Market-Consistent Replication of Insurance Liabilities in a Multiple Risk Economy

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Concept of Valuation Portfolio (VaPo)

VaPo construction

Idea: perfect replication of insurance contracts based on financial instruments (Wüthrich et al. (2010), Sections 3 & 5, Sandström(2011), Section 8.3)

VaPo = list of financial instruments with a specification of the number of units of each instrument that are needed to replicate the insurance liabilities

Given

- random vector $C = (C_0, C_1, \ldots, C_n)$ of **liability cash-flows** over time horizon n
- series of replicating **zero-coupon bonds** $Z^{(k)}$ that pay one unit at time k=0,…,n, with price structure $P(s,k), 0 \le s < k$
- series of **replicating risky units** $U^{(\ell)} = U^{(\ell)}(I_1, \ldots, I_m), \ell = 1, \ldots, p$ (financial instruments derived from m basic economic risks mapped by risky financial instruments $I_k, k = 1, \ldots, m$)

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VaPo & MCAV with state-price deflator (1)

- **Number of units** used to replicate the liability cash-flows $C_{(j)} = (0,\dots,0,C_j,\dots,C_n)$ with the series $Z^{(k)}$ respectively $U^{(\ell)}$: $\lambda_k^{(j)}, \eta_\ell^{(j)},\ j=0,\dots,n,\ k=0,\dots,n,\ \ell=1,\dots,p$

VaPo: set of linear combinations in the replicating financial instruments defined by

$$VaPo(C_{(j)}) = \sum_{k=0}^{n} \lambda_k^{(j)} \cdot Z^{(k)} + \sum_{\ell=1}^{p} \eta_\ell^{(j)} \cdot U^{(\ell)}, \quad j=0,\dots,n$$

MCAV at initial time (market value principle using state-price deflator $\{D_t\}_{0\le t\le n}$)

$$\mathrm{M}_0\left(VaPo(C_{(j)})\right) = \sum_{k=0}^{n} \lambda_k^{(j)} \cdot \mathrm{M}_0(Z^{(k)}) + \sum_{\ell=1}^{p} \eta_\ell^{(j)} \cdot \mathrm{M}_0(U^{(\ell)})$$

$$= \sum_{k=0}^{n} \lambda_k^{(j)} \cdot P(0,k) + \sum_{\ell=1}^{p} \eta_\ell^{(j)} \cdot E_0\left[\sum_{r=j}^{n} D_r \overline{U}_r^{(\ell)}\left(S^{(1)},\dots,S^{(m)}\right)\right], \quad j=0,\dots,n$$

with $\overline{U}^{(\ell)} = \overline{U}^{(\ell)}\left(S^{(1)},\dots,S^{(m)}\right),\ \ell=0,\dots,p$ the vectors of cash-flows associated to the risky units $U^{(\ell)}$, and $S^{(k)} = \left\{S_t^{(k)}\right\}_{0\le t\le n},\ k=1,\dots,m$ are the real-world prices of the risky financial instruments $I_k,\ k=1,\dots,m$

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VaPo & MCAV with state-price deflator (2)

MCAV of current and future accounting years

$$\mathrm{M}_s\left(VaPo(C_{(j)})\right) = \sum_{k=0}^{n} \lambda_k^{(j)} \cdot \mathrm{M}_s(Z^{(k)}) + \sum_{\ell=1}^{p} \eta_\ell^{(j)} \cdot \mathrm{M}_s(U^{(\ell)})$$

$$= \sum_{k=0}^{n} \lambda_k^{(j)} \cdot P(s,k) + \sum_{\ell=1}^{p} \eta_\ell^{(j)} \cdot D_s^{-1} \cdot E_s\left[\sum_{r=j}^{n} D_r \overline{U}_r^{(\ell)}\left(S^{(1)},\dots,S^{(m)}\right)\right],$$

$$0 \le s \le j-1,\ j = 1,\dots,n$$

fair value future liabilities at time $s \le j-1$ (after premium payment at time $j-1$)

State-Price Deflators

Consider a price process $S = \{S_t\}_{0\le t\le n}$ such that S_t represents the random value at time t of a financial instrument. A **state-price deflator** $D = \{D_t\}_{0\le t\le n}$ is a strictly positive process such that the market value of S_t at time $s < t$ is given by $S_s = D_s^{-1} \cdot E_s[D_t S_t],\quad 0 \le s < t$. In other words, the deflated or discounted price process $DS = \{D_t S_t\}_{0\le t\le n}$ is a **martingale**.

Black-Scholes-Vasicek (BSV) Deflator (1)

Black-Scholes-Vasicek (BSV) Deflator (see Hürlimann (2011a))

Consider a multiple risk economy with $m \geq 1$ risky assets, whose real-world prices follow lognormal distributions. Given the current prices of these risky assets at time $s \geq 0$ the future prices at time $t > s$ are described by

$$S_t^{(k)} = S_s^{(k)} \exp\left\{\left(m_k(s,t) - \tfrac{1}{2}\sigma_k^2\right)(t-s) + v_k \sqrt{t-s} \cdot W_{t-s}^{(k)}\right\}, \quad 0 \leq s < t, \quad k = 1,\ldots,m$$

where the $W_{t-s}^{(k)}$'s are correlated standard Wiener processes such that $E\left[dW_{t-s}^{(i)} dW_{t-s}^{(j)}\right] = \rho_{ij} dt$, $m_k(s,t)$, $v_k(s,t)$ are the mean and standard deviation per time unit of the return differences on these risky assets, and σ_k is a volatility. This representation includes two popular return models:

Black-Scholes return model: $dr_t^{(k)} = \mu_k dt + \sigma_k dW_t^{(k)}$

$$m_k(s,t) = \mu_k \qquad v_k(s,t) = \sigma_k, \quad 0 \leq s < t$$

Black-Scholes-Vasicek (BSV) Deflator (2)

Vasicek (Ornstein-Uhlenbeck) return model: $dr_t^{(k)} = a_k\left(b_k - r_t^{(k)}\right)dt + \sigma_k dW_t^{(k)}$

$$m_k(s,t) = \frac{\left(b_k - r_s^{(k)}\right)\left(1 - e^{-a_k(t-s)}\right)}{t-s} \qquad v_k(s,t) = \sigma_k \sqrt{\frac{1 - e^{-2a_k(t-s)}}{2a_k(t-s)}}$$

The economic model contains also a deterministic money market account with value $M_t = M_s \exp\left((t-s)R(s,t)\right), \quad 0 \le s < t$, where $R(s,t), 0 \le s < t,$ are the deterministic continuously-compounded spot rates. The zero-coupon bond prices are denoted by $P(s,t) = \exp\left(-(t-s)R(s,t)\right), \quad 0 \le s < t$. The **BSV deflator** of dimension m has the same form as the price processes of the risky assets:

$$D_t^{(m)} = D_s^{(m)} \exp\left\{\alpha^{(m)}(s,t)(t-s) - \beta^{(m)}(s,t)^T \sqrt{t-s} \cdot W_{t-s}\right\}, \quad 0 \le s < t,$$

Theorem 1. Given is a financial market with a risk-free money market account and m risky assets with the defined real-world prices. Assume a non-singular positive semi-definite correlation matrix $C = (\rho_{ij})$. Then, the BSV deflator is determined by

Black-Scholes-Vasicek (BSV) Deflator (3)

$$D_t^{(m)} = D_s^{(m)} \exp\left\{ \begin{array}{l} -R(s,t)(t-s) - \frac{1}{2}\sum_{j=1}^{m} \beta_j^{(m)}(s,t)^2 (t-s) \\ -\sum_{1\le i<j\le m} \rho_{ij}\beta_i^{(m)}(s,t)\beta_j^{(m)}(s,t)(t-s) - \sum_{j=1}^{m}\beta_j^{(m)}(s,t)\sqrt{t-s}\cdot W_{t-s}^{(j)} \end{array} \right\}, \ 0 \le s < t,$$

with

$$\beta_j^{(m)}(s,t) = \det(C)^{-1} \cdot \sum_{i=1}^{m} (-1)^{i+j} \det\left(C_j^{(i)}\right) \cdot \lambda_i(s,t),$$

$$\lambda_i(s,t) = \frac{m_i(s,t) - R(s,t) - \frac{1}{2}\left(\sigma_i^2 - v_i^2(s,t)\right)}{v_i(s,t)}, \quad 0 \le s < t,$$

where $C_j^{(i)}$ is the matrix formed by deleting the i-th row and k-th column of C.
The quantity $\lambda_i(s,t)$ is called market price of the i-th risky asset.
Proof. One must satisfy the martingale conditions (equivalent to a system of linear equations):

$$E_s\left[D_t^{(m)}\right] = D_s^{(m)} P(s,t) = D_s^{(m)} e^{-(t-s)R(s,t)},$$

$$E_s\left[D_t^{(m)} S_t^{(k)}\right] = D_s^{(m)} S_s^{(k)}, \quad 0 \le s < t, \quad k = 1,\dots,m.$$

Index Linked Inflation Protected Life Contracts (1)

Example: index linked cohort of n-year endowment contracts

Notations

n : contract term

x : age of an insured live at contract issue

ℓ_x : number of insured lives in the cohort at contract issue

$d_x = q_x \ell_x$: number of insured lives aged x who exit within one year

$\ell_{x+k} = \ell_{x+k-1} - d_{x+k-1},\ k = 1,\dots,n-1$: recursion for number of insured lives

i : technical interest rate, $r = 1+i$

π : market-consistent (or fair) pure risk premium per contract

MCAV follows a three steps algorithm.

Case 1: minimum interest guarantee only

Step 1: set of replicating financial instruments

$Z^{(k)},\ k = 0,\dots,n-1$: zero-coupon bonds

$U^{(1)} = S$: indexed fund with price process $S_t,\ S_0 = 1$

Index Linked Inflation Protected Life Contracts (2)

$U^{(\ell)} = P^{(\ell-1)}, \ell = 2,...,n+1$, with $P^{(k)} = P^{(k)}(S, r^k), k = 1,...,n$:

European put option on indexed fund with strike time k and strike price r^k

Step 2: number of units

$$\lambda_k^{(j)} = \begin{cases} -\pi \cdot \ell_{x+k}, & k = j,...,n-1, \\ 0, & else \end{cases}$$

$$\eta_1^{(j)} = \begin{cases} \ell_x, & j = 0, \\ \ell_{x+j-1}, & j = 1,...,n, \end{cases} \qquad \eta_\ell^{(j)} = d_{x+\ell-2}, \quad \ell = 2,...,n+1, \quad j = 0,...,n$$

Step 3: determination of market values (Black-Scholes formula)

a) Determine the fair premium

Applying the fair premium equivalence principle (law of one price) solve the equation

$$\mathrm{M}_0\left(VaPo(C_{(0)})\right) = -\pi \cdot \sum_{k=1}^{n} \ell_{x+k-1} P(0, k-1) + \ell_x$$

$$+ \sum_{k=1}^{n} d_{x+k-1}\left(r^k P(0,k) \cdot \overline{\Phi}(d_2(0,k)) - \overline{\Phi}(d_1(0,k))\right) = 0$$

Index Linked Inflation Protected Life Contracts (3)

b) Determine the market-consistent value of future liabilities

$$\mathrm{M}_s\left(VaPo(C_{(j)})\right) = -\pi \cdot \sum_{k=j}^{n-1} \ell_{x+k} P(s,k) + \ell_{x+j-1} S_s$$

$$+ \sum_{k=j}^{n} d_{x+k-1}\left(r^k P(s,k)\cdot \overline{\Phi}(d_2(s,k)) - S_s \cdot \overline{\Phi}(d_1(s,k))\right),$$

$$d_1(s,k) = \frac{-\ln\left\{r^k P(s,k)/S_s\right\} + \frac{1}{2}\sigma^2 (k-s)}{\sigma\sqrt{k-s}},$$

$$d_2(s,k) = d_1(s,k) - \sigma\sqrt{k-s}, \quad 0 \le s \le j-1,\ j = 1,\ldots,n,$$

Case 2: inflation protection only

Step 1: set of replicating financial instruments

The European put options on the indexed fund are replaced by European exchange options $EX^{(k)} = EX^{(k)}(I,S),\ k = 1,\ldots,n$ to exchange the inflation index with the index fund

Step 3: determination of market values

Index Linked Inflation Protected Life Contracts (4)

Instead of the Black-Scholes formula one applies an extended version of Margrabe's formula in a multiple risk economy (e.g. Hürlimann (2011a), Theorem 2):

$$\mathrm{M}_s\left(EX^{(k)}\right) = D_s^{-1} \cdot E_s\left[D_k (I_k - S_k)_+\right]$$

$$= I_s \cdot \Phi\left(\frac{\ln(I_s / S_s) + \frac{1}{2} v^2(s,k)(k-s)}{v(s,k)\sqrt{k-s}}\right) - S_s \cdot \Phi\left(\frac{\ln(I_s / S_s) - \frac{1}{2} v^2(s,k)(k-s)}{v(s,k)\sqrt{k-s}}\right),$$

$$v^2(s,k) = \sigma_S^2 + v_I^2(s,k) - 2\rho_{SI}\sigma_S v_I(s,k), \quad 0 \le s \le k-1, \quad k = 1,\dots,n.$$

Case 3: combined minimum interest guarantee and inflation protection

It is possible to combine the inflation protection and a guaranteed minimum death benefit, say T_k at time k (generalizing the preceding guarantee $T_k = r^k$). The required double-trigger option at time k has the contingent financial payoff

$$(I_k - S_k)_+ \cdot 1\{I_k > T_k\} + (T_k - S_k)_+ \cdot 1\{I_k \le T_k\}$$

Its market value is determined in Hürlimann (2011b).

Inflation Protected Non-Life Contracts (1)

Example: index linked cohort of non-life contracts (single accident year)

Notations (claims development model from Walhin et al.(2001))

n : run-off time horizon

$c_j,\ j=1,\ldots,n$: claims payment pattern

$d_j,\ j=1,\ldots,n$: reserve deviation pattern ($d_0=0$ by convention)

$f_j,\ j=0,\ldots,n,\ f_0=1$: expected inflation pattern

π : fair pure risk premium of the cohort per unit of expected ultimate nominal aggregate paid claims

Case 1: replication of nominal values only

Step 1: set of replicating financial instruments

$Z^{(k)},\ k=0,\ldots,n$: zero-coupon bonds

Step 2: number of units

$$\lambda_0^{(0)}=-\pi\quad,\qquad \lambda_k^{(j)}=\begin{cases}(1-d_{j-1})\cdot c_j, & k=j,\\ \Delta d_j\cdot c_k, & k=j+1,\ldots,n,\\ 0, & else\end{cases}\qquad j=1,\ldots,n$$

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Inflation Protected Non-Life Contracts (2)

Step 3: determination of market values

a) Determine the fair premium

Solve the equation $\mathrm{M}_0\left(VaPo(C_{(0)})\right)=0$ to get $\pi=\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\lambda_k^{(j)}\right)P(0,k)$

b) Determine the market-consistent value of future liabilities

$$\mathrm{M}_s\left(VaPo(C_{(j)})\right)=\sum_{k=j}^{n}\left(\sum_{j=1}^{n}\lambda_k^{(j)}\right)P(s,k) \qquad 0\le s\le j-1, \quad j=1,\dots,n$$

Case 2: replication with inflation protection

Step 1: set of replicating financial instruments

(i) zero-coupon bonds $Z^{(k)}, k=0,\dots,n$

(ii) replicating call options $U^{(\ell)}=C^{(\ell)}(I,f_\ell), \ell=1,\dots,n$ on the inflation index I with future random values $I_k, k=1,\dots,n$ and initial value $I_0=1$

Step 2: number of units

$$\lambda_0^{(0)}=-\pi \qquad \lambda_k^{(j)}=\begin{cases}(1-d_{j-1})\cdot c_j\cdot f_j, & k=j,\\ \Delta d_j\cdot c_k\cdot f_k, & k=j+1,\dots,n,\\ 0, & else\end{cases} \qquad j=1,\dots,n$$

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Inflation Protected Non-Life Contracts (3)

$$\eta_\ell^{(j)} = \begin{cases} (1-d_{j-1})\cdot c_j, & \ell = j, \\ \Delta d_j \cdot c_\ell, & \ell = j+1,\dots,n, \qquad\qquad j=1,\dots,n \\ 0, & else \end{cases}$$

Step 3: determination of market values

a) Determine the fair premium: solve the equation $\mathrm{M}_0\left(VaPo(C_{(0)})\right)=0$ to get

$$\pi = \sum_{k=1}^{n}\left(\sum_{j=1}^{n}\lambda_k^{(j)}\right)P(0,k) + \sum_{k=1}^{n}\left(\sum_{j=1}^{n}\eta_k^{(j)}\right)\mathrm{M}_0\left(C^{(k)}\right)$$

b) Determine the market-consistent value of future liabilities

$$\mathrm{M}_s\left(VaPo(C_{(j)})\right) = \sum_{k=j}^{n}\left(\sum_{j=1}^{n}\lambda_k^{(j)}\right)P(s,k) + \sum_{k=1}^{n}\left(\sum_{j=1}^{n}\eta_k^{(j)}\right)\mathrm{M}_s\left(C^{(k)}\right) \qquad 0 \le s \le j-1, \quad j=1,\dots,n$$

with

$$\mathrm{M}_s\left(C^{(k)}\right) = D_s^{-1}\cdot E_s\left[D_k\left(I_k - f_k\right)_+\right] = I_s\cdot\Phi\left(d_1(s,k)\right) - f_k P(s,k)\cdot\Phi\left(d_2(s,k)\right),$$

$$d_1(s,k) = \frac{\ln\left(I_s/f_k\right) + \left(R(s,k)+\frac{1}{2}v_I^2(s,k)\right)(k-s)}{v_I(s,k)\sqrt{k-s}},$$

$$d_2(s,k) = d_1(s,k) - v_I(s,k)\sqrt{k-s}, \quad 0\le s\le k-1, \quad k=1,\dots,n.$$

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Some references (Presentation only)

- *Hürlimann, W.* (2011a). An extension of the Black-Scholes and Margrabe formulas to a multiple risk economy. Applied Mathematics 2(4), 427-432.
- *Hürlimann, W.* (2011b). Analytical pricing of an insurance embedded option: alternative formulas and Gaussian approximation. Journal of Informatics and Mathematical Sciences.
- *Sandström, A.* (2011). Handbook of Solvency for Actuaries and Risk Managers – Theory and Practice. Chapman & Hall/CRC Finance Series. CRC Press, Taylor & Francis Group.
- *Walhin, J.F., Herfurth, L. and P. De Longueville* (2001). The practical pricing of excess-of-loss treaties: actuarial, financial, economic and commercial aspects. Belgian Actuarial Bulletin 1(1), 1-17.
- *Wüthrich, M.V., Bühlmann, H. and H. Furrer* (2010). Market-Consistent Actuarial Valuation. EAA Lecture Notes, vol. 1 (2nd revised, and enlarged edition), Springer-Verlag.