^{-1}Y_u du \Big/ \Im_s\right]}}$ represents the opportunity of an investment at time $s$.

it is calculated by comparing the present Sharpe Ratio to its expected quadratic average in the future.

The quantity, $\frac{\Gamma_t^{-1}Y_t}{\sqrt{\frac{1}{T-t}E\left[\int_t^T {}^tY_u\Gamma_u^{-1}Y_u du \Big/ \Im_t\right]}}$ represents the optimal mean-quadratic variation investment strategy for an average quadratic variation consumption of one.

The quadratic variation consumed at time $t$ with the optimal trading strategy equals

$$\int_0^t \langle Z_s, Z_s\rangle_s ds = \int_0^t \frac{\gamma^2 T}{T-s}\exp\left(-\int_0^s \frac{f(u)}{T-u}du\right) f(s)ds$$

$$= \gamma^2 T \int_0^t \frac{f(s)}{T-s}\exp\left(-\int_0^s \frac{f(u)}{T-u}du\right)ds$$

$$= \gamma^2 T\left[1-\exp\left(-\int_0^t \frac{f(u)}{T-u}du\right)\right]$$

So, at time $t$ the quadratic variation remaining to consume per unit time equals

$$\frac{\gamma^2 T}{T-s}\exp\left(-\int_0^s \frac{f(u)}{T-u}du\right). \qquad (3.1)$$

Finally, the optimal trading strategy we obtained can be decomposed into two terms. The first one is the quadratic consumption control term and the second one is the renormalised instantaneous mean- variance optimal portfolio.

In order to define an "average Sharpe Ratio": expected earnings per unit time, divided by the square of the quadratic variation per unit time of the constraint, we expect the constraint to saturate, this can be obtained very simply with a constraint of continuity and non nullity at time $T$ of the instantaneous Sharpe Ratio.

**Proposition 3.1**

*If the instantaneous Sharpe Ratio is continuous and doesn't vanish at time T, the quadratic consumption constraint in the, mean-quadratic variation, optimisation problem saturates.*

*Proof of proposition 3.1*

By continuity of the Sharpe ratio and as it doesn't vanish, $\lim_{s \to T} f(s) = 1$

and, $\lim_{s \to T} \exp\left( -\int_0^s \frac{f(u)}{T-u} du \right) = 0.$

The financial interpretation of this result is the following: as the Sharpe ratio doesn't vanish at time $T$, there is an incentive to handle a position, and to finish to consume all the remaining quadratic variation allowed.

## *4. Conclusion*

We derive in this article an explicit formulation of the optimal trading strategy in a mean-quadratic variation framework. The problem of optimaly trading in this context appears as a problem of managing the consumption of quadratic variation through time and with the best successive instantaneous Markowitz portfolios. The problem solved here can be extended to include stop loss constraints on the P&L and will be discussed, with these additional constraints, in a forthcoming publication with C.Lopez (*forthcoming*).

**References**

Bhattacharya R.N. and Waymire E.C. (1990) *Stochastic Processes with Applications.* Wiley Interscience.

Bayeux I. and Portait.R. (1993) *Dynamic Asset Allocation in a Mean-Variance Framework.* AFFI.

Bellman R. (1957) *Dynamic Programming.* Princetown University Press.

Brugière P. and Lopez C. (forthcoming) *Optimal Trading with Stop Loss Constraints.*

Cox J.C. and Huang C.F. (1992) *A continuous Time Portfolio Turnpike Theorem.* Preprint MIT.

Duffie D. and Richardson H.R. (1991) *Mean Variance Hedging in Continuous Time.* The Annals of Applied Probability. Vol 1, N°1, 1-15.

Fleming W.H. and Rishell R.W. (1975) *Deterministic and Stochastic Optimal Control.* Springer Verlag Berlin.

Jacquillat B. and Solnik B. (1990) *Gestion de Portefeuille et des Risques.* Dunod.

Karatzas I., Lehoczky J. and Schreve S. (1987) *Optimal Portfolio and Consumption Decisions for a "Small Investor on a Finite Horizon".* SIAM Journal of Control and Optimization 25, 1157-1186.

Lintner (dec 1969) *Security Prices, Risk and Maximal Gains from Diversification.* Journal of Finance.

Markowitz H.M. (march 1959) *Portfolio Selection.* Journal of Finance.

Markowitz H.M. (1959) *Efficient Diversification of Investments.* John Wiley New York.

Merton R. (1990) *Continuous Time Finance.* Blackwell.

Mossin J. (1968) *Optimal Multiperiod Portfolio Policies.* Journal of Business 31, 215-229.

Richardson H. (1989) *A Minimum Variance Result in Continuous Trading Portfolio Optimization.* Management Science, Vol 35, N°9 September, 1045-1055.

Sharpe W.F. (1970) *Portfolio Theory and Capital Markets.* Mac Graw Hill New York.

Schweizer M. (1992) *Mean Variance Hedging for General Claims.* The Annals of Applied Probability. Vol 2, N°1, 171-179.