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**Risk Measures and the Role of Derivatives in Risk Minimization**

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## CONTENT

- Background about risk measures (properties, types, representation, examples, etc.).
- Background about pricing (arbitrage, stochastic discount factor, etc.).
- Can a risky security be less risky than the riskless asset? What are the implications?
- Optimal reinsurance, optimal investment of CR, portfolio choice and other problems leading to solutions that are derivatives.
- The absence of short selling restrictions and the presence of caveats.
- Compatibility between prices and risks and the notion of good deal.



## BACKGROUND ABOUT RISK MEASURES

Mainly, a risk measure is a real-valued function on a space of random variables. The random variable  $y$  represents a final wealth or pay-off (or loss, in some actuarial applications) at a future date  $T$ , and we attempt to measure its associated risk

$$\rho : R.V. \rightarrow \mathcal{R}$$
$$y \rightarrow \rho(y)$$

Examples: VaR, CVaR=ES=WCE, DPT, Wang, WCVaR, etc.



## RISK MEASURES: EXAMPLES

### Value at Risk

$$VaR_{\mu_0}(y) = -\text{Inf}\{r; \mu(y \leq r) > 1 - \mu_0\}$$

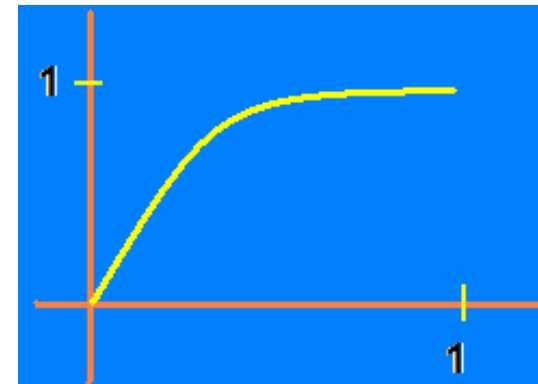
For losses we must use

$$VaR_{\mu_0}(y) = \text{Sup}\{r; \mu(y \geq r) > 1 - \mu_0\}$$

VaR is defined in  $L^0$

**Weighted Value at Risk** ( $g$  concave and non-d)

$$WVaR_g(y) = \int_0^1 VaR_{1-t}(y) dg(t)$$

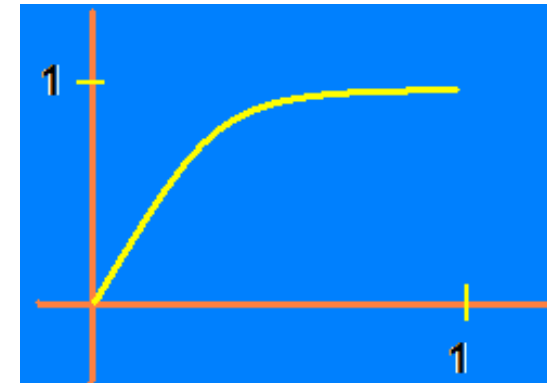




## RISK MEASURES: EXAMPLES

### Conditional Value at Risk

$$CVaR_{\mu_0}(y) = \frac{1}{1-\mu_0} \int_0^{1-\mu_0} VaR_{1-t} dt$$



### Dual Power Transform and Wang Measure

$$g(t) = 1 - (1-t)^a \quad g(t) = \Phi(\Phi^{-1}(t) + a)$$

CVaR is defined in  $L^1$ , DPT is defined in  $L^1$ , Wang is defined in  $L^2$

**Weighted Conditional Value at Risk** ( $g$  is only non decreasing)

$$WCVaR_g(y) = \int_0^1 CVaR_{1-t}(y) dg(t)$$



## RISK MEASURES: PROPERTIES AND KINDS

a) Translation invariant  $\rho(y + k) = \rho(y) - k$

b) Homogeneous  $\rho(ky) = k\rho(y), k > 0$

c) Sub-additive  $\rho(y + y') \leq \rho(y) + \rho(y')$

d) Decreasing  $y \leq y' \Rightarrow \rho(y) \geq \rho(y')$

e) Mean dominating  $\rho(y) \geq -E(y)$

**Coherent:** a), b), c), d) (CVaR, DPT, Wang WCVaR)

**Expectation Bounded:** a), b), c), e) (CVaR, DPT, Wang WCVaR)

**VaR** does not satisfy c) or e)

**There are also convex m, consistent m, satisfying m, risk indices, etc.** 6



## RISK MEASURES: REPRESENTATION

For continuous expectation bounded risk measures we have

$$\rho(y) = \text{Max}\{-E(yz); z \in \Delta_\rho\}$$

where the sub-gradient is given by

$$\Delta_\rho = \{z \in L^q; \rho(y) \geq -E(yz), \forall y \in L^p\}. \quad (1/p) + (1/q) = 1.$$

and is weakly\*-compact.

For instance, for CVaR we have

$$\Delta_{CVaR} = \{z \in L^\infty; E(z) = 1, 0 \leq z \leq 1/(1 - \mu_0)\}$$



## BACKGROUND ABOUT PRICES

In a frictionless market the absence of arbitrage implies the presence of a Stochastic Discount Factor (SDF) ( $p=2$ ).

The SDF is the pay-off (wealth at T) of a particular portfolio. It is attainable and unique

The pricing rule becomes  $\pi(y) = e^{-r_f T} E(yz_\pi), \quad E(z_\pi) = 1.$

For a market with frictions the pricing rule becomes sub-additive and homogeneous, and may be represented by its sub-gradient

$$\pi(y) = \text{Max}\{E(yz); z \in \Delta_\pi\}, \quad \Delta_\pi = \{z \in L^2; E(yz) \leq \pi(y), \forall y \in L^2\}$$

The sub-gradient becomes a singleton in the frictionless case

$$\Delta_\pi = \{e^{-r_f T} z_\pi\}$$





## CAN $y$ BE LESS RISKY THAN THE RISKLESS ASSET?

Suppose that for translation invariant homogeneous (VaR, for instance) risk measure and a risky asset

$$\rho(y) < -\Pi(y).$$

$y$  is less risky than the riskless asset. Then  $\Pi(\alpha(y - \Pi(y))) = 0$

$$\rho(\alpha(y - \Pi(y))) = \alpha(\rho(y) + \Pi(y)) \rightarrow -\infty,$$

If we add the portfolio above and a  $k$ -priced riskless asset we will reach the risk minus infinite for every price.

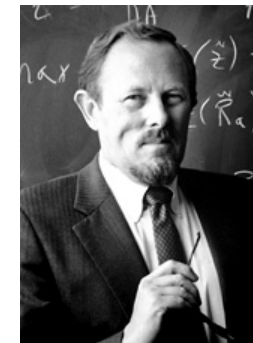
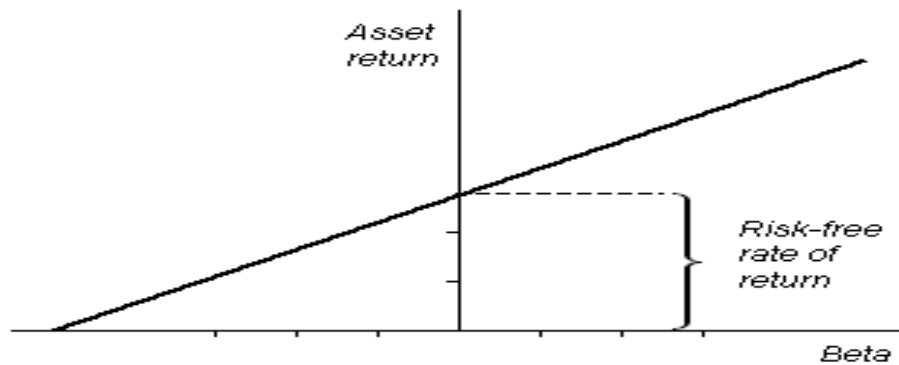
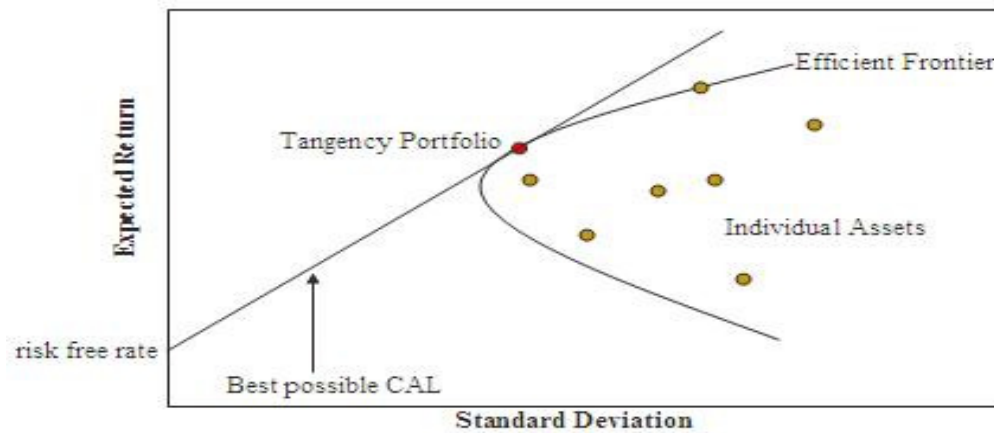
If the risk measure is mean dominating (CVaR, DPT, WANG, WCVaR, etc) then

$$E(\alpha(y - \Pi(y))) \geq -\rho(\alpha(y - \Pi(y))) \rightarrow \infty$$

Where is the Capital Market Line of the CAPM model? There should not exist securities less risky than the riskless asset.



## CLASSICAL PORTFOLIO THEORY AND CAPM (Risk = Standard Deviation)





## **CAN y BE LESS RISKY THAN THE RISKLESS ASSET (II)?**

- As said above, there should not exist investment opportunities less risky than the riskless asset.
- However, many portfolio selection problems lead to solutions that are not the riskless asset.
- Many optimal reinsurance problems dealing with VaR or CVaR and the Expected Value Premium Principle lead to stop-loss or closely related optimal solutions.
- Stop-loss reinsurance contracts are closely related to options in finance.
- Is there any caveat?



## OPTIMAL REINSURANCE

- The optimization problem may be

$$\left\{ \begin{array}{l} \text{Min} \quad \rho(S_0 - y - \Pi(y_0 - y)) \\ \quad \quad \quad \Pi(y_0 - y) \leq S_1 \\ \quad \quad \quad 0 \leq y \leq y_0 \end{array} \right.$$

- The dual problem becomes

$$\left\{ \begin{array}{l} \text{Max} \quad \text{Inf}_y \{ -S_0 - S_1\tau + E(yz) + (1 + \tau)\Pi(y_0 - y) \} \\ \quad \quad \quad \tau \geq 0 \\ \quad \quad \quad z \in \Delta_\rho \end{array} \right.$$



## OPTIMAL REINSURANCE

- With the Expected Value Principle

$$\Pi(y) = kE(y), \quad k > e^{r_f T}$$

the necessary and sufficient optimality conditions are

$$\left\{ \begin{array}{l} E(y^* z^*) \geq E(y^* z), \quad z \in \Delta_\rho \\ E(y^* z^*) + (1 + \tau^*) \Pi(y_0 - y^*) \leq \\ E(yz^*) + (1 + \tau^*) \Pi(y_0 - y), \quad 0 \leq y \leq y_0 \\ \tau^* (S_1 - \Pi(y_0 - y^*)) = 0 \end{array} \right.$$



## OPTIMAL REINSURANCE

- With the CVaR the solution becomes a stop-loss reinsurance

$$y^* = \begin{cases} y_0, & y_0 \leq \alpha \\ \alpha, & y_0 \geq \alpha \end{cases}$$

A stop-loss reinsurance contract remains optimal if CVaR is replaced by a “large enough” expectation bounded risk measure, even if we consider the simultaneous optimization of several risk functions (vector or multi-objective optimization problems).



## OPTIMAL INVESTMENT OF CAPITAL REQUIREMENTS

- Problem below may lead to the optimal investment of Capital Requirements

$$\left\{ \begin{array}{l} \text{Min} \quad \rho(y + y_0 - E(yz_\pi)) \\ E(yz_\pi) \leq C \\ y \geq 0 \end{array} \right.$$

The **dual** problem is **LINEAR**

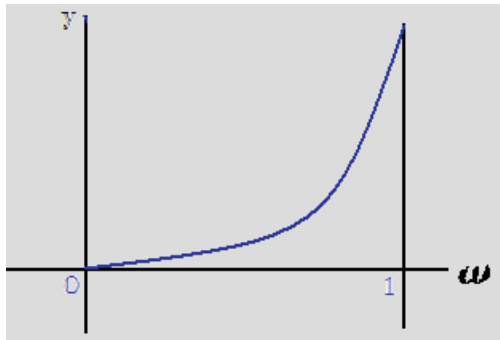
$$\left\{ \begin{array}{l} \text{Max} \quad -C\lambda - E(y_0z) \\ z \leq (1 + \lambda)z_\pi \\ \lambda \geq 0, z \in \Delta_\rho \end{array} \right.$$

Once again we can give n.s. optimality conditions

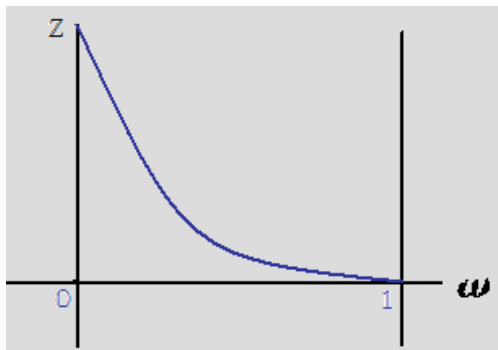


## EXAMPLE: THE BLACK AND SCHOLES MODEL

- Under the Black and Scholes model the SDC satisfies the Log-Normal distribution



$$y = y_0 \text{Exp} \left( \left( r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} \omega \right).$$



$$z_{\pi}(\omega) = \text{Exp} \left( -\frac{a^2}{2} - a\Phi^{-1}(\omega) \right), \quad a = \frac{r - r_t}{\sigma} \sqrt{T}.$$





## EXAMPLE: THE BLACK AND SCHOLES MODEL

- Under the Black and Scholes model and the CVaR the optimal solution is a combination of three put options.

$$y^* = \begin{cases} 0, & y_0 \geq \alpha \\ y_0 - \alpha, & \beta \leq y_0 \leq \alpha \\ 0, & y_0 < \beta \end{cases}$$

- This is not a riskless asset. As said above, if there is no limit to borrow money, the sequence below will lead to **(risk, return) = (minus infinite, plus infinite)**

**We can construct a  
GOOD DEAL (for both  
VaR and CVaR)**

$$y^* = \begin{cases} -\gamma_n, & y_0 \geq \alpha_n \\ y_0 - \alpha, & \beta_n \leq y_0 \leq \alpha_n \\ -\gamma_n, & y_0 < \beta_n \end{cases}$$



## EXAMPLE: GENERAL DISCRETE TIME DYNAMIC PRICING MODEL

- For every discrete time pricing model with negative correlation between the underlying asset and the SDF and short enough minimum trading period we can give a **general explicit expression of the GOOD DEAL**

$$y^* = \begin{cases} 0, & \omega \leq \omega_0 \\ C + \frac{\sum_{\omega=\omega_0+1}^{\omega_1-1} y_0(\omega) z_\pi(\omega) \mathbf{P}(\omega)}{\sum_{\omega=\omega_0+1}^{\omega_1-1} z_\pi(\omega) \mathbf{P}(\omega)} - y_0, & \omega_0 < \omega < \omega_1 \\ 0, & \omega \geq \omega_1 \end{cases}$$

- The expression above holds for VaR and CVaR, and may be also extended for many Stochastic Volatility Pricing Models.



## WHAT ABOUT MODIFYING THE RISK MEASURE? PORTFOLIO CHOICE PROBLEMS

$$\left\{ \begin{array}{l} \textit{Min} \quad \rho(y) \\ \pi(y) \leq 1 \\ E(y) \geq R \\ y, \textit{reachable} \end{array} \right.$$

We minimize the risk level with a required minimum expected return. If the risk measure is the Standard Deviation then this is an extension of the Markowitz model.

A linear dual problem and necessary and sufficient optimality conditions may be derived



## WHAT ABOUT MODIFYING THE RISK MEASURE? PORTFOLIO CHOICE PROBLEMS

**Theorem** Suppose that

$$\Pr\{z_\pi < \delta\} > 0$$

holds for every positive number. Then, if the risk measure is coherent and expectation bounded, the caveats

$$(\text{risk}, \text{return}) = (-\text{infinite}, \text{infinite})$$

$$(\text{risk}, \text{return}) = (0, \text{infinite})$$

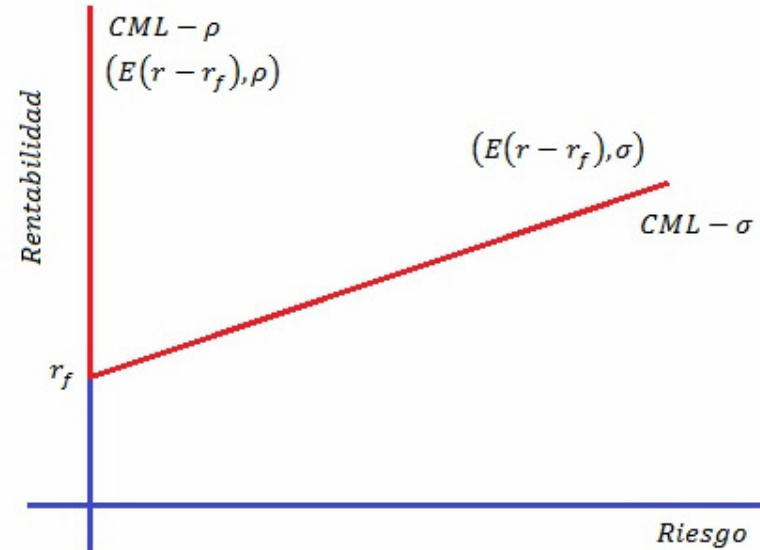
will hold. This result remains true if we incorporate transaction costs.

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**LOG-NORMAL or HEAVIER TAILED distributions provoke a caveat**



## WHAT ABOUT MODIFYING THE RISK MEASURE? PORTFOLIO CHOICE PROBLEMS (II)

Unbounded from below Stochastic Discount Factors lead to **VERTICAL** Capital Market Lines.





## Conclusions

- Many actuarial and financial classical problems may be revisited with risk measures.
- This is an interesting exercise because most of these measures are compatible with the second order stochastic dominance, and the risk level is more closely related to possible losses of capital (money).
- Very often the solution of the problem is given by a derivative (optimal reinsurance, pricing, investing capital requirements, portfolio choice).
- However, there are caveats, since the riskless asset may be less riskier than risky assets
- If short sales are allowed then meaningless results may be obtained, since many problems are unbounded. The RISK may become MINUS INFINITE AND PRICES REMAIN CONSTANT (even negative). These “GOOD DEALS” are often a limit case reachable by a SEQUENCE OF DERIVATIVES.
- We have introduced two different types of “GOOD DEAL”.
- Complete (stochastic volatility) pricing models with log normal or heavier tailed SDF are
- Not Compatible with VaR, CVaR and DPT (among others).
- Not Strongly Compatible with every coherent and expectation bounded risk measure
- These results remain true for discrete time models and models with frictions.



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**Thanks for your attention**

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