An Actuarial Approach to Determine the Required Capital for Portfolios of Options with Default Risk

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Abstract

We adress the problem of calculating the required risk based capital (RBC) of a portfolio of options subject to default risk. In order to solve this problem, an actuarial approach is adopted in specifying a (*Markov*) default process, modelling the (discounted) credit loss and finally calculating RBC in a way to keep the loss probability under control. As an alternative we propose a calculation of RBC restricting the expected excess value of discounted credit loss.

Résumé

Nous analysons le problème de la détermination du montant du capital basé du risque (RBC) qu'il faut retenir pour une portefeuille des options qui sont vulnerable au risque de crédit. Afin de résoudre ce problème, une approche actuarielle est adoptée par la spécification d'un processus de solvabilité *markovien* modelisant la perte de crédit discontée et finalement calculant le capital nécessaire d'une manière garantassant que la probabilité d'une perte est bien contrôlée. Comme alternative nous proposons le calcul du capital nécessaire de manière que l'espérance mathématique de la valeur d'excès de la perte de crédit discontée est limité à un certain niveau.

Keywords

Default risk, risk based capital, discounted credit loss, shortfall probability, shortfall expectation, return on risk based capital.

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1 Introduction

The present paper adresses the problem of calculating the required risk based capital (*RBC*) of a portfolio of options subject to default risk. In order to achieve this objective[1)] an actuarial approach is adopted in specifying a (*Markov*) default process, modelling the (discounted) credit loss and finally calculating *RBC* in a way to keep the loss probability under control, i.e. maintaining a tolerance level ε. A small simulation study illustrates the approach proposed. As an alternative a control criterion is adopted which amounts to restricting the expected excess value of discounted credit loss.

2. Modelling the Discounted Credit Loss Distribution

2.1 The Default Process

Let $I(t) = I_R(t)$ denote the *default process* of a counterparty belonging to rating class R. The deafult process is modelled as a time-discrete inhomogeneous two-state *Markov* chain $\{I(t);\ t = 0, 1,\ldots\}$. The random variable $I(t)$ indicates the status of solvency of the counterparty at time t. We have

$$I(t) = \begin{cases} 1 & \text{counterparty is insolvent in } t \\ 0 & \text{else.} \end{cases} \tag{2.1}$$

The process starts with $I(0) \equiv 0$. In addition we assume, that state 1 is an absorbing state of the chain, i.e. there is no probability of recovering from a default. The one-period transition probabilities of the process are given by

$$\begin{aligned} q_t &:= Pr[I(t) = 1 \mid I(t-1) = 0], \\ p_t &:= Pr[I(t) = 0 \mid I(t-1) = 0] = 1-q_t\ . \end{aligned} \tag{2.2}$$

We call q_t the *one-period default probability*. Let now $Q_t := Pr[I(t) = 1]$ denote the *cumulative default probability*. We then have

$$1 - Q_t = Pr[I(t) = 0] = \prod_{j=1}^{t} p_j \, , \quad t \geq 1 \, . \tag{2.3}$$

On the other hand we have

$$1 - q_t = \frac{1 - Q_t}{1 - Q_{t-1}} \, . \tag{2.4}$$

Central to the analysis of the default process of a counterparty is the (here: integer valued) *default time* $\tau = \tau_R$, a stopping time of the process. The default time is the time of the first passage from state 0 to state 1 and we have

$$\tau = \min_{t \geq 1} \{t; \, I(t) = 1 \mid I(0) = 0\} \, . \tag{2.5}$$

The first passage probabilities $f_t = Pr(\tau = t)$, $t \geq 1$ are given by

$$\begin{aligned} Pr(\tau = t) &= Pr[I(t) = 1, \, I(t) = 0 \text{ for } 1 \leq j < t \mid I(t) = 0] \\ &= \left[\prod_{j=1}^{t-1} p_j \right] q_t \, . \end{aligned} \tag{2.6}$$

In addition, we obviously have

$$Q_t = \sum_{j \leq t} f_j \, . \tag{2.7}$$

For the purpose of simulation the introduction of an additional quantity is useful. Let $J(t) := I(t) - I(t\text{-}1)$, then we have

$$J(t) = \begin{cases} 1 & \tau = t \\ 0 & \text{else} \end{cases} \tag{2.8}$$

and $Pr[\, J(t) = 1] = f_t$, the corresponding first passage probability.

For the purpose of a statistical identification of the default process we use the results of *Lucas* (1995). *Lucas* calculates cumulative default probabilities $Q_t = Q_t(R)$ depending on rating class R and period t on the basis of data of *Moody's*

Investor Service on occurrences of default and changes of credit quality. $Q_t(R)$ gives the cumulative probability of default until time t of a counterparty belonging to rating class R at time 0. Table 1 gives the cumulative default probabilities as calculated by *Lucas*.

	1	2	3	4	5
Aa3	0,04	0,08	0,15	0,24	0,37
A3	0,09	0,28	0,59	1,01	1,56
Baa3	0,75	1,73	2,97	4,46	6,17
Ba3	4,27	9,02	13,99	18,98	23,88
B1	6,80	13,73	20,49	26,91	32,87

Table 1: Cumulative default probabilities of selected rating classes in per cent

Table 2 gives the corresponding one-period default probabilities calculated on the basis of (2.4).

	1	2	3	4	5
Aa3	0,04	0,04	0,07	0,09	0,13
A3	0,09	0,19	0,31	0,42	0,56
Baa3	0,75	0,99	1,26	1,54	1,79
Ba3	4,27	4,96	5.46	5,8	6,05
B1	6,80	7,40	7,70	8,20	8,90

Table 2: One-period default probabilities of selected rating classes in per cent

2.2 The Amount of Credit Loss

The credit exposure of a derivative product transaction at any time over the life of the transaction is equal to the maximum of zero and its mark-to-market value (which is assumed to be the *cost of replacing* a transaction if the counterparty defaults on an in-the-money position). As we only consider option positions in this paper this means that only long positions in options are subject to credit risk and - because the price process of an option is always non-negative - the replacement cost is identical to the price of the option at the time of default.

2.3 Discounted Credit Loss

2.3.1 The One-Counterparty / One-Option Case

We fix a counterparty with rating R and we concentrate on the case of one option position (a long call, a long put or a long position in an exotic option) on one unit of the underlying position, which will be a stock or a stock index in the framework of the present paper. Let T denote the expiration date of the option and $\{OP(t);\ 0 \leq t \leq T\}$ the price process of the option.

The central quantity for evaluating credit loss exposure now is the discounted credit loss $DCL = DCL(R)$ given by

$$DCL = \begin{cases} \exp(-r\tau)\ OP(\tau), & \tau \leq T \\ 0, & \tau > T. \end{cases} \tag{2.9}$$

The discounted credit loss denotes the discounted replacement cost in case of default of the fixed counterparty. For the purpose of keeping the definition simple, we have worked with a flat term structure of interest rates (in a time continuous setting). This can easily be generalized and the term structure used should be commensurate with the option pricing model applied.

Employing quantitiy $J(t)$ from (2.8) an equivalent expression for the discounted credit loss is

$$DCL = \sum_{t=1}^{T} \exp(-rt)\ J(t)\ OP(t)\ , \tag{2.10}$$

which is more suitable for the purpose of simulation.

To prepare a further point in the analysis we are interested in $Pr(DCL = 0)$, as already from (2.9) it is obvious, that DCL possesses a mass point in $x = 0$. There are two mutually exclusive events, which imply $DCL = 0$:

1) $\tau > T$

2) $\tau = T$ and the option expires worthless.

If we denote the probability that the option expires worthless by w_T we obtain

$$\begin{aligned} Pr(DCL = 0) &= Pr(\tau > T) + Pr(\tau = T)\ w_T \\ &= 1 - Q_T + Pr(\tau = T)\ w_T \\ &= 1 - \sum_{t \le T} f_t + f_T\ w_T. \end{aligned} \tag{2.11}$$

In case of e.g. a put option with exercise price X we have $w_T = Pr(S_T > X) = 1 - F_T(X)$, where S_T is the price of the underlying at time T and F_T is the distribution function of S_T.

2.3.2 The Multi-Counterparty / Multi-Option Case

The general case can be modelled as follows. Let there be N rating classes R_n, $n = 1,\ldots, N$. In rating class R_n there are $I(n)$ counterparties $i_n = 1,\ldots, I(n)$. The default time of counterparty i_n is denoted by τ_{i_n}. With counterparty i_n there are $j(i_n)$ long option positions of different type, $j(i_n) = 1,\ldots, J(i_n)$. These option positions have volume a_{ij} (number of underlyings in one option contract), expiration dates T_{ij} and price processes $\{OP_{ij}(t);\ 0 \le t \le T_{ij}\}$. The discounted credit loss of one particular position is

$$DCL_{ij} = \begin{cases} \exp(-r\tau_{i_n})\, a_{ij}\, OP_{ij}\,(\tau_{i_n}), & \tau_{i_n} \leq T_{ij} \\ 0, & \tau_{i_n} > T_{ij} \end{cases} \tag{2.12}$$

The total discounted credit loss *TDCL* of the considered portfolio of options then is given by

$$TDCL = \sum_{n=1}^{N} \sum_{i_n=1}^{I(n)} \sum_{j(i_n)=1}^{J(i_n)} DCL_{ij} . \tag{2.13}$$

3 Required Risk Based Capital

3.1 Approach I: Controlling the Probability of Discounted Credit Loss

3.1.1 The One-Counterparty / One-Option Case

A traditional risk theoretical approach to determine solvency capital is to restrict the loss probability. This is essentially the idea that we translate to the present case. If *RBC* denotes the risk based capital to be determined the postulated requirement is $Pr(DCL > RBC) \leq \varepsilon$. The probability that the discounted replacement costs exceed the available risk based capital should be under control and small. We present a slightly modified approach, which is virtually identical, but can be directly generalized (§ 3.2).

We define the discounted return on risk based capital (*RORBC*) by

$$RORBC = \frac{RBC - DCL}{RBC} = 1 - \frac{DCL}{RBC} \tag{3.1}$$

and choose as control criterion

$$Pr(RORBC < 0) = \varepsilon, \tag{3.2a}$$

i.e. a negative *RORBC* may only occour with a small probability ε. As $Pr(RORBC < 0) = Pr(RBC - DCL < 0) = Pr(DCL > RBC)$, (3.2a) is equivalent to

$$Pr(DCL > RBC) = \varepsilon, \tag{3.2b}$$

the capital requirement criterion as mentioned above. Let $F_{DCL}(x)$ denote the cumulative distribution function of DCL. If we define as usually $F_{DCL}^{-1}(y) := \inf\{x; F_{DCL}(x) \geq y\}$, then we have

$$RBC = F_{DCL}^{-1}(1-\varepsilon), \tag{3.3}$$

i.e. RBC is the $(1-\varepsilon)$-quantile of the distribution of DCL.

3.1.2 The Multi-Counterparty / Multi-Option Case

The general case is as follows. Let $TDCL$ be the total discounted credit loss according to (2.13) with distribution function $F_{TDCL}(x)$. Then the total risk based capital $TRBC$ required is

$$TRBC = F_{TDCL}^{-1}(1-\varepsilon). \tag{3.4}$$

3.1.3 The One-Counterparty Case: Zero Risk Based Capital

Beginning with the one-option case it is easy to see in which cases the required risk based capital amounts to zero. As $Pr(DCL > RBC) \leq \varepsilon$ is equivalent to $Pr(DCL \leq RBC) \geq 1-\varepsilon$ and there is a probability mass of DCL in $x = 0$ according to (2.11), we have the relation

$$RBC = 0 \quad \Leftrightarrow \quad Pr(DCL = 0) \geq 1-\varepsilon. \tag{3.5}$$

As $Pr(DCL = 0) = Pr(\tau > T) + Pr(\tau = T)\, w_T$, a *sufficient* criterion - which is independent of the type of option contract (but on its expiration date) and only is depending on the default process of the counterparty - for this is:

$$Pr(\tau > T) \geq 1-\varepsilon \quad \Leftrightarrow \quad Pr(\tau \leq T) \leq \varepsilon. \tag{3.6}$$

As long as the probability of default of the counterparty within the expiration period of the option contract is not exceeding the required safety level ε the risk based capital will be zero.

We now take a look at the case of J different option positions j, ($j = 1,\ldots, J$) held with one counterparty. Let the expiration date of option j be T_j and the volume of the position be a_j. Then we have $DCL = \sum_{j=1}^{J} DCL_j$, where

$$DCL_j = \begin{cases} \exp(-r\tau)\ a_j\ OP_j(\tau), & \tau \leq T_j \\ 0, & \tau > T_j. \end{cases} \tag{3.7}$$

Obviously if $\tau > T_j$ for all $j = 1,\ldots, J$, then $DCL_j = 0$ for all j and thus $DCL = 0$. Define $T = \max(T_1,\ldots, T_J)$. In case $Pr(\tau > T) \geq 1\text{-}\varepsilon$ again the required risk based capital amounts to zero. The constellation of a risk based capital with amount zero therefore crucially depends on the required safety level ε, the rating class R of the counterparty and the lifetimes of the options.

3.2 Approach II: Controlling the Expected Excess Value of Discounted Credit Loss

Stating control criterion (3.2a) alternatively, it controls the shortfall probability of the (discounted) return *RORBC* with respect to the target return $r_T = 0$. As demonstrated by *Albrecht* (1994, p. 93) a general class of shortfall measures of a return R with respect to a target return r_T is given by

$$LPM_n(r_T) = \mathrm{E}[\max(r_T - R,\ 0)^n], \tag{3.8}$$

the n-th order lower partial moments of R with respect to the target return r_T. The case $n = 0$ gives the shortfall probability, the control criterion of paragraph 3.1, the case $n = 1$ gives the shortfall expectation, which will be the basis for an alternative control criterion considered now. This new control criterion implies the restriction of the shortfall expectation of *RORBC* with respect to the target $r_T = 0$ to a small amount α, (e.g. 1% or 5%). Formally we have

$$E[\max(-RORBC,\ 0)] = \alpha, \tag{3.9a}$$

resp. equivalently

$$E\left[\max\left[\frac{DCL}{RBC} - 1,\ 0\right]\right] = \alpha. \tag{3.9b}$$

As $E[\max(kX, 0)] = k\, E[\max(X, 0)]$ this in turn is equivalent to

$$E[\max(DCL - RBC,\ 0)] = \alpha\ RBC. \tag{3.9c}$$

Relation (3.9c) has the interpretation, that the expected excess value of the discounted credit loss *DCL* over the available risk based capital is restricted to a given fraction of the risk based capital.

We use the version (3.9b) of the control criterion for the further analysis. Define $H_{DCL}(z) = E\left[\max\left[\frac{DCL}{z} - 1,\ 0\right]\right]$ for $z > 0$. First we want to show, that $H(z)$ is monotonically decreasing in z. To see this, note that for $z_1 < z_2$ we have $\max\left[\frac{x}{z_1} - 1,\ 0\right] \geq \max\left[\frac{x}{z_2} - 1,\ 0\right]$ for all x and from that $\max\left[\frac{DCL}{z_1} - 1,\ 0\right] \geq \max\left[\frac{DCL}{z_2} - 1,\ 0\right]$ a.s., which in turn implies $E\left[\max\left[\frac{DCL}{z_1} - 1,\ 0\right]\right] \geq E\left[\max\left[\frac{DCL}{z_2} - 1,\ 0\right]\right]$ and thus the stated behaviour of $H(z)$. Now define $H^{-1}(y) = \inf\{z;\ H(z) \leq y\}$ and we can conclude

$$RBC = H_{DCL}^{-1}(\alpha). \tag{3.12}$$

This equation is properly defined, if $\alpha \in \operatorname{im} H_{DCL}(z)$ is fulfilled.

The suggested approach clearly deserves further consideration as an alternative to restricting the loss probability (shortfall probability) with the calculated risk based capital. This however will be done in a different contribution. In the simulation analyses presented in the remaining part of the paper we will restrict to the "traditional" criterion.

4 Preliminary Simulation Results

In this section we present the results of a small simulation study in order to demonstrate that the approach described in § 3.1 does work in principle. We only consider long put positions, assume that the process $\{S_t;\ t \geq 0\}$ driving the underlying object is a geometric *Wiener* process with drift μ and diffusion σ and that the option price is the *Black-Scholes*-price of a Euro-pean put option.

Our starting point is the difference equation

$$S(t) = S(t-1) \exp\left[(\mu - \tfrac{1}{2}\sigma^2) + \sigma\, \Delta W(t)\right], \tag{4.1}$$

where $W(t)$ denotes the standard *Wiener* process at time t.

This means that the increments $\Delta W(t) = W(t) - W(t\text{-}1)$ are stochastically independent and normally distributed with parameter $\mu = 0$ and $\sigma = 1$. Departing with a starting value $S(0)$ we generate in the context of a *Monte-Carlo*-simulation H paths[2)] of the process of the underlying stock for a time of T periods[3)]. At each time $t = 1, 2, \ldots, T$ we calculate on the basis of the realizations of the stock value process the corresponding *Black-Scholes*-values which represent by assumption the price - demand of the short position on the OTC-market.

For the stochastic process driving the development of the option prices the following model specification is chosen where we intend to preserve the nominal amount of the initial capital ($X = S(0) = 100$):

exercise price X	diffusion σ	drift μ	stock price $P(0)$	life time option T	interest rate r
100	0,13	0,1	100	5	0,05

Table 3: Model specification of the process driving the option price

This model specification implies (without regarding the default risk) an option price an time zero $P(0) = 2{,}75302$ DM.

After running the simulation we get the estimated distribution function of the discounted credit loss DCL for selected initial rating classes R. Figure 1 shows as an example for rating class Baa3 the simulated distribution function $F(C_i)$ of the discounted credit loss of the i-th counterparty. From the graph of the distribution function we see on the one hand the rareness of insolvency, on the other hand one can recognize the big loss potential, especially compared to the OTC price.

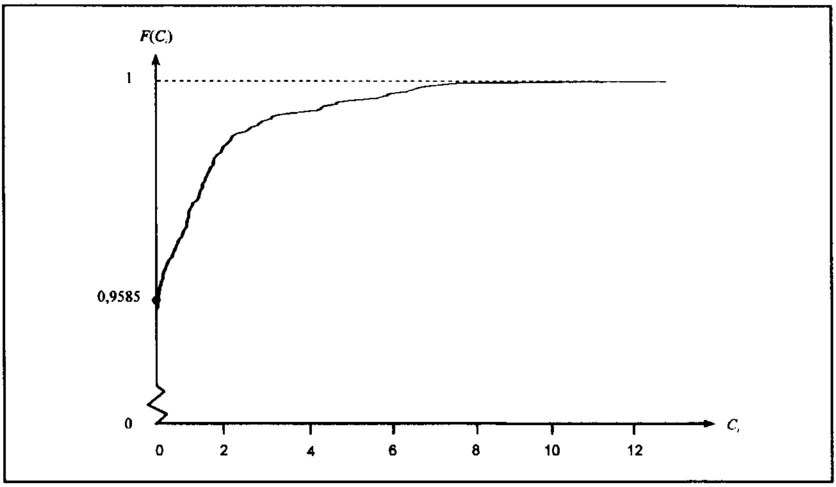

Figure 1: Distribution function of the discounted credit loss for rating class Baa 3

The following Table 4 shows the calculated default probabilities and risk based capital requirements for selected rating classes when regarding a single counterparty.

Rating	Default probability	RBC (95%-percentile)	RBC (99%-percentile)
Aa3	0,0025	0	0
A3	0,0111	0	0
Baa1	0,0142	0	0,0119
Baa2	0,0274	0	0,6055
Baa3	0,0443	0	1,7450
Ba1	0,0868	0,1400	3,0073
Ba3	0,1991	1,7875	5,8851
B3	0,5071	3,9229	8,9327

Table 4: Default probabilities and risk based capital requirements

Assuming that the risk measure is given by the excess probability and that the exogeneously determined safety level is 95% (99%), the results for selected rating classes can be interpreted in the following way:

In both cases (95% and 99% safety level), counterparties belonging to rating classes A3 or higher do not require any risk based capital. For example an Aa3 counterparty possesses a discounted credit loss probability of 0,0025; in this case, without any risk based capital requirement the safety level amounts to 99,75%. In other words, the buyer of the option is safe to an extent of more than 99% even without an underlying RBC.

Counterparties which are rated Baa3 to Ba1 require risk based capital only in the case of a desired safety level of 99%. If the target level is the 95%-percentile, we do not need any underlying capital. As one can see in the case of a counterparty rated Baa3 the discounted credit loss probability runs up to 0,0443; the resulting safety level of the option buyer without consideration of RBC amounts to 95,57%;

thus, the resulting safety level is higher than 95% but lower than 99%. In order to reach a safety level of 99%, it is necessary to acquire a risk based capital amounting to 1,745 DM.

Within the third group consisting of rating classes Ba1 and lower risk based capital is required for both safety levels. With a counterparty rated Ba3, the corresponding safety level without consideration of RBC is 80,37%. In order to achieve a safety level of 95% (99%), consequently a risk based capital amounting to 1,7875 DM (5,8851 DM) is necessary.

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Endnotes

1) A similar approach is presented in *Iben/Brotherton-Ratcliffe* (1984) for a portfolio of swaps. However, the authors concentrate more on the simulation process and less on the analysis of the structural properties of the model, as it is done here.

2) We have $H = 3250$.

3) The present analysis concentrates on $T = 5$ periods.