Variations on a Theme of Bruno Dupire

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Abstract

This paper considers the problem of existence, uniqueness and construction of diffusion processes compatible with observed option prices in a continuous time setting. Following the approach of Bruno Dupire we study the option prices resulting from price diffusions $dX_t = b(X_t,t)dW_t$, investigate the coherence of two partial differential equations for the call option prices $C(x,t,K,T)$ resulting from the Kolmogorov forward respectively backwards equations and give in the case of the standard reference model $b(x,t) = v\ x$ certain functional equations for the call option prices.

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Introduction:

The Black-Scholes option pricing formula has had an immense influence on the traded options markets in the past. Most of the option-traders apply the mentioned formula in a strange looking way: Normally one would expect, that brokers invest some effort to estimate the historical variance of stock returns. But the usual procedure is different: Since the current call option´s price à la Black-Scholes increases monotonically with volatility v, there is a one-to-one correspondence between volatility and option prices. More precisely, when the stock price $X = X(t)$, exercise price K, time to maturity T of the option and the riskless interest rate are known, one can "invert" the formula to obtain the so called implied volatility,implied by the observed call price. And in this sense most of the option´s traders sell and buy "volatility" when they deal on the option´s market based on "homemade" variance estimations. Unfortunately the implied volatilities do not appear to coincide with the historically observed volatilities (usually the implied volatilities are higher than the historically observed) and even worse, if one tries to estimate the market view of volatility using more than one option price, for instance a variety of expiry dates or alternatively consider them across exercise prices one observes, that implied volatilities strongly depend on the maturity and does not appear to be constant across exercise prices. The last phenomenon is traditionally called the "smile" effect (for details compare P. Willmott et al. 1993, B. Dupire 1993,1994) . Furthermore empirical studies by Latané and Rendleman (1976) and Beckers (1981) have shown that implied volatilities predict better than historically observed volatilities the future variability of stock prices, therefore it looks to be obvious to discuss at least the possibility that option prices respectively the "Black-Scholes-traders" which simply apply the classical Black-Scholes recipe influence the pricing process of the underlying security. Following the approach of B. Dupire this note investigates the following type of question:
Which arbitrage free call-option-prices corresponding to a diffusion of type

(1) $$dX = a(X,t)\,dt + b(X,t)\,dW_t$$

coincide with the given prices, given by a market controlled by Black-Scholes-traders ?

1.) Security price processes compatible with prescribed option prices

Bruno Dupire (1993,1994) considered the problem of existence, uniqueness and construction of a diffusion process compatible with the observed option prices in a continuous time setting. Here for notational simplicity the riskless interest rate is assumed to be zero (this amounts to change the original price process $x = X(t)$ to $X(t)/e^{rt}$. together with the obvious changes in the corresponding option price formula). This transformation of the price process is assumed tacitly for the rest of this paper. Suppose that for a given maturity T   (i.e. the remaining time to maturity is T-t)   the collection of call prices C(x,t,K,T) at time t of different strikes K are given by the conditional expectation

(2) $C(x,t,K,T) = E((x - K)^+|x(t)) = \int_0^\infty (y - K)^+ \cdot p(x,t,y,T)dy$

with transition probability density function p(x,t,y,T) such that
$\lim_{t \uparrow T} p(x,t,y,T) = \delta(x - y)$ where δ denotes the Dirac-function. Since then the equation

(3) $p(x,t,K,T) = \dfrac{\partial^2 C}{\partial K^2}(x,t,K,T)$ holds,

the conditional densities p - if they exist - are determined by the knowledge of the function C. As Bruno Dupire (1993) pointed out, there are examples of distinct diffusion processes for $x = X(t)$ which generate the same transition probability distributions p. However under additional assumptions on the diffusion process $x = X(t)$ one can recover a unique diffusion process from the density function p.

These are the assumptions:

(i) The price process $x = x(t)$ is given by a risk-neutral diffusion of the form
     $dx = b(x,t) \, dW_t$ i.e. in equation (1) $a(x,t) = 0$ holds,

(ii) the transition probabilities of the stochastic process x(t) are given by a continuous density function p, such that for the fair price $C = C(x,t,K,T)$ of European call options equation (2) holds,

(iii) the partial derivations $\dfrac{\partial p}{\partial T}, \dfrac{\partial^2 (b(y,T)p(x,t,y,T))}{(\partial y)^2}$ exist and are continuous

functions and the Fokker-Planck-equation

(4) $\dfrac{1}{2} \dfrac{\partial^2 (b^2 p)}{(\partial y)^2} = \dfrac{\partial p}{\partial T}$ holds,

(iv) for arbitrary but fixed x, t,T the asymptotic relation

$$\lim_{K\to\infty} \frac{\partial C}{\partial T}(x,t,K,T) = 0 \quad \text{holds,}$$

(v) furthermore the following growth conditions are fulfilled:

$$b(y,t) = O(y) \quad \text{for} \quad y \to \infty \quad \text{and} \quad \frac{\partial C}{\partial T} \leq K^2 \frac{\partial^2 C}{\partial K^2} \quad \text{for large K.}$$

Under the mentioned assumptions B. Dupire (1993) gets the following result:

(5) For X(t), t fixed , the partial differential equation $\quad \dfrac{\partial C}{\partial T} = \dfrac{b^2(K,T)}{2}\dfrac{\partial^2 C}{\partial K^2}$

holds, and this equation allows the determination of the function b and hence the determination of the diffusion process (i) for known call-option prices C = C(x,t,K,T).

Notice, that for zero interest rate the Black-Scholes partial differential equation reads as follows:

(6) $$-\frac{b^2(x,t)}{2}\frac{\partial^2 C}{\partial x^2} = \frac{\partial C}{\partial t}$$

(usually with the assumption b(x,t) = v x , with the constant v called volatility).

Warning: The "instantaneous volatility" defined by v(x,t):= b(x,t) / x generally does not coincide with the implied volatility corresponding to the given option price C(x,t,K,T). This implied volatility $v_{impl}(x,t,K,T)$ is implicitly defined by the equation   C(x,t,K,T) = $C_{BS}(x,t,K,T;v_{impl}(x,t,K,T))$.

In the following paragraphs we
- recapitulate the well known calculation of option prices for risk neutral diffusions of type (1) with a(x,t) ≡ 0 (this leads in particular to an equation of type (6), even in the case where b(x,t) is not necessarily of the form v x with a constant v, which allows the determination of the diffusion from known option prices)
- investigate the coherence of the two partial differential equations (5) and (6),

— discuss in the special case of Black-Scholes traders the connection with the classical Black-Scholes equation and certain functional equations for call-option prices.

2.) The calculation of option prices for martingal diffusions

We start with a stochastic differential equation

(7) $d\,x\,=\,b(x,t)\,dW_t \qquad 0 \le t \le T$,
      $x(0) = c$ a random variable independent of $W_t - W_0$ for $0 \le t \le T$

describing the price fluctuation of say a stock, where (W_t) is a Brownian motion under the measure P, the probability measure of the underlying probability space $(\Omega\,,\,\mathcal{F}\,,P\,)$.
Recall for convenience that under the following assumptions :

(7a) there exist a constant $L > 0$ such that for all $t \in [0,T]$, $x,y \in R$:

$$|b(x,t)-b(y,t)| \le L\,|x-y| \quad \text{and the polynomial growth condition}$$

$$|b(x,t)|^2 \le L^2\,(1+x^2) \qquad \text{hold , and furthermore}$$

(7b) the function b: $R \times [0,T] \to R$ is measurable and continuous in t

there exist a unique solution of (7), which is a diffusion $X = X_t$ with drift zero and diffusion coefficient b (L.Arnold, 1973).

Remark 1 : By the well known method " transformation of drift " together with the corresponding change of the probability measure P one can pass over from a diffusion of type (1) to a diffusion of type (7) if one assumes that there exist a constant $d > 0$ such that $b(x,t) > d$ for all x,t holds. For the details compare for instance N. Ikeda, S. Watanabe (1981) Chapter IV,4, compare also condition (A1) below.

Let $D = \frac{1}{2} b^2(x,t) \frac{\partial^2}{\partial x^2}$ denote the partial differential operator belonging to the diffusion $X = X_t$ and consider the Cauchy boundary value problem

(8) $(D + \frac{\partial}{\partial t}) f = 0$, $f(x,T) = (x - K)^+$.

There exist a unique solution for (8) under the following assumptions (A. Friedman (1975):

(A1) There is a positive constant μ such that $b^2(x,t) \geq \mu$ for all $(x,t) \in R \times [0,T]$.
(Compare remark 1 above)

(A2) The function $b(x,t)$ is bounded continuous in $R \times [0,T]$, and continuous in t, uniformly with respect to (x,t) in $R \times [0,T]$.

(A3) The function $b(x,t)$ is Hölder continuous in x (say with exponent $\alpha \neq 0$) uniformly with respect to (x,t) in $R \times [0,T]$.

Using this solution f together with Itô´s formula one obtains the following Itô representation of the contingent claim "call-option" with strike price K (cf. Föllmer, 1991, Föllmer, Schweizer 1989, Harrison, Pliska 1981):

(9) $(x - K)^+ = f(X_0,0) + \int_0^T f_x(X_s,s) b(X_s,s) dW_s$.

Therefore a selffinancing duplication portfolio for the call-option is given by

(10) $\xi_t := f_x(X_t,t)$, $\eta_t := f(X_0,0) + \int_0^t \xi_s\, b(X_s,s) dW_s - \xi_t\, X_t$,

and the fair value $C(x,0,K,T)$ of the call option at the initial time t=0 with starting price $X_0 = x$ is given by:

(11) $C(x,0,K,T) = \xi_0\, X_0 + \eta_0 = f(X_0,0) = f(x,0)$,

and at time t>0 given $X_t = x$ the call option price amounts to

(12) $C(x,t,K,T) = f(x,t) = E((X_T - K)^+ | X_t = x)$,

            f the solution of the Cauchy problem (8).

Remark 2: For important applications the conditions (A1) - (A3) are only fullfilled in smaller domains $G \times [0,T] \subseteq R \times [0,T]$. If the boundary of this domain and the coefficients are smooth enough, again a corresponding solution exist and it is unique.

Corollary 1: Under the assumptions (7), (7a),(7b) and (A1),(A2),(A3) for the price process $x = X_t$ of the underlying security , the call-option price C(x,t,K,T) satisfies the generalized Black-Scholes partial differential equation

(6)
$$-\frac{b(x,t)}{2} \frac{\partial^2 C}{\partial x^2} = \frac{\partial C}{\partial t} \quad .$$

In the special case where the diffusion coefficient b does not depend on the time parameter t , the call option price C(x,t,K,T) depends only on the remaining time T-t, hence can be written

(13)
$$C(x,t,K,T) = C(x,T-t,K) .$$

In that case the following equation

(14)
$$\frac{\partial C}{\partial t} = -\frac{\partial C}{\partial T}$$

holds and hence one gets:

Corollary 2: Suppose that the assumptions of corollary 1 are fulfilled and assume equation (14) holds (- this is true if the diffusion coefficient b(x,t) does not depend on the t-variable, i.e. b(x,t) = b(x) -); furthermore assume equation (5) holds, then the call option price
C(x,T-t,K) is a solution of the "wave equation"

(15)
$$b^2(x,t) \frac{\partial^2 C}{\partial x^2}(x,T-t,K) = b^2(K,T) \frac{\partial^2 C}{\partial K^2}(x,T-t,K) \quad .$$

Remark 3: Corollary 2 expresses a certain symmetry of the call option prices $C(x,T-t,K)$ with respect to the variables x and K. These symmetries in the classical Black-Scholes case $b(x,t) = b(x) = v x$ (v the constant volatility) are investigated in more detail in the following paragraph, while more general cases of equation (15) will be discussed in a forthcoming paper. In any case equation (15) shows the limitations of the model used by B. Dupire: For given option prices $C(x,t,K,T)$ it does not suffice to define the function $b(x,t)$ according to equation (5), the prices C has to fulfill extra conditions in order to fit in a coherent model.

3. x-K-symmetries and the coherence of the partial differential equations (5) and (6) in the classical case of constant volatilities.

Let us assume that the option market follows the standard option price recipe, that means the call option prices are given by the following formula:

(16) $C(x,T-t,K) =$

$$= x N(\frac{1}{v\sqrt{T-t}}(\log\frac{x}{K} + \frac{1}{2}v^2(T-t))) - K N(\frac{1}{v\sqrt{T-t}}(\log\frac{x}{K} - \frac{1}{2}v^2(T-t))) ,$$

where N denotes the distribution function of the standardized normal distribution (recall we assumed for notational simplicity an external interest rate zero !).

Which price diffusion process X_t produce these call option prices ?
As a consequence of equation (5) as well as (6) one can conclude the answer: The only diffusion (7) compatible with call option prices (16)
has diffusion coefficient $b(x,t) = v x$.

Lemma 1: For the call option prices $C(x,T-t,K)$ according to equation (16) the relations

$$\frac{\partial C}{\partial t} = -\frac{\partial C}{\partial T} \quad \text{and} \quad K^2\frac{\partial^2 C}{\partial K^2}(x,T-t,K) = x^2\frac{\partial^2 C}{\partial x^2}(x,T-t,K) \text{ hold.}$$

Proof : Tedious calculation.

In the case of equation (16) lemma 1 shows the coherence of the two partial differential equations (5) and (6): They coincide and can be considered as two different views of the same factual situation.

Concerning the symmetries with respect to the variables x and K inherent in formula (16) we prove:

Lemma 2:   For the option prices according to equation (16) the relation

$$C(K,T\text{-}t,x) = K\text{-}x + C(x,T\text{-}t,K) \qquad \text{holds.}$$

Corollary 3: In the case of a price diffusion $dX_t = b(X_t,t)\,dW_t$ with
$b(x,t) = v\,x$ the European put option price P(x,T-t,K) for the strike price K is equal to the call option price with interchanged x - K - arguments:
(17) $P(x,T\text{-}t,K) = C(K,T\text{-}t,x)$,

here C(x,T-t,K) denotes the call option price corresponding to the strike K with actual value x of the underlying security.

Proof of Lemma 2:

Using formula (16) one has

$$C(K,T\text{-}t,x) = K\,N(-d_2) - x\,N(-d_1) \quad \text{with } d_1 = \frac{\log(\tfrac{x}{K}) + \tfrac{1}{2}v^2(T-t)}{v\sqrt{T-t}}$$

$$\text{and} \qquad d_2 = \frac{\log(\tfrac{x}{K}) - \tfrac{1}{2}v^2(T-t)}{v\sqrt{T-t}}.$$

Using the relation   N(-d) = 1- N(d)   one has

$$C(K,T\text{-}t,x) = K\text{-}x + x\,N(d_1) - K\,N(d_2) = K\text{-}x + C(x,T\text{-}t,K) \text{ , q.e.d.}$$

Proof of Corollary 3:

The well known Put-Call-Parity relationship for European options on nondividend-paying stocks reads for interest rate zero as follows
(compare for instance R. Gibson 1991):

$P(x,T\text{-}t,K) = C(x,T\text{-}t,K) - x + K.$ Hence Lemma 2 completes the proof.

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