Proper Discounting when Tax Payments are Postponed

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Summary

Asset/liability-managers in financial institutions as well as designers of insurance products for private and for corporate customers must take into account taxation. When investments, or cash flows respectively, are judged by their present values, then this amounts to, firstly, calculating cash flows after rather than before tax, and secondly, applying discount factors net of taxes. It ist not uncommon that tax payments are not due until one or more periods after tax liabilities are originated. Deriving the correct after tax flows in this case is trivial. However, it is not obvious how a pre-tax discount factor should be transformed into a post-tax discount factor. Simply multiplying by one minus the tax rate (as seems to be common practice) ignores the delay in payments. We present an approach which allows the derivation of proper post-tax discount factors for the case of postponed tax payments. For important cases we calculate explicit results.

Résumé

L´aspect des impôts doit être tenu compte aussi bien par les gérants des institutions financières que par ceux qui s´occupent de la réalisation des produits d´assurance pour la clientèle privée et commerciale. Notamment, les impôts sur le revenu ont des implications sur la calculation des valeurs actuelles. Pour cette calculation il est important, premièrement, d´escompter les cash flows après-taxe au lieu des cash flows pré-taxe, et, deuxièmement, d´utiliser un taux d´escompte après-taxe. Souvent les impôts viennent être payés une periode après la réalisation de la dette fiscale. Dans ce cas la soustraction des paiements fiscaux au niveau des cash flows est trivial. Pourtant ce n´est pas évident comment le taux d´escompte pré-taxe doit être transformé dans un taux d´escompte après-taxe. On voit souvent que le taux pré-taxe ist simplement multiplié par un facteur "1 moins taux d´imposition". Ceci ignore le décalage des paiments fiscaux. Nours présentons une methode qui permet la dérivation correcte des taux d´escompte après-taxe au cas ou les paiements fiscaus seront remis à plus tard. Pour certains cas importants nous avons calculé les résultats explicits.

Keywords

Taxation, present value, discount factor.

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1. Introduction

Taxes are an important matter for financial institutions in general and insurance companies in particular. For instance, insurance products derive much of their attractiveness for the customer from advantages stemming from tax-deductibility as compared to other financial products.[1] Nevertheless, asset/liability-managers in these institutions most often rely on pre-tax valuation methods and also do not fully use the information contained in the (possibly) non-flat term structure of interest rates.[2]

Applying any present value rule means comparing an investment with its opportunities. This is done by transforming the stream of cash flows of either an alternative investment or some means of financing[3] into a single figure, the discount rate. To integrate income taxation features into present value calculations, one has to take into account the tax payments which arise for the considered investment as well as for the alternatives. While the former leads to a post-tax stream of periodic cash flows the latter calls for an adjusted discount rate. It is common practice[4] to take an interest paying financial investment as opportunity and to assume a time invariant market interest rate i.[5] This amounts to the post-tax discount rate[6] being $i \cdot (1-s)$ where s is the marginal tax rate under consideration (also assumed to be constant over time).[7] This means that a flat post-tax term structure is used to discount the tax-corrected cash flows.

Frequently, tax liabilities are to be paid at times different from their origination. For example the German *Körperschaftsteuer* embodies time lags between cash flows and corresponding tax payments as well as between cash flows and corresponding tax advances (*Steuervorauszahlungen*).[8] In this paper we explicitly model the effects of tax postponement on investment evaluation.

While in the case of postponed tax payments deriving the correct post-tax stream of cash flows is basically trivial, it is far from being obvious how the opportunities' stream of tax-adjusted cash flows is to be transformed into a single figure, i.e. how we should determine a post-tax discount rate. Surprisingly enough, this question has been rarely

addressed in the past. Among the exceptions are the contributions by *Franks/Hodges* (1979) and *Jennergren* (1993).

We present an approach that is based on the duplication of cash flows and taxable incomes.[9] An analytical solution is being derived for our central case of a one-period delay and a flat term structure. Given time invariant tax rates, we show that in this case a flat pre-tax term structure transforms into a flat post-tax term structure as has been conjectured by *Franks* and *Hodges*[10]. Our approach also allows for non-flat (pre-tax) term structures[11] as well as for tax postponements of more than one period. While analytical solutions cannot always be derived, approximate numerical solutions generally can be computed for non-flat term structures and for delays of more than one period.

The remainder of the paper is organized as follows. In Section 2, we formalize our duplication problem. Sections 3 specifies the one-period tax delay and flat term structure case formally. The example in Section 4 illustrates this case. Section 5 discusses an extension of the results to longer tax delays (but still flat term structure cases). Finally, the results are summarized in Section 6.

2. The fundamental cash flow duplication problem

Suppose we want to value an n-period investment that is characterized by its price, its future cash flows and its impact on the firm's periodic income, which is subject to taxation. Moreover, the firm has n unlimited and linearly independent borrowing and lending opportunities that are characterized by the same kind of data. Any opportunities are feasible. They may consist of marketable securities with periodic interest payments. Since securities markets are free of riskless arbitrage opportunities, any n linearly independent securities can be chosen as duplication base if there are more than n (linearly independent) securities.

Formally, the valuation is being achieved by solving a set of linear equations

(1) $$\mathbf{K} \cdot \mathbf{x}_j = \mathbf{v}_j .$$

On the right hand side, $\mathbf{v}_j' = (-p_j, \mathbf{cf}_j', \mathbf{ew}_j')$, p_j represents investment j's price, $\mathbf{cf}_j$ denotes its periodic cash flows from t = 1 to n and $\mathbf{ew}_j$ denotes its impact on periodic taxable income from t = 1 to n. The left hand side is the product of a ($2n$+1 x $2n$+1) matrix **K** and a vector $\mathbf{x}_j$ where $\mathbf{x}_j$ indicates the portfolio equivalent to $\mathbf{v}_j$ and

(2) $$\mathbf{K} = \begin{pmatrix} 1 & \mathbf{p}' & \mathbf{0}' \\ \mathbf{0} & \mathbf{Z} & \mathbf{S} \\ \mathbf{0} & \mathbf{G} & \mathbf{E} \end{pmatrix}.$$

In **K**, the matrices **Z**, **S**, **G**, and **E** are all n x n. **0** is a vector of n zeros. The middle column of **K** contains information about the opportunities: **p** denotes the opportunities' price vector, the columns of **Z** contain their cash flows, and those of **G** their impacts on periodic taxable income. The right column of **K** describes the modalities of taxation: **E** is the identity matrix, and **S** contains the tax payment entries corresponding to the period indicated by 1 in the identity matrix . The first column in **K** indicates the numeraire of calculation, i.e. state prices will be expressed as present values. Equation (1) allows for a non-zero net present value of investment $\mathbf{v}_j$ which appears as the first element in $\mathbf{x}_j$.

The solution of the duplication problem can be found by solving (1) for $\mathbf{x}_j$. The inverse matrix $\mathbf{K}^{-1}$ is

(3) $$\mathbf{K}^{-1} = \begin{pmatrix} 1 & -\mathbf{p}'\tilde{\mathbf{Z}}^{-1} & \mathbf{p}'\tilde{\mathbf{Z}}^{-1}\mathbf{S} \\ \mathbf{0} & \tilde{\mathbf{Z}}^{-1} & -\tilde{\mathbf{Z}}^{-1}\mathbf{S} \\ \mathbf{0} & -\mathbf{G}\tilde{\mathbf{Z}}^{-1} & \mathbf{E}+\mathbf{G}\tilde{\mathbf{Z}}^{-1}\mathbf{S} \end{pmatrix} \text{ with } \tilde{\mathbf{Z}}=\mathbf{Z}-\mathbf{SG}.$$

$\mathbf{K}^{-1}$ exists if $\begin{pmatrix} \mathbf{Z} & \mathbf{S} \\ \mathbf{G} & \mathbf{E} \end{pmatrix}$ is regular. This is the case if and only if $\tilde{\mathbf{Z}}$ is regular. Multiplying (1) by $\mathbf{K}^{-1}$ yields

$$\mathbf{x}_j = \mathbf{K}^{-1}\mathbf{v}_j = \mathbf{K}^{-1}(-p_j, \mathbf{cf}_j, \mathbf{ew}_j).$$

The first of these equations is of particular interest. It states that the net present value equals the first row vector of $\mathbf{K}^{-1}$ multiplied by the investment's vector $\mathbf{v}_j$. This first row vector of $\mathbf{K}^{-1}$ contains the desired state prices:

(4) $$NPV_j = (1, \mathbf{q}', \mathbf{g}') \cdot (-p_j, \mathbf{cf}_j', \mathbf{ew}_j')'$$

where $\mathbf{q}' = -\mathbf{p}'\widetilde{\mathbf{Z}}^{-1}$ denotes the vector of discount factors applicable to pre-tax payments and $\mathbf{g}$ denotes the discount factors applicable to periodic taxable income. Hence, this discounting method values the gross cash flows and the taxable income streams. It can be transformed into a net discounting method. Note that $\mathbf{g}' = \mathbf{p}'\widetilde{\mathbf{Z}}^{-1}\mathbf{S} = -\mathbf{q}' \cdot \mathbf{S}$, implying

$$NPV_j = -p_j + \mathbf{q}' \cdot \mathbf{cf}_j - \mathbf{q}' \cdot \mathbf{S} \cdot \mathbf{ew}_j = -p_j + \mathbf{q}' \cdot (\mathbf{cf}_j - \mathbf{S} \cdot \mathbf{ew}_j).$$

This means that alternatively the net present value can be computed by discounting the tax adjusted stream of cash flows applying the discount factors $\mathbf{q}$.

The fact of tax postponement is being expressed by entries in $\mathbf{S}$ below the principal diagonal. As a matter of fact, if taxes are postponed for (at least) one period, cash flows in period n (or even earlier cash flows) usually have an impact on later periods' taxable income. This impact cannot be duplicated because the instruments to compensate the induced future tax payments themselves will have tax consequences in even later periods and so forth.[12] Generally, tax postponement imposes a fundamental obstacle to duplication based valuation methods. The impact on income after the nth period is ignored in our duplication approach. Hence the discount factors calculated in (3) are not necessarily exact.[13] Nevertheless, one can improve the precision of the discount factors (provided they converge) by increasing the time horizon n of the duplication framework.

3. One-period tax delay and flat term structure

We now present the simple case of a one year tax delay in which the opportunities consist of fixed income securities paying a time invariant market interest rate i. This section's formal analysis is illustrated by an example in Section 4.

We make the following assumptions:

(A1) At each point in time exactly one security matures.
(A2) All securities pay positive interest i each period and have a principal payment of 1 at the maturity date.
(A3) The securities' impact on taxable income consists of their interest payments.
(A4) There is a constant non-negative time-invariant marginal tax rate s that is less than 1.
(A5) Taxes on one period's income are paid at the end of the subsequent period.
(A6) The term structure is flat at i, that is the one-period market forward rates are identical and equal to i.
(A7) Every security's price is 1.

Assumptions (A1) and (A2) simplify the analytical derivations but could be relaxed. In absence of taxes, the securities' prices would be 1 since i is the appropriate discount rate and 1 is the securities' principal payment. Since the securities do not offer any depreciation allowances they can be said to be „tax neutral". It is well known that under tax neutrality, taxes leave the pre-tax present values (here: 1) unchanged.[14] Hence (A7) is in line with tax neutrality.

Let $\mathbf{\Delta}$ denote an upper „1" triangular matrix (on and above the principal diagonal the entries are 1, and 0 otherwise) and $\mathbf{\Lambda}_U$ be a lower „1" diagonal matrix (on the diagonal below the principal diagonal the entries are 1, and 0 otherwise). Furthermore let $\mathbf{1}$ denote the unity vector. From (A1), (A2), (A3), (A6), and (A7) it follows that

(5) $$\mathbf{Z} = \mathbf{E} + i\mathbf{\Delta},$$

(6) $$\mathbf{p} = -\mathbf{1},$$

(7) $$\mathbf{G} = i\mathbf{\Delta}.$$

Also from (A4) and (A5) one can conclude that

(8) $$\mathbf{S} = s\mathbf{\Lambda}_U.$$

We now want to look at the resulting discount factors found in **q** and **g** within the inverse matrix $\mathbf{K}^{-1}$. The following theorem is the central result of our paper and is proved in the appendix.

***THEOREM 1**: Given assumptions (A1) to (A7), the post-tax discount factors q_k are for all k=1,...*

(9) $$q_k = \left(\frac{1}{1+i_s^*}\right)^k$$

with the periodic post-tax interest rate

(10) $$i_s^* = \frac{i-1+\sqrt{(1+i)^2-4is}}{2}.$$ [15]

The discount factors applicable to periodic income are for all k=1,...

(11) $$g_k = -sq_{k+1}.$$

Thus, the post-tax term structure is flat at i_s^*. This is not surprising. An alternative and straightforward manner to compute i_s^* would be to correct a one-year security's stream of cash flows for tax payments and solve the resulting polynomial

$$1 = \frac{1+i}{1+i_s^*} - \frac{i \cdot s}{\left(1+i_s^*\right)^2}$$

(12) $$\Leftrightarrow\ i_s^* = i - \frac{i \cdot s}{1+i_s^*}$$

for i_s^* as has been suggested by *Franks* and *Hodges*[16]. Nevertheless one has to *assume* the flatness of the post-tax term structure to obtain this result[17], whereas we are able to infer it as a consequence of our other assumptions (see Appendix).

4. An illustrative example

In this section we present an example that illustrates our method as well as the derived post-tax discounting rule for the tax-adjusted cash flows.

Suppose a firm considers two alternative financial investments. These investments are assumed to be riskless and do not give rise to depreciation. Whereas investment A's principal is paid back in two equal installments in t=1 and t=2, investment B's principal is paid back in a lump sum at t=2. The firm's opportunities are fixed income securities paying an annual interest of 9,5% independent of their maturity. The firm's marginal tax rate is 50%, taxes are to be paid one year later.

Table 1 contains the data where $\mathbf{v_A}$ and $\mathbf{v_B}$ denote the investments' vectors of cash flows and impacts on taxable income. The matrix **K** contains the securities' vectors and the tax modality vectors.

Table 1: Duplication example with one year tax delay and n=5

	Matrix **K**											Alt. A	Alt. B
	PV	s_1	s_2	s_3	s_4	s_5	tm_1	tm_2	tm_3	tm_4	tm_5	$\mathbf{v_A}$	$\mathbf{v_B}$
cf_0	1	-1	-1	-1	-1	-1	0	0	0	0	0	-100	-100
cf_1	0	1,095	0,095	0,095	0,095	0,095	0	0	0	0	0	60	0
cf_2	0	0	1,095	0,095	0,095	0,095	0,5	0	0	0	0	60	125,1
cf_3	0	0	0	1,095	0,095	0,095	0	0,5	0	0	0	0	0
cf_4	0	0	0	0	1,095	0,095	0	0	0,5	0	0	0	0
cf_5	0	0	0	0	0	1,095	0	0	0	0,5	0	0	0
inc_1	0	0,095	0,095	0,095	0,095	0,095	1	0	0	0	0	10	0
inc_2	0	0	0,095	0,095	0,095	0,095	0	1	0	0	0	10	25,1
inc_3	0	0	0	0,095	0,095	0,095	0	0	1	0	0	0	0
inc_4	0	0	0	0	0,095	0,095	0	0	0	1	0	0	0
inc_5	0	0	0	0	0	0,095	0	0	0	0	1	0	0

The method described yields the solutions $\mathbf{x_A}$ and $\mathbf{x_B}$. The first elements in $\mathbf{x_A}$ and $\mathbf{x_B}$ indicate the investments' net present values. Investment A should be preferred to investment B since its net present value $NPV_{A5} = 2,74$ is greater than $NPV_{B5} = 2,67$.[18]

Table 2 : Solution to duplication example with one year tax delay and n=5

x_A	x_B	Inverse Matrix K^{-1}										
2,74	2,67	1	0,9526	0,9075	0,8644	0,822	0,7506	-0,4537	-0,4322	-0,411	-0,3753	0
50,24	-9,75	0	0,9095	-0,0862	-0,0821	-0,0781	-0,0713	0,0431	0,0411	0,039	0,0357	0
54,89	119,30	0	0,0412	0,9487	-0,0489	-0,0465	-0,0424	-0,4744	0,0244	0,0232	0,0212	0
-2,28	-6,56	0	0,0019	0,0429	0,9504	-0,0472	-0,0431	-0,0215	-0,4752	0,0236	0,0215	0
-0,10	-0,30	0	0,0001	0,0019	0,043	0,9506	-0,0452	-0,001	-0,0215	-0,4753	0,0226	0
0	-0,01	0	0	0,0001	0,002	0,0431	0,9526	0	-0,001	-0,0216	-0,4763	0
0,24	-9,75	0	-0,0905	-0,0862	-0,0821	-0,0781	-0,0713	1,0431	0,0411	0,039	0,0357	0
5,01	14,42	0	-0,0041	-0,0944	-0,0899	-0,0855	-0,0781	0,0472	1,045	0,0428	0,039	0
0,23	0,65	0	-0,0002	-0,0043	-0,0946	-0,0899	-0,0821	0,0021	0,0473	1,045	0,0411	0
0,01	0,03	0	0	-0,0002	-0,0043	-0,0944	-0,0862	0,0001	0,0021	0,0472	1,0431	0
0	0	0	0	0	-0,0002	-0,0041	-0,0905	0	0,0001	0,002	0,0452	1

As has been pointed out in Section 2, the discount factors and hence the derived net present values are only approximately correct. Alternatively, the post-tax discounting rule for the tax-adjusted cash flows can be applied. Investment A generates a taxable income of 10 in t=1 and t=2. The resulting taxes amount to 5 and are paid in t=2 and t=3 so that the tax-adjusted cash flows from investment A are (-100 / 60 / 55 / -5). Similarly, investment B generates a taxable income of 25,1 in t=2. The resulting taxes (12,55) are paid in t=3 so that the tax adjusted cash flows from investment B are (-100 / 0 / 125,1 / -12,55).

Computation of the periodic post-tax interest rate yields $i_s^* = 4{,}975\%$. Discounting the investments' tax-adjusted cash flows at this rate we obtain $NPV_A = 2{,}74$ and $NPV_B = 2{,}67$. Note that these present values are (almost) the same as those resulting from the duplication approach with cut-off horizon for deferred taxes at $n = 5$.

It seems to be common practice (for the sake of simplicity) to discount the tax-adjusted cash flows at the post-tax interest rate that would result in the absence of tax delays, that is $i_s = (1-s)\cdot i = 4{,}75\%$.[19] Applying this rate to the investments' tax-adjusted cash flows yields $NPV_A = 3{,}05$ and $NPV_B = 3{,}09$. This suggests that - contrary to the correct calculation - investment B should be preferred to investment A. Generally, discounting

by i_s instead of i_s^* systematically overprices pure investments[20] because $i_s^* > i_s$. The impact of altering the discount rate on present value is higher for later than for earlier periods since discount factor computation involves raising a constant to the power of the period under consideration. As investment B has more „weight" in the later period it benefits more from the reduction in the discount rate. Thus, our example shows that the use of no-delay post-tax interest rates may be misleading.

5. More than one-period tax delay and flat term structure

The duplication scheme described above can easily be modified to represent more than one-period tax postponements. This is achieved by simply shifting downward any entries of **S** according to the number of periods for which tax payments are delayed.

Numerical solutions to the modified duplication scheme can then be found for any reasonable situation. Nevertheless an analytical solution, i.e. the matrix inversion, may be cumbersome. Computation of the i_s^*-formula is difficult and may often even be impossible. Additionally, the flatness of the post-tax term structure needs to be proved in every single case.

If we *assume* flat pre- and post-tax term structures, however, we can use our duplication approach to derive further formal results. Introducing

(A5′) Taxes on one period's income are paid d periods later (d=0, 1, ...).
(A8) The post-tax term structure is flat with interest rate $i_s^* > 0$.

we can state:

THEOREM 2: Given assumptions (A1) to (A4), (A5'), and (A6) to (A8), the post-tax discount factors q_k are for all k=1,...

(9) $$q_k = \left(\frac{1}{1+i_s^*}\right)^k$$

and the discount factors applicable to periodic income are for all k=1,...

(11') $$g_k = -sq_{k+d}.$$

The periodic post-tax interest rate i_s^* is the unique positive fixed point of the function

(13) $$h(x) = i \cdot \left[1 - \frac{s}{(1+x)^d}\right]$$

and it is (for i and s constant, and d=0, 1, ...) strictly increasing from $i_s = i \cdot (1-s)$ to its limit i.

This theorem is also proved in the Appendix. It has a number of important messages. Firstly, there always exists a discount rate which correctly embodies the tax postponement and this rate is also unique. Secondly, it can be calculated from a rather simple function by any standard fixed point algorithm. Thirdly, for increasing tax delay the proper discount rate increases monotonically from the no-delay case, $i_s^* = i_s = i \cdot (1-s)$, to the no-tax case, $i_s^* = i$.

6. Conclusions

We have presented a duplication approach to evaluate investment alternatives that are subject to income taxes. Applying some linear algebra made it obvious that our method is equivalent to present value calculation of tax-adjusted cash flows when the appropriate discount rate is being used. The tax-adjusted discount rate has to reflect that not only the income from the investment alternatives but also the income resulting from opportunities is subject to tax delay. Simply applying the standard no-delay discount rate may yield wrong investment decisions.

We then focussed on the one-period delay and flat term structure case. A flat pre-tax term structure at i transforms into a flat post-tax term structure at $i_s^* = \frac{i-1+\sqrt{(1+i)^2-4is}}{2}$. Application of the standard no-delay discount rate $i_s = i \cdot (1-s)$ leads to pure investments' present values that are systematically too high since $i_s^* > i_s$. The resulting effects are not negligible as was shown by a simple example.

For flat pre-tax term structures, assuming also flat post-tax term structures, we have derived a simple condition from which the correct i_s^* can be calculated for arbitrary delays d = 0,1, ... numerically. For non-flat term structures, approximate numerical solutions can be found.

A more detailed analysis of the actual tax code seems to be important for future research. One question that should be addressed within our framework is how far tax advances compensate the effects of tax postponement.

Appendix

PROOF OF THEOREM 1:

Starting from (3) and using (5), (8) , and (7), it follows that

$$\tilde{\mathbf{Z}} = \mathbf{E} + i\boldsymbol{\Delta} - is\boldsymbol{\Lambda}_U\boldsymbol{\Delta} = \mathbf{E} + i\cdot(\mathbf{E} - s\boldsymbol{\Lambda}_U)\boldsymbol{\Delta}.$$

Let $\boldsymbol{\Lambda}_O$ denote an upper „1" diagonal matrix (on the diagonal above the principal diagonal the entries are 1, and 0 otherwise). Since $\boldsymbol{\Delta}^{-1} = \mathbf{E} - \boldsymbol{\Lambda}_O$, the above equation can be transformed to

$$\tilde{\mathbf{Z}} = \left[(\mathbf{E} - \boldsymbol{\Lambda}_O) + i(\mathbf{E} - s\boldsymbol{\Lambda}_U)\right]\boldsymbol{\Delta} = \left[(1+i)\mathbf{E} - \boldsymbol{\Lambda}_O - is\boldsymbol{\Lambda}_U\right]\boldsymbol{\Delta}.$$

The product of regular matrices is regular. It can easily be verified that the matrix $\left[(1+i)\mathbf{E} - \boldsymbol{\Lambda}_O - is\boldsymbol{\Lambda}_U\right]$ is regular. The inverse matrix $\tilde{\mathbf{Z}}^{-1}$ is therefore

(14) $$\tilde{\mathbf{Z}}^{-1} = \boldsymbol{\Delta}^{-1}\left[(1+i)\mathbf{E} - \boldsymbol{\Lambda}_O - is\boldsymbol{\Lambda}_U\right]^{-1} = (\mathbf{E} - \boldsymbol{\Lambda}_O)\left[(1+i)\mathbf{E} - \boldsymbol{\Lambda}_O - is\boldsymbol{\Lambda}_U\right]^{-1}.$$

Let $\mathbf{e}_1$ denote the first unit row vector. From the definition of $\mathbf{q}'$ we can derive, observing (6) and (14),

(15) $$\mathbf{q}' = \mathbf{1}'\cdot(\mathbf{E} - \boldsymbol{\Lambda}_O)\left[(1+i)\mathbf{E} - \boldsymbol{\Lambda}_O - is\boldsymbol{\Lambda}_U\right]^{-1}$$

$$= (\mathbf{1}'\cdot\mathbf{E} - \mathbf{1}'\cdot\boldsymbol{\Lambda}_O)\left[(1+i)\mathbf{E} - \boldsymbol{\Lambda}_O - is\boldsymbol{\Lambda}_U\right]^{-1}$$

$$= \mathbf{e}_1\cdot\left[(1+i)\mathbf{E} - \boldsymbol{\Lambda}_O - is\boldsymbol{\Lambda}_U\right]^{-1}.$$

An equivalent set of equations is

(16) $$\mathbf{q}'\cdot\left[(1+i)\mathbf{E} - \boldsymbol{\Lambda}_O - is\boldsymbol{\Lambda}_U\right] = \mathbf{e}_1.$$

Giving up the matrix notation,

(16a) $$q_1(1+i) - q_2 is = 1,$$

(16b) $$-q_{k-1} + q_k(1+i) - q_{k+1}is = 0 \quad \text{for } 2 \le k \le n-1,$$

(16c) $$-q_{n-1} + q_n(1+i) = 0,$$

for every fixed n.

We now continue to prove equation (9), the formula for the post-tax discount factors q_k, and equation (10) - for the post-tax interest rate.

We define the no-arbitrage one period forward factor as $f_k = \frac{q_{k-1}}{q_k}$.

Hence, (16b) can be reexpressed as

(16b') $$f_k = 1 + i - \frac{is}{f_{k+1}} \quad \text{for } 2 \le k \le n-1.$$

Similarly, (16c) transforms to

(16c') $$f_n = 1 + i.$$

(This indicates that the last period's tax consequence is omitted for fixed n.)[21]

Furthermore, we define the number f as $f = \frac{1+i+\sqrt{(1+i)^2 - 4is}}{2}$.

We now prove that f_k is a strictly decreasing sequence for $k=n,n-1,\ldots,2$, and that f_k has f as a lower bound. Together this implies that f_k converges.

Since $i > 0$ is assumed and $0 < s < 1$, it follows that

(17) $$f_n = 1 + i > f > 0.$$

LEMMA: For all $k \in \{2,\ldots,n-1\}$

(18) $$f_{k+1} > f \Rightarrow f_{k+1} > f_k > f.$$

holds.

PROOF OF THE LEMMA:

According to (16b') for $k \in \{2,\ldots,n-1\}$

$$f_{k+1} - f_k = f_{k+1} - (1+i) + \frac{is}{f_{k+1}} = \frac{1}{f_{k+1}} \cdot \left[\left(f_{k+1} - \frac{1+i}{2} \right)^2 - \left(\frac{1+i}{2} \right)^2 + is \right].$$

f can be substituted for f_{k+1} in the term in brackets. Since $f_{k+1} > f > \frac{1+i}{2}$, we obtain a lower bound of $f_{k+1} - f_k$:

$$f_{k+1} - f_k > \frac{1}{f_{k+1}} \cdot \left[\frac{(1+i)^2 - 4is}{4} - \left(\frac{1+i}{2}\right)^2 + is \right] = 0 .$$

This proves $f_{k+1} > f_k$. From $f_{k+1} > f$, one can also derive estimate a lower bound for f_k from (16'b) by substituting f for f_{k+1}:

$$f_k > 1+i - \frac{is}{\frac{1+i+\sqrt{(1+i)^2 - 4is}}{2}} = \frac{1+i}{2} + \frac{(1+i)\left(1+i+\sqrt{\cdot}\right) - 4is}{2\left(1+i+\sqrt{\cdot}\right)}$$

$$= \frac{1+i}{2} + \frac{(1+i)\left(1+i+\sqrt{\cdot}\right) - \left(1+i+\sqrt{\cdot}\right)\sqrt{\cdot} + \left(1+i+\sqrt{\cdot}\right)\sqrt{\cdot} - 4is}{2\left(1+i+\sqrt{\cdot}\right)}$$

$$= \frac{1+i}{2} + \frac{\left(1+i-\sqrt{\cdot}\right)\left(1+i+\sqrt{\cdot}\right) + \left(1+i+\sqrt{\cdot}\right)\sqrt{\cdot} - 4is}{2\left(1+i+\sqrt{\cdot}\right)}$$

$$= \frac{1+i}{2} + \frac{(1+i)^2 - \left[(1+i)^2 - 4is\right] + \left(1+i+\sqrt{\cdot}\right)\sqrt{\cdot} - 4is}{2\left(1+i+\sqrt{\cdot}\right)} = \frac{1+i}{2} + \frac{\sqrt{(1+i)^2 - 4is}}{2} = f .$$

This proves $f_k > f$, and thus the Lemma. •

From (17) and the Lemma it follows that $f_{k+1} > f_k > f \; \forall \; k \in \{2,\ldots,n-1\}$. For fixed n, the sequence's values $f_n,\ldots, f_2$ are strictly decreasing, starting from $1+i$, and strictly greater than f. To avoid mistakes resulting from the tax payment cut-off in the last period, we would have to consider infinite dimensional matrices. Since the time of the cut-off is arbitrary we can choose n arbitrarily high. We then can conclude from the Lemma that the sequence's values for every fixed k converge with increasing n. (For example, f_2 is the forth value starting from n=5, but the ninth value starting from n=10. Starting from n=10, f_7 has the same value as f_2, when computation starts from n=5. A more precise notation would therefore require an additional subscript n.) This implies that the post-tax term structure is flat in the limit.

For each $k \geq 2$ the limit l for $n \to \infty$ can be calculated from (16b')

(19) $$l = 1 + i - \frac{is}{l}.$$

Rearranging shows that the limit is equal to the lower bound f.[22] Choosing an ever greater n moves every f_k closer to the limit and improves the precision of calculation.

We now know that for k=2,...

$$q_k = \frac{q_{k-1}}{f}$$

where $f = 1 + i_s^*$ with

$$i_s^* = \frac{i - 1 + \sqrt{(1+i)^2 - 4is}}{2}.$$

Equation (16a) can therefore be rearranged as

$$q_1(1+i) - q_1 \frac{is}{f} = q_1\left(1 + i - \frac{is}{f}\right) = 1,$$

which - according to equation (19) (with l instead of f) - yields

(20) $$q_1 = \frac{1}{f} = \frac{1}{1 + i_s^*}.$$

We now have proved equations (9) and (10).

From (3), (4), and (8) we can conclude that

$$\mathbf{g}' = -\mathbf{q}' \cdot s\mathbf{\Lambda}_U.$$

It is evident that

$$g_k = -sq_{k+1},$$

i.e. equation (11) holds. Note that for fixed n, the discount factor corresponding to the last period's taxable income is zero: $g_n = 0$.

PROOF OF THEOREM 2:

First of all, notice that (A8) implies

(9) $$q_k = \left(\frac{1}{1+i_s^*}\right)^k$$

with some i_s^* (if there exists such a positive i_s^*). Furthermore, $\mathbf{g}' = -\mathbf{q}' \cdot \mathbf{S}$ implies

(11') $$g_k = -sq_{k+d}.$$

Throughout the remainder of this proof we will ignore the trivial cases s=0 (implying $i_s^* = i$) and d=0 (implying $i_s^* = i \cdot (1-s)$), i.e. s>0, d>0, and i_s^*>0 are assumed from now on.

The opportunity with maturity 1 (like all the others) has a net present value of 0. Therefore, by (4),

$$0 = -1 + q_1 \cdot (1+i) + g_1 \cdot i$$

must hold which can be written, using (9) and (11), as

$$1 = \frac{1+i}{1+i_s^*} - \frac{i \cdot s}{\left(1+i_s^*\right)^{1+d}}.$$

Upon rearranging terms, this becomes

(21) $$i_s^* = i - \frac{i \cdot s}{\left(1+i_s^*\right)^d}$$

which is a generalization of the *Franks/Hodges* idea shown in (12). It is obvious that the claim i_s^* is a fixed point of h (cf. (13)). Existence of a unique fixed point i_s^*>0 is easily shown via the following steps:

- h is strictly increasing in i_s^* and continuous,
- $h(0) = i \cdot [1-s] > 0$,
- $h(x) = i \cdot \left[1 - \frac{s}{(1+i)^d}\right] < i$.

Likewise, the remaining assertions are simply a matter of algebra. •

References

Atkinson, A.B./Stiglitz, J.E (1980), Lectures On Public Economics.

Brealey, R.A./Myers, S.C. (1991), Principles of Corporate Finance, 4th edition.

Buhl, H.U. (1993), Finanzanalyse von Entscheidungsalternativen bei der Software-Vertragsgestaltung, in: Zeitschrift für betriebswirtschaftliche Forschung **45**, pp. 911-932.

Buhl, H.U./Sandbiller, K./Will, A. (1994), Vorteilhaftigkeit und Systemunterstützung kombinierter Allfinanzprodukte, Discussion Paper Nr. 48/1994, Professur für BWL-Wirtschaftsinformatik, Universität Gießen.

Dedner, M./Günther, R. (1980), Zur Ertragssteuerplanung unter Berücksichtigung des zeitlichen Auseinanderfallens von Steuerschuld und Steuerzahlung auf der Grundlage modellexogener Daten, in: Zeitschrift für Betriebswirtschaft **50**, pp. 853-874.

Dietrich, J.K. (1996), Financial Services and Financial Institutions.

Franks, J.R./Hodges, S.D. (1979), The Role of Leasing in Capital Investment, National Westminster Bank Quarterly Review, pp. 20-31.

Gründl, H. (1995), Anwendungsgebiete und Grenzen einer marktzinsorientierten Ergebnisrechnung in der Lebensversicherung, in: Zeitschrift für die gesamte Versicherungswirtschaft **84**, pp. 229-246.

Hirshleifer, J. (1958), On the Theory of Optimal Investment Decision, in: Journal of Political Economy **66**, pp. 329-352.

Jennergren, L.P. (1993), Discounting when Taxes are Paid One Year Later: A Finance Application of Linear Programming Duality, in: Flavell, R. (Ed.), Modelling Reality and Personal Modelling, pp. 178-188.

Schneider, D. (1990), Investition, Finanzierung und Besteuerung, 6th edition.

Sick, G.A. (1990), Tax-Adjusted Discount Rates, in: Management Science **36**, pp. 1432-1450.

Stiglitz, J.E. (1973), Taxation, Corporate Financial Policy, and the Cost of Capital, in: Journal of Public Economics **2**, pp. 1-34.

Notes

1 See *Dietrich* (1996), pp. 357-358; *Buhl/Sandbiller/Will* (1994).

2 See *Gründl* (1995).

3 If the discount rate is computed from means of financing-data, it is usually referred to as the cost of capital. The cost of capital depends in general on the firm's financial policy and corporate and personal tax rates as well as on taxation features such as interest deductibility. For a comprehensive treatment of cost of capital questions see *Stiglitz* (1973) and *Atkinson/Stiglitz* (1980), ch. 5, pp. 128-159.

4 See for example *Hirshleifer* (1958), p. 331; *Brealey/Myers* (1991), pp. 11-24.

5 See *Hirshleifer* (1958), p. 344.

6 See *Sick* (1990), p. 1436; *Schneider* (1990), p. 189.

7 This may be either the investor's personal or the firm's corporate tax rate depending on the firm's financial policy and the investor's tax bracket. See *Atkinson/Stiglitz* (1980), p.147.

8 For a detailed discussion see *Dedner/Günther* (1980).

9 *Jennergren* (1993) has suggested a very similar approach. He uses a linear programming technique and differential equations to derive dual variables which represent the desired disount factors. The differences between *Jennergren's* and our approach will be pointed out below.

10 See *Franks/Hodges* (1979), p.30.

11 As *Gründl* (1995), p. 230, points out it would be particularly interesting to calculate life insurances on the basis of the actual market term structure instead of assuming a constant discount rate.

12 One way to cut off this infinite stream of induced payments is to introduce cash holdings, i.e. zero interest rate security holdings, as duplication devices so that the ignored tax payment becomes zero (see *Jennergren* (1993), p. 181).

13 *Jennergren* (1993) classifies the resulting set of discount factors as correct and calls a method based on a limiting argument on these discount factors only an „approximate valuation rule" (p. 184). We regard our discount factors only as approximately correct and in certain cases a limit of these approximately correct disount factors yields an exact valuation rule, as will be pointed out in the proof (see Appendix).

14 See *Schneider* (1990), p. 189. For a discussion of tax neutrality see for example *Atkinson/Stiglitz* (1980), pp. 145.

15 Note that for s=0 it follows that $i_s^* = i$, the no-tax case.

16 See *Franks/Hodges* (1979), p.30.

17 Furthermore, for delays of more than one period there may exist no analytical solution for the resulting polynomial. See also below section 5.

18 The „5" in the subscripts denotes the cut-off horizon for deferred taxes.

19 See for example *Buhl* (1989), p. 928.

20 This is true if „pure investments" comprise only positive cash flows after the initial negative cash flow (expenditure): See also *Schneider* (1990), p. 26.

21 Following the approach in *Jennergren* (1993), f_n would be 1, since he sets last year's interest rate equal to zero.

22 The second solution to the quadratic equation is smaller than f and can therefore be neglected according to the Lemma.