Consistent, Dynamic Affine Model for Longevity Risk Applications

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AFIR Colloquia, Madrid, Spain, 19-22 June, 2011

Modelling Framework

- Develop a pricing and risk management framework for mortality
- Affine Term Structure Framework - Model all ages
- Multiple stochastic factors
- Consistent Forecasts
  - Bjork, T., and B. J. Christensen, 1999
  - Forecast survival curves have the same parametric form as the estimated survival curves
- State-Space Representation
  - Impose structure to the factor dynamics
  - Efficient estimation with the Kalman Filter
  - Solve all ages simultaneously
  - Separate factor volatility from measurement variance
  - Optimisation to find maximum likelihood
  - Bootstrap state-space residuals for distribution of parameter estimates
- Swedish Mortality Data - Human Mortality Database
  - Raw Death and Exposure data
  - Year range 1910-2007
  - Age range 50-99

Swedish Mortality Data (Males 50-99)

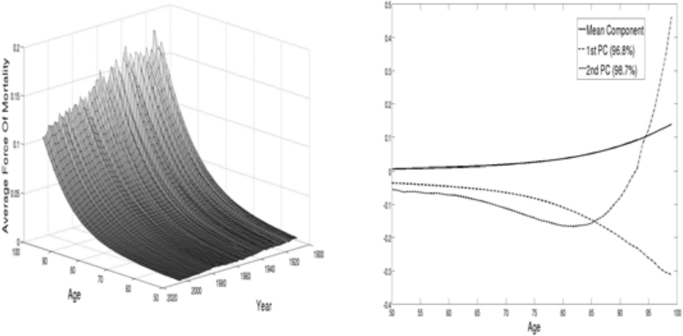


Figure: Average force of mortality

The average force of mortality:

$$z_{x,t}(\tau) = -\frac{1}{\tau} ln[S_{x,t}(\tau)] = -B(\tau)X_t + C(\tau)$$

Consistent 3-Factor Affine Mortality Model

- The risk adjusted survivor curve

$$S_{x,t}(\tau) = E_t^Q\left[e^{-\int^{\tau}\mu_{x+u,t}du}\middle|\mathbb{F}_t\right] = e^{-B(\tau)X_t+C(\tau)}$$

- 3-Factor Dependent Model

$$\begin{pmatrix} dX_{1,t} \\ dX_{2,t} \\ dX_{3,t} \end{pmatrix} = \begin{pmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & \alpha_{32} & \alpha_{33} \end{pmatrix} \left[\begin{pmatrix} \theta_1^Q \\ \theta_2^Q \\ \theta_3^Q \end{pmatrix} - \begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{pmatrix}\right] dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_{1,t}^Q \\ dW_{2,y}^Q \\ dW_{3,y}^Q \end{pmatrix}$$

- 3-Factor Independent Model

$$\begin{pmatrix} dX_{1,t} \\ dX_{2,t} \\ dX_{3,t} \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \left[\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} - \begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{pmatrix}\right] dt + \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} dW_{1,t}^Q \\ dW_{2,t}^Q \\ dW_{3,t}^Q \end{pmatrix}$$

- 3-Factor Nelson-Siegel Model

$$\begin{pmatrix} dX_{1,t} \\ dX_{2,t} \\ dX_{3,t} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & -\alpha & \alpha \end{pmatrix} \left[\begin{pmatrix} \theta_1^Q \\ \theta_2^Q \\ \theta_3^Q \end{pmatrix} - \begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{pmatrix}\right] dt + \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} dW_{1,t}^Q \\ dW_{2,y}^Q \\ dW_{3,y}^Q \end{pmatrix}$$

Solution to Ordinary Differential Equations

- Instantaneous Force of Mortality

$$\mu_{x,t}(0) = \rho_1 X_{1,t} + \rho_2 X_{2,t} + \rho_3 X_{3,t}$$

- 3-Factor Dependent Model

$$B(\tau) = -\begin{pmatrix} \rho_1 \frac{(1-e^{-\alpha_{11}\tau})}{\alpha_{11}} \\ \rho_2 \frac{(1-e^{-\alpha_{22}\tau})}{\alpha_{22}} \\ \rho_2 \frac{\alpha_{32}}{\alpha_{22}-\alpha_{33}} \left(\frac{1-e^{-\alpha_{22}\tau}}{\alpha_{22}} - \frac{1-e^{-\alpha_{33}\tau}}{\alpha_{33}} \right) + \rho_3 \frac{1-e^{-\alpha_{33}\tau}}{\alpha_{33}} \end{pmatrix}$$

- 3-Factor Independent Model

$$B_i(\tau) = \frac{1 - e^{-\alpha_i \tau}}{\alpha_i}$$

- 3-Factor Nelson-Siegel Model

$$B(\tau) = -\begin{pmatrix} -\tau \\ \frac{1-e^{-\delta\tau}}{\alpha} \\ \frac{1-e^{-\alpha\tau}}{\alpha} - \tau e^{-\alpha\tau} \end{pmatrix}$$

Change of Measure

- Relationship between risk neutral $\mathbb{Q}$ and real world measure $\mathbb{P}$

$$dW_t^Q = dW_t^P + \Gamma_t dt$$

- "Essentially Affine" risk premium specification

$$\Gamma_t = \gamma^0 + \gamma X_t$$

- Choose any drift under the $\mathbb{P}$ measure and retain $\mathbb{Q}$ structure dynamics

$$dX_t = K^P(\delta^P - X_t)dt + \Sigma dW_t^P$$

- Transition system

$$\begin{pmatrix} dX_{1,t} \\ dX_{2,t} \\ dX_{3,t} \end{pmatrix} = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix} \left[\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} - \begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{pmatrix} \right] dt + \Sigma \begin{pmatrix} dW_{1,t}^P \\ dW_{2,t}^P \\ dW_{3,t}^P \end{pmatrix}$$

- Measurement system

$$z_{x,t}(\tau) = \frac{B_1(\tau)X_{1,t}}{\tau} + \frac{B_2(\tau)X_{2,t}}{\tau} + \frac{B_3(\tau)X_{3,t}}{\tau} - \frac{C(\tau)}{\tau} + \eta_t \; \eta \sim N(0, R)$$

Consistency

- The average force of mortality curve

$$z_{x,t}(\tau) = -\frac{B_1(\tau)}{\tau}X_{1,t} - \frac{B_2(\tau)}{\tau}X_{2,t} - \frac{B_3(\tau)}{\tau}X_{3,t} + \frac{C(\tau)}{\tau}$$

- Determine the force of mortality

$$\mu_{x,t}(\tau) = -\frac{d\Big[\tau \times z_{x,t}(\tau)\Big]}{d\tau}$$

- Force of mortality dynamics

$$d\mu_{x,t}(\tau) = \mu_{x,t}(\tau)dt + \Big[\Xi_1 e^{-\alpha_{11}\tau}\Big]dX_{1,t} + \Big[\Xi_2 e^{-\alpha_{22}\tau}\Big]dX_{2,t} + \Big[\Xi_3 e^{-\alpha_{22}\tau} + \Xi_4 e^{-\alpha_{33}\tau}\Big]dX_{3,t}$$

- Constant drift condition

$$\mu_\tau(\tau, z) + \sigma(\tau)\left(\int_0^\tau \sigma(y)dy\right)^T \in Im[\mu_z(\tau, z)]$$

State-space Specification

- State-space formulation

$$
\begin{aligned}
z_t &= BX_t - C + \eta_t \quad \eta \sim N(0, R) \\
X_t &= (I - e^{-\kappa})\,\delta + e^{-\kappa} X_{t-1} + \epsilon_y \quad \epsilon \sim N(0, Q) \\
Q &= \int_0^1 e^{-\kappa s} \Sigma e^{-\kappa s} ds, \quad \Sigma = \sigma\sigma'
\end{aligned}
$$

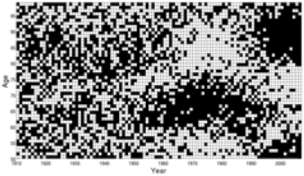
- Measurement error variance
  - Time invariant and Independent between ages
  - Simple 3 parameter model (exponentially increasing)
- For a given parameter set - Kalman filter returns the Likelihood
- Non-linear optimisation - find parameter set that maximises the Likelihood
- Bootstrap residuals for distribution of parameter estimates

Fit Comparison

	3-Factor Dependent	3-Factor Independent	3-Factor Nelson-Siegel	2-Factor Dependent	2-Factor Independent
Log Likelihood	31672	31805	31707	29222	29276
RMSE	0.00088	0.00090	0.00094	0.00195	0.00221
No. of Model Parameters	20	15	13	15	11
No. of Factors Estimated	294	294	294	196	196
Parameter Restrictions	-	5	7	203	207
Likelihood Ratio Test	-	<0	<0	4900	4792
ΔAICb	-	<0	<0	5355	4983

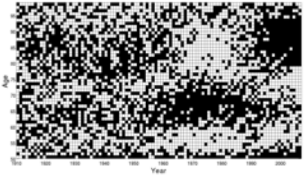
Table: Comparison of Log Likelihood, RMSE and the number of parameters estimated for each model.

Residuals - Force of Mortality

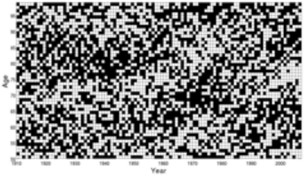


(a) 2-factor dependent model

(b) 3-factor dependent model



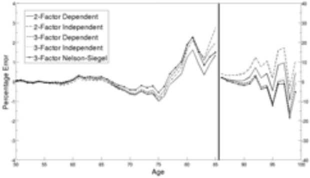
(c) 2-factor independent model



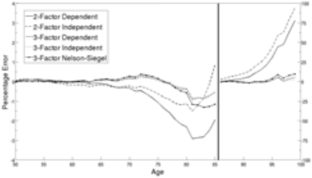
(d) 3-factor independent model

Figure: 2-Factor and 3-Factor Residuals

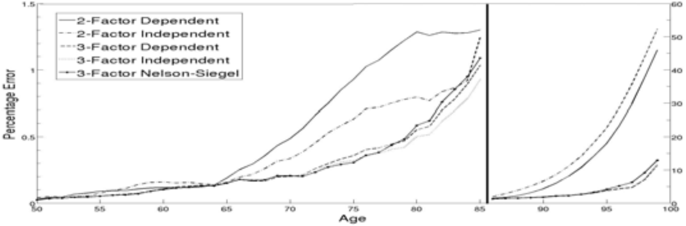
Survival Curve - In-sample Fit



(a) Relative Error 1940



(b) Relative Error 2000



(c) Survival Curve Mean Absolute Relative Error (MARE)

Factors and Factor Loadings

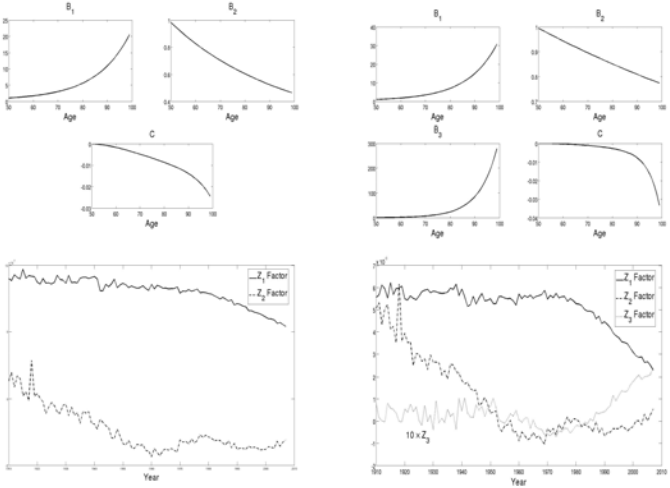
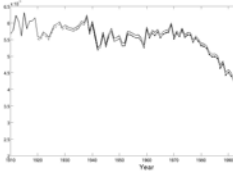
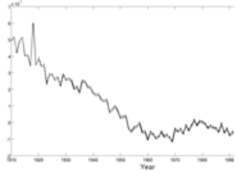


Figure: 2- & 3-Factor Independent Model Factor Loadings and Factors

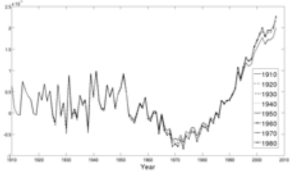
Robust Test



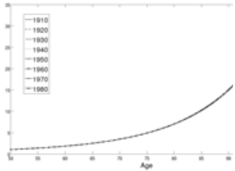
(a) Factor Z_1



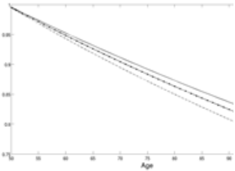
(b) Factor Z_2



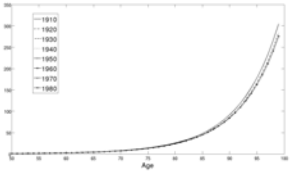
(c) Factor Z_3



(d) Factor B_1



(e) Factor B_2



(f) Factor B_3

Figure: 3-Factor Independent Model

Conclusion

Main Contribution:

- Fit an Affine Term Structure Model with Consistent forecasts
- Separate factor volatility and measurement variance
- Derive Force of Mortality dynamics - age dependant volatility function
- Bootstrap gives a distribution of fitted parameters
- Identify factors driving mortality
- Excellent in-sample fitting
- Risk Management - forecast confidence intervals for all ages
- Pricing - Framework for longevity products

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