Optimizing Investment and Contribution Policies of a Defined Benefit Pension Fund

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Abstract

The management of pension funds financially encompasses asset allocation and the control of the future flows of contributions. A high proportion of stocks in the portfolio has the benefit of a lower mean contribution level, but at the price of a higher time varation of contribution flows. This paper models the trade-off in an inter-temporal framework and uses stochastic control to obtain an optimal asset allocation - between a risky asst and a riskless asset - and the contribution policy. In case of a defined benefit scheme we add a constraint on the maximum contribution to obtain a more realistic framework than a previous one, in which a closed form solution was obtained. This constraint modifies significantly the optimal solution. It is no longer in closed form, and the stock allocation is no longer a decreasing function of the relative wealth. The solution is depicted on graphs for various values of the input parameters.

Résumé

La gestion financière d'un fonds de pension comprend l'allocation d'actif et l'évolution des contribution et/ou des pensions. Pour un fonds à pensions fixées, l'augmentation de l'allocation en actions permet en moyenne de baisser les contributions mais au prix d'une plus grande variabilité de celles-ci. L'objectif de ce papier est de modéliser dans un cadre intertemporel la gestion financière et de trouver la politique optimale en terme d'allocation entre actions et fonds monétaire et en terme d'évolution des contributions. Dans le cas d'un fonds à pension définie nous imposons une borne supérieure pour la cotisation. La forme de la solution est profondément modifiée par cette contrainte: ce n'est plus une solution analytique, et l'allocation en actions n'est plus une fonction monotone décroissante de la richesse du fonds. La solution est illustrée pour une gamme de paramètres.

Keywords

Pension fund, asset allocation, financial policy optimization, stochastic control.

Mots clefs

Fonds de pension, allocation d'actifs, politique financière optimale, contrôle stochastique.

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I. Introduction

Pension funds are managed provisions made by corporations and their employees in order to fund the future payments of pensions to the later. This systems differs from the pay as you go system to the extent that the contributions paid are invested in securities for very long periods of time, typically forty years. Therefore the provisions made will depend non solely on the level of contribution but also on the return of the investment portfolio. This paper examines the investment policies and contributions scheme that provide the best possible outcome for those who will benefit from it.

The principles underlying pension funds are quite simple, even if the variety of actual schemes from one country or one industry to another is vast and complex. Workers and corporations pay contributions to a pension fund, which invests them over a very long period of time and releases them when the workers retire, in the form of pensions. Obviously, the more the contribution, the higher should be the pension. Nevertheless, asset allocation also comes into play, in so far as that even a slight improvement of the asset portfolio mean return, say one or two percent, may result after fourthy years of accumulation, in a sizeable increase –by 40% to 100%– of the pensions. On the other hand, too much exposure to stock market fluctuations could, in the absence of careful management of the asset portfolio, severely damage the asset value and impose an undesirable increase of the contributions.

In this context, portfolio management and the contributions scheme are clearly interdependant. Moreover, the decisions made over one year have certainly consequences in the future. Therefore multiple horizon optimization seems to be appropriate. Because stock returns are uncertain in efficient markets, stochastic control would help in finding the optimal investment policy, as well as the adequate level of contribution.

Up to our knowledge, until Boulier et al (1995) denoted BFT in the remainder the use of stochastic control for pension fund financial management has not been reported in the literature. However, Merton (1971) described the basic framework of intertemporal optimization and showed how the Bellman function can provide a solution to the asset allocation of an investor, given his objectives and risk tolerance. On the other hand, asset and liability management has been invoked by several recent studies in order to determine the asset allocation of pension funds. Sharpe and Tint (1990) proposed to optimize $A-kL$, where A and L stand for asset and liability respectively and k is a positive constant less than one. Surprisingly, as remarked by Sharpe and Tint, investment policies of pension funds was hardly a subject of interest before the early 90's. A few authors, such as Tepper and Affleck (1974) and Black and Jones (1988) had made attempts to propose solutions to the asset allocation problem, given the liabilities of the pension fund. Nevertheless as asset and liability management techniques improve and are put into practice, an increasing number of papers have addressed the issue, now seen as important. Among them, Leibowitz et al. (1993) showed how to cope with a number of conflicting constraints and to come up with an appropriate and yet simple optimization. Griffin (1993) has also presented a methodology which he has applied to Dutch and English pension funds globally, in an attempt to explain why the Dutch invest less than 25% in stocks, whereas the British invest more than 80%. All these studies stick to the asset allocation problem in a one period framework. However Boender et al. (1993) have tried to investigate the financial management problem in a more general setting, making use of a scenario approach.

In BFT, an attempt was made to model the financial management of a pension fund with defined benefits (i.e: pensions are fixed but contributions may vary). In this context, the optimal choice corresponds to the maximum expected utility. Under simplistic but realistic assumptions they modelled the allocation decision between two assets, a risky (stocks) and a risk free (cash) by means of a utility that was a decreasing and concave function of the future stream of contributions. BFT solved the dynamic control problem in the special case when no upper bound is set to the contributions. A closed form solution was obtained in which the contribution and the proportion of stocks held in the portfolio are functions of the difference between a maximum necessary wealth and the actual wealth. As a result, when the pension fund is sufficiently funded, the contribution rate can be lowered and the portfolio more invested in cash. When the pension fund is not enough funded, the solution leads to a rise in contributions, which is understandable, but it also leads to higher proportion in stocks, which seams unreasonable.

The present paper aims at imposing a maximum level to the contributions and at comparing the results with the previous solution. The mathematical solution will be shortly presented in an appendix, so that more attention can be drawn to the financial discussion of the result in an illustrative example.

The outline of the paper is the following. Section II recalls the financial framework of BFT, section III describes the solution and recalls the previous unconstrained solution. Illustrations are provided in the next section which comments the main financial features. Last section concludes and indicates directions for further research.

II. Setting the problem

Let us first recall the notation and hypothesis of BFT.

1°) Definition of variables

In the rest of the paper the following variables will be used:

P_t	pension payments
c_t	contributions
x_t	portfolio market value
S_t	market value of the risky asset
u_t	investment in the risky asset, as a proportion of portfolio value

All of them are functions of time t, either stochastic or deterministic, p_t.and c_t are supposed to be positive. On the other hand, the following variables are supposed to be constant:

r	risk free rate
λ	risk premium (positive)
σ	volatility of the risky asset
α	pension growth rate
β	psychological discount rate
δ	risk aversion parameter

Finally, we shall refer to (c^*, u^*) as the optimal policy is x_m as the maximum necessary wealth, defined below and to $w = x/xm$ as the relative wealth.

2°) *Main hypothesis*

We consider the financial management of the aggregated pension fund position and assume pension flows in the future are known. For sake of simplicity, their growth rates are taken as constant, α. This growth rate may account for a demographic trend, an inflation scenario, a purchasing power evolution or any kind of combination of these factors as long as it leads to a deterministic growth rate. Thus, in the defined contribution pension fund that we consider here:

$$dp_t = \alpha p_t$$

Although more general assumptions are clearly possible, we restrict this study to the case of two assets investigated by Merton (1972):

– a riskless asset whose return is r, the risk-free rate, assumed constant;

– a risky asset whose price S_t follows the standard geometric brownian motion

$$dS_t / S_t = (\lambda + r)dt + \sigma dW_t$$

where dW_t denotes the usual differential of brownian motion. The expected return of this risky asset, $r + \lambda$, is therefore higher than the riskfree rate, the difference being the constant risk premium λ. On the other hand the future returns of the risky asset are not known with certainty because of the volatility (assumed to be constant) and the stochastic process W.

In this simple setting the portfolio management consists in allocating a proportion u_t of the value x_t into the risky asset. Typically the portfolio is composed of stocks and short term bills. Again, generalization is possible and will be discussed later.

Suppose that all the wealth is invested in the riskless asset then the future wealth is deterministic. In order to pay all the pension in the future, the necessary wealth is:

$$x_m = \int_o^\infty p_o \, exp(\alpha t). exp(-rt)dt = p_o / (r - \alpha)$$

This wealth is therefore the maximum wealth necessary to fund the futur payments.

3°) *Optimization*

We suppose that contributors are reluctant to pay higher contributions either today or in the future, but that they have their own judgement as to the discount rate, which we have denominated the psychological discount rate β. For sake of mathematical tractability we have also assumed that their disutility V is a power function of the contribution c, δ being the exponent. Under these circumstances, a rational pension fund manager would try to minimize:

$$V = \int_0^\infty \exp(-\beta s)\mu(c_s)ds$$

with $\mu(c) = c + \delta c^2$

The optimum policy must satisfy the following constraints:

– payment of the pension p_t

– positive value for x_t

Therefore we seek the policy *(c_t, u_t)* which minimizes *E(V)* under the two preceding constraints. For sake of computational tractability, the integration interval will be limited to T, a very large number of years, typically T = 150 years.

Now the difference with BFT comes from an additional constraint:

$$c_t \le kp_t$$

where $c_m = kp_t$ is a maximum contribution level set by the contributors for understandable reasons. It is equal to the product of a constant k and the pension level at that date in order to take into account the inflation rate. This problem is denoted (P_o).

III. Solving the problem

Let start with a change of variables.

$$d_t = C_t / p_t$$
$$X_t = x_t / p_t$$
$$A = u / p$$

The problem to be solved is:

$$\left(P_o^{'}\right) \quad Max\ \mathrm{J(d,A)}$$
$$X_t \ge 0$$
$$d \le k$$

$$with\ \mathrm{J(d,A)} = E\left(\int_o^T \exp(-\beta s)U(d_s)ds\right)$$

As a first step we seek the solution of (P_1) which is identical to $(P_o^{'})$ with the exception of the first contraint that is temporarily overlooked.

We denote I(x) the function defined by:

$$
\begin{array}{ll}
I(x) = 0 & \text{if } x \le u'(o) \\
I(x) = u'^{-1}(x) & \text{if } u'(o) \le x \le u'(k) \\
I(x) = k & \text{if } u'(k) \le x
\end{array}
$$

and $\phi(y) = E\left(\int_o^T \exp(-(r-\alpha)s) I(y\Gamma_s) ds\right)$

where $\Gamma_s = \exp((\beta - r)s Z_s)$

and $Z_s = \exp\left(-\int_o^s \theta dW_s - \frac{1}{2}\int_o^s \theta^2 ds\right)$

where $\theta = \lambda / \sigma$

If ψ is the inverse function of ϕ Michel and Wisnia (1994) showed that:

$$d(x_o, s) = I\left(\Gamma_s \psi\left(\int_o^T \exp(-(r-\alpha)s) ds - x_o\right)\right)$$

is now solution of (P_1).

Now, an asset allocation $\tilde{A}$ associated with $d(x_o, s)$ can be obtained in such a way that $x_t \ge 0$. $(\tilde{A}, d(x_o, s))$ is then the solution. Nevertheless a feed back solution is also derived in Michel and Wisnia (1994) in order to have a tractable means to compute it.

IV. Illustrations and comments

Having looked at the mathematical solution, that is no longer in a closed form formula when contribution rates are bounded upwards, we would like to give some illustrations of the optimal allocation and contribution policies. This will help to pitpoint the financial aspects of the solution and to draw practical implications. Nevertheless one should bear in mind that all these conclusions will be tied to the selection of values for the parameters.

1°) Framework for illustration

Among our primary objectives, we would like to analyse links between asset allocation and contribution policy. How the new relationship compare to the previous unconstrained case? Also we would like to analyse the sensitivity of stock allocation and contribution policy to the various parameter specially, r, σ and λ but also to some extent to others like β. For obvious reasons returns and wealth quantities were computed in real term rather than in nominal term. In addition, we set some long term value T = 170 years in all integrals for sake of computational practicability. The following table shows the values taken for the parameter with, in the second column, the central value for which results will be discussed in IV.2, then in the third the other analysis (IV.3)

	Central value (%)	Other values (%)
$r-\alpha$	2.5	0.5 - 3
λ	2	5, 8
σ	15	25
β	2.5	5

δ is set to 0.5.

2°) Optimal allocation and contribution policies

Figure 1 below depicts (in case of central value set) the optimal allocation of the pension fund portfolio in stocks (continued line) and the corresponding optimal policy (lashed line) as functions of the relative wealth $w = x/x_m$ of the pension fund. Indeed, when portfolio wealth reaches x_m (so that the relative wealth is 1) there is no need to raise any more the contributions and to take risk; then the portfolio is fully invested in riskless asset that have a return r. Although it can be reached, such a situation is obviously very unlikely.

Now in usual situations, the relative wealth lies between 0 and 1. Let us examine the contribution policy first; there are three situations that can be observed in figure 1.

A - if relative wealth is between 0 and w_1, then the contribution level is equal to the maximum.

B - if w lies between w1 and w2, then the contribution is gradually decreasing almost linearly function w down to zero for $w = w_2$.

C - between w_2 and 1, no more contributions are necessary.

Compared to the unconstrained case, the situation has not radically changed: contributions are still a decreasing function of relative wealth. But of course, to satisfy the additional constraint, in situation A, the contribution reach its maximum value. Note also that the contributions vanish in situation C while they were nil only when $w = 1$ in the unconstrained case; its seems thus that the overall shape of the function was changed by the addition of a new constraint.

When it comes to the allocation in stocks, we observe a radical change from the previous unconstrained case. Whereas in the later the investment in stocks was decreasing function of w, it is now bell shaped with a maximum reached in situation B. Note also that when the relative wealth is too low only bonds are purchased. Another feature worth mentioning is that in situation C, investment in stocks is still significant although there are no more contributions provided by the employees and the corporation. Finally, the maximum exposure to stock, which is reached in section B, remain less than 100%.

3°) Sensivity analysis

The sensitivity of the stock allocation policy to the risk premium λ and market volatility is shown in figure 2. The influence of the volatility is striking: a higher volatility leads to a sharp reduction in the stock allocation even in case of high risk premium. Here the effect is very similar to the unconstrained case, for which the stock allocation is proportional to λ / σ^2. On the other hand the influence of the risk premium λ is less pronounced although a wide range of values was considered. It seems that a higher λ leads to a maximum in stocks that is reach for lower values of the relative wealth w. But the height of this maximum is apparently first increasing from $\lambda = 2$ to $\lambda = 5$ but then decreases from $\lambda = 5$ to $\lambda = 8$ whatever the volatility value is.

A higher λ means a better profitability for the portfolio (at least when it is invested in stocks) so the resulting contribution rate can be easily deduced from the previous features. The value w_1 of the relative wealth below which contributions must remain at their maximum, is now a decreasing function of λ, as can be seen in figure 3. As a consequence of the volatility on asset allocation, this value is also increasing with σ, because the portfolio return is too low to ensure that pensions will be paid.

Finally figure 4 and 5 show the impact of change in the value of r (still kept constant over time) on the contributions and the return of the portfolio respectively. These effects are clear cut and economically logic. Indeed given à fixed β (preference for the present) the lower the interest rate r the smaller the contributions, simply because the advantage of in terms of utility of investing now do not balance the preference for consumption expressed in the value of β. On the other hand, it is obvious that the return of the portfolio is a decreasing with respect to r. As can be seen in figure 5, this effect is slightly dependent on the portfolio relative wealth, which shows that when λ is kept constant, the asset allocation is also a function of r.

Comments and conclusions

Real pension funds are complex financial entities, the management of which requires a disciplined approach, in order to maximize the utility of the contributors. We have presented a simple model for a defined contribution scheme in order to stress the main effect on financial policy of various parameter describing the economy.

Firstly the problem was posed in a multiperiod framework, as an optimization problem. As the wealth accumulated by the pension fund will serve to pay known pensions ex-ante, the utility considered here was the discounted value of future contributions, the discounting factor being tied to the preferences of the contributors. This leads to stochastic control problem, in which contribution rates and asset allocations are interdependant. In the simplified case of single risky asset and also a stable population (of pensioners and of contributors) the optimal policy was derived analytically.

The illustrative results have shown that the pension fund financial policy was mainly tied to its relative wealth. Whether it is low, medium or large, both contribution rate and asset allocation are changed. For low wealth, contributions should be kept to a maximum and risky allocation should be selected, with an exposure to the risky asset which is increasing with respect to wealth. For a medium wealth the contribution start to decrease and the allocation in the risky asset reaches a maximum, then decreasing as long as the wealth is above a threshold. Finally if the pension fund is rich enough, these is no longer a need for neither contributions nor risky investments. This last situation was already obtained by BFT in a simpler case, when contributions were unbounded.

The effect of this constrain is a reduction in stock holding at low funding ratio (low wealth relative to the value of the liability).

Refinements of this model are still necessary to guide pension fund in practice: what is the effect of having interest rate risk, for example with a random r? Do we have a similar two fund separation theorem as in Merton (1974) when more than one risky asset are available? What are the modifications induced by changes in the population, for a pension fund that begins or that is terminating? Of course refinement will oblige one to rest on numerical solutions, that require careful analysis of numerical errors. We think that this model could also help asserting the quality of numerical solutions.

REFERENCES

Fisher Black and Robert Jones, 1988
Simplifying portfolio insurance for corporate pension plans
Journal of Portfolio Management, Summer

C.G.E. Boender, C.L. van Aalst, D. Bams and Heemskerk, 1993
Scenario approaches for asset-liability management
Inquire, April

Jean-François Boulier, Danièle Florens et Etienne Trussant, 1995
A dynamic model for pension fund management
AFIR Conference, Brussels

Mark Griffin, 1993
A new rationale for the different asset allocation of Dutch and UK pension funds
3rd AFIR Conference, Rome, April

Martin L. Leibowitz, Stanley Kogelman and Lawrence N. Bader, 1993
Asset performance and surplus control : A dual-shortfall approach
Journal of Portfolio Management, Winter

R. Merton, 1971
Optimum consumption and portfolio rules in a continuous-time model
Journal of Economic Theory, 3 (p 373-413)

William F. Sharpe and Lawrence G. Tint, 1990
Liabilities - A new approach
Journal of Portfolio Management p. 5-10, Winter

Irwin Tepper and A.R.P. Affleck, 1974
Pension plan liabilities and corporate financial strategies
The Journal of Finance, December

FIGURE 1

Portfolio composition and contributions as functions of relative wealth

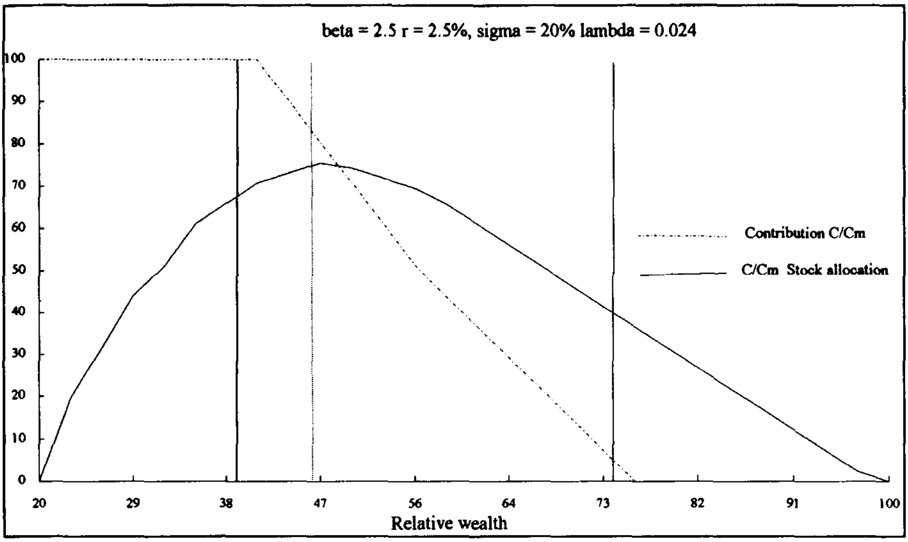

FIGURE 2

Stock allocation as a function of relative wealth for various volatilities (σ) and risk premium (λ)

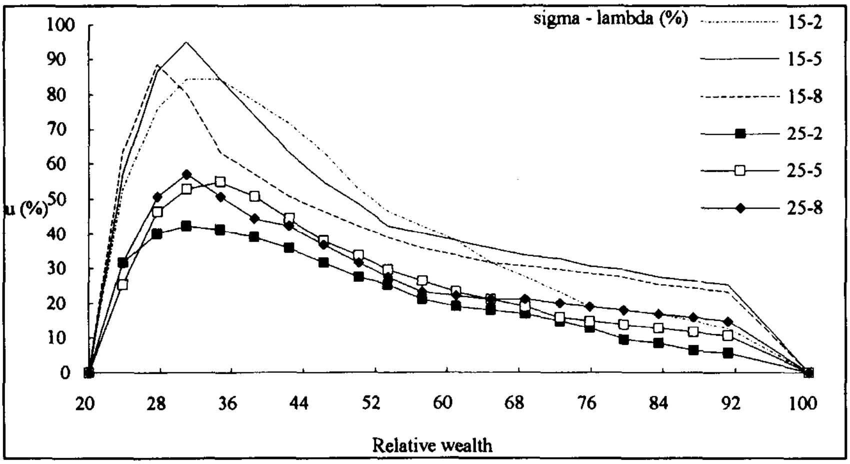

FIGURE 3

Influences of volatility and risk premium on contributions

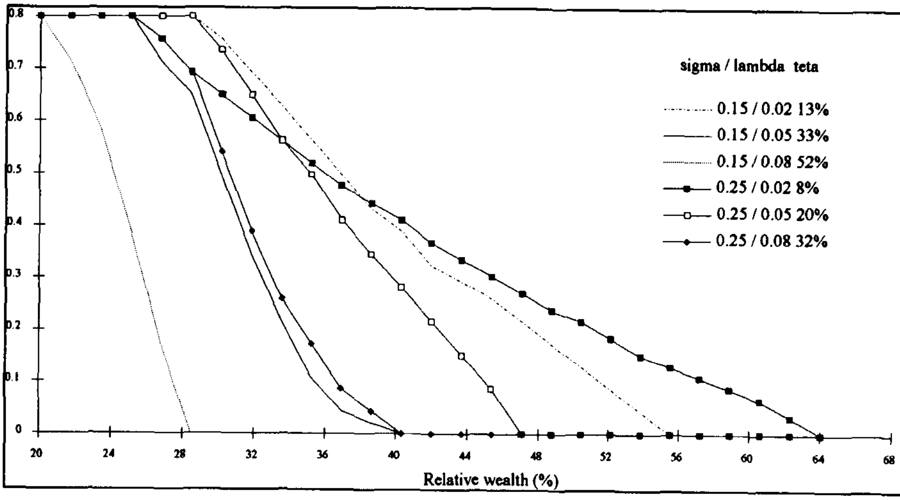

FIGURE 4

Influences of the relative wealth and real riskless rate on contributions

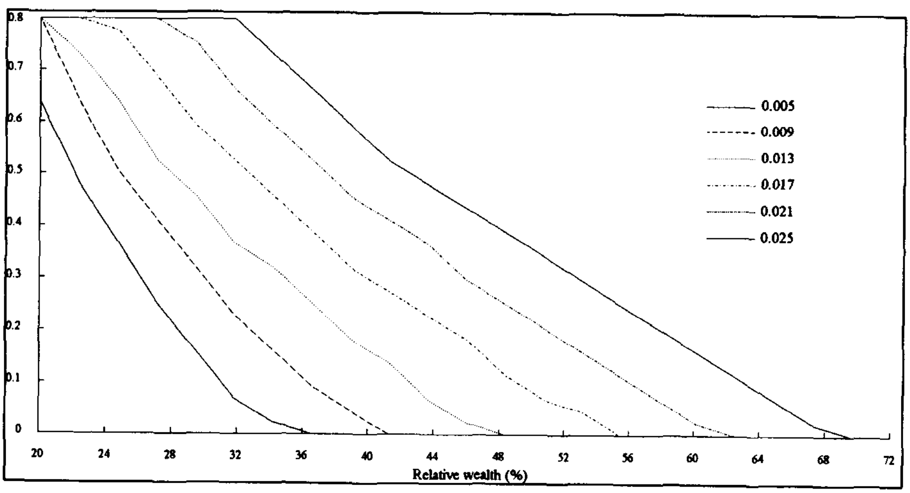

FIGURE 5

Influence of relative wealth and of real riskless return on portfolio return