

**WHY THE LONG TERM REDUCES THE RISK OF INVESTING IN SHARES**

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**Summary and Conclusions**

The question of whether a risk averse investor might be the more willing to hold shares rather than cash the longer his investment horizon is a vexed one, and recently articles have appeared putting forward both sides of the case.

In this article I make suitable assumptions about how one might discount future payments before taking expected utilities, and about the distribution of returns on shares and risk-free cash, then assume investment only in such shares or cash over various suitable time periods, then use a kinked linear utility function, and thus investigate whether an investor would prefer shares, cash or a mixture, over the relevant time periods.

The conclusion is that it depends on the parameters. This is not unexpected, but it gives the lie to those who assert either that shares are never preferable to cash even for longer time horizons, always preferable to cash for longer time horizons, or that the time horizon never makes any difference.

My assumption about share returns is that total returns in successive years are independent and lognormally distributed with the same parameters each year. I do not use any autoregressive assumption (which could well make shares preferable in more circumstances than under the assumption of independence).

The simple model of a kinked linear utility function means that the investor maximises his expected utility by a portfolio that is either wholly in cash or wholly in shares, and not by a mixture of the two. The assumptions also mean that, if the parameters are such that shares are ever preferred to cash, they are preferred for longer durations rather than shorter, so the assertion that a longer time horizon is a justification for holding more shares is true in many circumstances.

Cash is always preferred to shares in some rather obvious circumstances, such as when the expected return on shares equals that on cash, or is less than it, but there is still some risk. Typically cash is preferred to shares over short time horizons, shares to cash over longer. But if shares are sufficiently attractive (high mean premium over cash, low standard deviation), or if the investor's risk-aversion is sufficiently low, then shares may be preferred to cash over periods from one year upwards (I do not investigate what happens during the first year). On the other hand, certain combinations of parameters mean that shares may be preferred to cash only if the horizon is extremely long, such as over one hundred years.

## 1. *Introduction*

There have been many articles discussing whether or not shares are somehow less risky if they are held for longer periods, with proponents of both sides, e.g. Thorley (1995), Bodie (1995).

In this article I use a utility theory approach which, shows conclusively that the answer depends on the utility function, but that, yes, in many (but not all) circumstances, the longer the holding period the more reasonable it is to favour shares. My model is very simplified, but not so simplified as that of some other authors.

## 2. *The investment model*

I assume that there are two investments available, which I call cash and shares. Cash provides a risk free return, the same for all time periods. I denote the return on cash by  $r$  per time period (and I often call a time period a year, though it could be any time period, until we get to numerical examples). Thus 1 invested at time 0 provides  $r$  at the end of the period, and  $r^n$  over  $n$  periods. I shall also put  $r = e^\rho$ .

Shares provide a random return over  $n$  periods which I denote as  $R_n$ . I also denote it relative to the return on cash, by putting  $R_n = r^n \cdot S_n$ .  $S_n$  has distribution function  $F_n(x)$  and density function  $f_n(x)$ . Later I shall assume that  $S_n$  is lognormally distributed, independent from year to year, so that  $\ln(S_n)$  is normally distributed with mean  $n(\nu - \frac{1}{2}\sigma^2)$  and variance  $n\sigma^2$ . This means that  $S_n$  is lognormally distributed with mean  $\exp(n\nu)$ . (This parameterisation has certain advantages.) It also means that  $R_n$  is also lognormally distributed, so that  $\ln(R_n)$  is normally distributed with mean  $n(\nu + \rho - \frac{1}{2}\sigma^2)$  and the same variance as  $\ln(S_n)$ ,  $n\sigma^2$ .

An investor invests a proportion  $p$  of his assets in shares and  $q = 1 - p$  in cash, and leaves it unchanged for the relevant holding period. There is no reallocation of investments during the period of investment.

## 3. *The utility function*

The utility function approach has, in my view, been hampered by three things. First, there is no agreement about what utility function to use; many of the utility functions that have been proposed have mathematical disadvantages that make them inconvenient. Second there is no agreement about how to compare utilities over time. Third, the dependence of the utility function on current wealth does not seem to have been tackled. I consider all three of these in my solution, in which I use a 'kinked linear' utility function.

First, the dependence of utility function on current wealth: many authors write as if the utility function of an investor (economic agent) did not depend on their current wealth. I think it helps if we make this dependence explicit. An individual with £10,000 to invest may not only have a different utility function from a different investor (who may be more or less risk-averse), but may change his utility function if he suddenly finds himself with £1,000,000 to invest. Likewise, those who control an institution, such as a pension fund, with liabilities or assets of say £1,000,000 may have a quite different attitude (in absolute terms) from those who control an institution with liabilities or assets of say £1,000,000,000, even if they are the same group of people. Yet proportionately their attitude may be the same. Certain popular utility functions are scale invariant (e.g. the exponential; but this has other disadvantages). The kinked linear utility function is able to take scale into account.

Second, the comparison of utilities over time. It is generally assumed that individuals prefer money now to an equal amount of money at some later time, and this justifies the requirement for interest on the supply side of money for investment. The risk free case has been well documented (see e.g. Hirschleifer, 1970, or Wilkie, 1986). French (1988) shows how a uniform personal discount rate is a consequence of certain quite reasonable assumptions. However, French goes on to assume independence of returns in different years, and discounts expected utilities so that he uses a formula of the form:

$$EU = \sum_t v^t E[u(X_t)].$$

This does not take into account the possibility that returns in different years may be correlated. It is therefore necessary to discount first before taking expectations. Cuthbertson (1996) moves the expectation operator outside the summation, to give:

$$EU = E[\sum_t v^t u(X_t)].$$

However, I feel uncomfortable with the idea of discounting utilities. I have no feel for the appropriate numerical values. It seems to me better to discount money first, and then to sum before taking utilities. so I use:

$$EU = E[u(\sum_t v^t X_t)].$$

I refer to  $v$  as the personal discount factor, and also put  $v = e^{-\delta}$ .

What value should one use for  $v$ ? Well, if an individual is not going to earn enough interest to increase his utility by investing he will not invest but just spend, so if there is only certainty the personal discount rate (force)  $\delta$  must be no greater than the risk free rate (force)  $\rho$ . It is possible that an individual might have a personal discount rate that was greater than  $\rho$  but less than the expected return on shares, combined with a utility function that was sufficiently little risk averse so that he was willing to

invest in shares as opposed to consuming now, but would prefer to consume now than to invest in cash. Later I shall assume for simplicity that  $\delta = \rho$ , so that  $vr = 1$ .

#### 4. *The kinked linear utility function*

The kinked linear utility function consists of two straight lines, of different slopes:

$$\begin{aligned} u(w) &= (w - w_0) & \text{if } w \geq w_0 \\ &= a(w - w_0) & \text{if } w < w_0 \quad \text{with } a > 1 \end{aligned}$$

Thus above  $w_0$  utility goes up with any extra return pound for pound. Below  $w_0$  a pound lost is treated as a loss of  $a (> 1)$ . This is consistent with treating shortfalls below  $w_0$  as having a penalty cost (recourse cost) of  $a$  per pound deficit.

If  $a = 1$  we have simply a linear utility function.

If  $w_0$  is either very large or very small compared with the likely returns, we also get something very close to a linear utility function, so  $w_0$  needs to be somewhere well within the normal range of results. It may be convenient to think of it as the value of the liability of the investor. If the proceeds are enough to pay off the liability then a pound of surplus can be treated as a pound. If the proceeds are too small, then the shortfall may need to be made up, and this is treated as incurring a penalty.

For simplicity I take the present value of the investible assets as 1, and later also put  $w_0 = 1$ .

If  $u(w)$  is any utility function, then  $v(w) = b + cu(w)$  with  $c > 0$  is an equivalent utility function, i.e. the vertical scale and location is not relevant. The kinked linear utility function can therefore be expressed in other ways, but it is convenient to take the slope above  $w_0$  as unity, and the value at  $w_0$  as zero.

This kinked linear function is increasing, but the derivative,  $u'(w)$ , is not defined at  $w = w_0$ . I discuss this problem in Section 13.

#### 5. *The solution*

We are now able to proceed. We invest assets of 1, with  $p$  in shares and  $q = 1 - p$  in cash, for a period of  $n$  years. The proceeds are:

$$P_n = q.r^n + p.R_n = q.r^n + p.r^n.S_n.$$

The present value of this, discounted at the personal discount rate, is:

$$W_n = v^n P_n = v^n \{q.r^n + p.r^n.S_n\} = (vr)^n \{1 - p + p.S_n\} = (vr)^n \{1 + p.(S_n - 1)\}.$$

The utility is:

$$\begin{aligned} U_n = u(W_n) &= (W_n - w_0) &= (vr)^n \{1 + p.(S_n - 1)\} - w_0 & \quad W_n \geq w_0 \\ &= a(W_n - w_0) &= a\{(vr)^n \{1 + p.(S_n - 1)\} - w_0\} & \quad W_n < w_0 \end{aligned}$$

$$W_n \geq w_0 \quad \text{if } (vr)^n \{1 + p.(S_n - 1)\} \geq w_0,$$

i.e.  $S_n \geq s_0$ , where:  $s_0 = 1 + \{w_0.(vr)^{-n} - 1\}/p$

We can now take expectations, and also write  $S_n = x$ :

$$\begin{aligned} EU_n = E[U_n] &= \int_{-\infty}^{s_0} a\{(vr)^n \{1 + p.(x - 1)\} - w_0\} f_n(x).dx \\ &+ \int_{s_0}^{\infty} \{(vr)^n \{1 + p.(x - 1)\} - w_0\} f_n(x).dx \\ &= \int_{-\infty}^{s_0} a\{(vr)^n - w_0\} f_n(x).dx + \int_{s_0}^{\infty} \{(vr)^n - w_0\} f_n(x).dx \\ &+ p.\{ \int_{-\infty}^{s_0} a\{(vr)^n(x - 1)\} f_n(x).dx \\ &+ \int_{s_0}^{\infty} \{(vr)^n(x - 1)\} f_n(x).dx \} \end{aligned} \quad (1)$$

We wish to choose the optimum value of  $p$  to maximise  $EU_n$ . To do this we would usually differentiate  $EU_n$  with respect to  $p$  and set the derivative to zero. However,  $s_0$  depends on  $p$ , so we shall first study a simplified case, by putting  $\delta = \rho$ , so  $vr = 1$ , and by putting  $w_0 = 1$ . We can return to (1) later.

If  $vr = 1$  and  $w_0 = 1$  then  $s_0 = 1$ , and we get:

$$EU_n = p.\{ \int_{-\infty}^1 a(x - 1) f_n(x).dx + \int_1^{\infty} (x - 1) f_n(x).dx \}$$

This is of the form  $EU_n = ph$ , which is proportional to  $p$ . This is intuitive. If  $p = 0$ , so that all assets are invested in cash, earning  $r$ , and if the proceeds are then discounted at  $v$  with  $vr = 1$ , then the present value is 1. But if  $w_0 = 1$ , the utility of 1 is zero. Given all our assumptions, any proportion invested in shares gives a proportionate extra return, dependent on  $\nu$  and  $\sigma$ , but proportionate to  $p$ .

Now if  $h > 0$ ,  $EU_n$  increases with increasing  $p$ ; if  $h < 0$ ,  $EU_n$  decreases with increasing  $p$ . We now need to use the proposition that  $p$  must lie in the range  $[0, 1]$ , i.e. we cannot borrow money to buy shares, and we cannot sell shares short to invest in cash.

The optimal value of  $p$  is then either 0 or 1, depending on the sign of  $h$ . If  $h > 0$ , we optimise by investing as much as we can, 100%, in shares. If  $h < 0$ , we optimise by investing as little as we can in shares, with 100% in cash. If  $h = 0$  we are at the boundary, and we are indifferent about how we divide our assets between cash and shares.

$$\begin{aligned} \text{Now } h &= \int_{-\infty}^{-1} a(x-1)f_n(x).dx + \int_{1}^{\infty} (x-1)f_n(x).dx \\ &= a. \int_{-\infty}^{-1} x.f_n(x).dx + \int_{1}^{\infty} x.f_n(x).dx - \{a. \int_{-\infty}^{-1} f_n(x).dx + \int_{1}^{\infty} f_n(x).dx\} \end{aligned}$$

So far we have kept the formulae general, but we now use the fact that  $S_n$  is lognormally distributed, with  $E[\ln(S_n)] = n(\nu - \frac{1}{2}\sigma^2)$  and  $\text{Var}[\ln(S_n)] = n\sigma^2$ . This means that:

$$\int_a^b f_n(x).dx = N((\ln b - n(\nu - \frac{1}{2}\sigma^2))/\sigma\sqrt{n}) - N((\ln a - n(\nu - \frac{1}{2}\sigma^2))/\sigma\sqrt{n})$$

and

$$\int_a^b x.f_n(x).dx = \exp(n\nu) \cdot \{N((\ln b - n(\nu - \frac{1}{2}\sigma^2))/\sigma\sqrt{n} - \sigma\sqrt{n}) - N((\ln a - n(\nu - \frac{1}{2}\sigma^2))/\sigma\sqrt{n} - \sigma\sqrt{n})\}$$

where  $N(\cdot)$  is the normal distribution function.

Using these results we get:

$$\begin{aligned} h &= \exp(n\nu) \cdot \{a.N(-\sqrt{n}(\nu - \frac{1}{2}\sigma^2)/\sigma - \sigma\sqrt{n}) + [1 - N(-\sqrt{n}(\nu - \frac{1}{2}\sigma^2)/\sigma - \sigma\sqrt{n})]\} \\ &\quad - \{a.N(-\sqrt{n}(\nu - \frac{1}{2}\sigma^2)/\sigma) + [1 - N(-\sqrt{n}(\nu - \frac{1}{2}\sigma^2)/\sigma)]\} \\ &= \exp(n\nu) \cdot \{(a-1).N\{-\sqrt{n}(\nu/\sigma + \frac{1}{2}\sigma)\} + 1\} \\ &\quad - \{(a-1).N(-\sqrt{n}(\nu/\sigma - \frac{1}{2}\sigma)) + 1\} \end{aligned}$$

## 6. Behaviour of $h$

The factor  $h$  is a function of  $\nu$ ,  $\sigma$ ,  $a$  and  $n$ . How does it behave? We consider first the intuitive results. We would expect  $h$  to be larger, and therefore we would be more inclined to favour shares, if shares give better mean returns (relative to cash), i.e.  $\nu$  is larger. We would expect  $h$  to be smaller, and therefore we would be less inclined to favour shares if  $\sigma$  is larger. We would expect  $h$  to favour shares less if  $a$ , the penalty cost, is larger.

To check these intuitions we calculate the derivatives of  $h$  with respect to each of these.

$$\begin{aligned} \text{First } dh/d\nu &= n \cdot \exp(n\nu) \cdot \{(a-1) \cdot N\{-\sqrt{n}(\nu/\sigma + 1/2\sigma)\} + 1\} \\ &\quad - \exp(n\nu) \cdot (a-1) \cdot f\{-\sqrt{n}(\nu/\sigma + 1/2\sigma)\} \cdot \sqrt{n}/\sigma \\ &\quad + (a-1) \cdot f\{-\sqrt{n}(\nu/\sigma - 1/2\sigma)\} \cdot \sqrt{n}/\sigma \end{aligned}$$

where  $f(x)$  is the unit normal density function  $= 1/\sqrt{(2\pi)} \cdot \exp(-x^2/2)$ .

$$\begin{aligned} \text{Next } dh/d\sigma &= \exp(n\nu) \cdot (a-1) \cdot f\{-\sqrt{n}(\nu/\sigma + 1/2\sigma)\} \cdot (-\sqrt{n}) \cdot (-\nu/\sigma^2 + 1/2) \\ &\quad - (a-1) \cdot f\{-\sqrt{n}(\nu/\sigma - 1/2\sigma)\} \cdot (-\sqrt{n}) \cdot (-\nu/\sigma^2 - 1/2) \\ &= \exp(n\nu) \cdot \{(a-1) \cdot f\{-\sqrt{n}(\nu/\sigma + 1/2\sigma)\} \cdot \sqrt{n} \cdot (\nu/\sigma^2 - 1/2) \\ &\quad - (a-1) \cdot f\{-\sqrt{n}(\nu/\sigma - 1/2\sigma)\} \cdot \sqrt{n} \cdot (\nu/\sigma^2 + 1/2)\} \end{aligned}$$

$$\text{Then } dh/da = \exp(n\nu) \cdot N\{-\sqrt{n}(\nu/\sigma + 1/2\sigma)\} - N\{-\sqrt{n}(\nu/\sigma - 1/2\sigma)\}$$

The \$64 dollar question is: how does  $h$  behave as we increase  $n$ ?

To check this we calculate  $dh/dn =$

$$\begin{aligned} &= \nu \cdot \exp(n\nu) \cdot \{(a-1) \cdot N\{-\sqrt{n}(\nu/\sigma + 1/2\sigma)\} + 1\} \\ &\quad - \exp(n\nu) \cdot (a-1) \cdot f\{-\sqrt{n}(\nu/\sigma + 1/2\sigma)\} \cdot (\nu/\sigma + 1/2\sigma)/(2\sqrt{n}) \\ &\quad - (a-1) \cdot f\{-\sqrt{n}(\nu/\sigma - 1/2\sigma)\} \cdot (\nu/\sigma - 1/2\sigma)/(2\sqrt{n}) \end{aligned}$$

These all look rather messy, but we can tackle them numerically, to get a feel for how they behave.

## 7. Equal expected returns on shares and cash

First, however, we explore a special case. We would generally assume that  $\nu > 0$ , otherwise shares would not be attractive at all relative to cash. However, consider the position where  $\nu = 0$ , so that the expected returns on cash and shares are equal. If  $\nu = 0$ , then  $\exp(\nu) = 1$ , and

$$\begin{aligned} h &= \{(a-1) \cdot N(-1/2\sqrt{n}\sigma) + 1\} - \{(a-1) \cdot N(1/2\sqrt{n}\sigma) + 1\} \\ &= (a-1) \cdot N(-1/2\sqrt{n}\sigma) - (a-1) \cdot N(1/2\sqrt{n}\sigma) \\ &= -(a-1) \cdot \{N(1/2\sqrt{n}\sigma) - N(-1/2\sqrt{n}\sigma)\} \end{aligned}$$

which is negative, unless either  $a = 1$  or  $\sigma = 0$ . Thus if the investor is at all risk-averse, and if there is any risk in shares, the investor chooses to invest wholly in cash. This is as expected.

If  $a = 1$ , so that the investor has a linear utility function, he is indifferent between shares and cash, since their expected returns are the same. We would expect that  $h = 0$ , as it is.

If  $\sigma = 0$  shares are no longer risky. Since they are assumed to have the same expected returns as cash, they are identical to cash, and again we would expect that  $h = 0$ , as it is. So far, so good.

Let us consider the derivatives at this point.

$$\begin{aligned} dh/dv &= n \cdot \{(a-1) \cdot N(-\frac{1}{2}\sqrt{n\sigma}) + 1\} \\ &\quad - (a-1) \cdot f(-\frac{1}{2}\sqrt{n\sigma}) \cdot \sqrt{n}/\sigma + (a-1) \cdot f(\frac{1}{2}\sqrt{n\sigma}) \cdot \sqrt{n}/\sigma \\ &= n \cdot \{(a-1) \cdot N(-\frac{1}{2}\sqrt{n\sigma}) + 1\} > 0, \end{aligned}$$

so it has the expected sign at this point.

Note that  $f(-x) = f(x)$ .

$$\begin{aligned} \text{Next } dh/d\sigma &= (a-1) \cdot f(-\frac{1}{2}\sqrt{n\sigma}) \cdot \sqrt{n} \cdot (-\frac{1}{2}) - (a-1) \cdot f(\frac{1}{2}\sqrt{n\sigma}) \cdot \sqrt{n} \cdot (+\frac{1}{2}) \\ &= -(a-1) \cdot f(\frac{1}{2}\sqrt{n\sigma}) \cdot \sqrt{n} < 0, \end{aligned}$$

So it too has the expected sign.

$$\text{Then } dh/da = N(-\frac{1}{2}\sqrt{n\sigma}) - N(\frac{1}{2}\sqrt{n\sigma}) < 0$$

which again is as expected.

$$\text{Now } dh/dn = -(a-1) \cdot f(-\frac{1}{2}\sqrt{n\sigma}) \cdot \frac{1}{2}\sigma/(2\sqrt{n}) - (a-1) \cdot f(\frac{1}{2}\sqrt{n\sigma}) \cdot (-\frac{1}{2}\sigma)/(2\sqrt{n}) = 0,$$

which implies that, in this case, the term makes no difference; shares are equally unattractive at all terms.

### 8. *Another special case*

If we set  $\sigma = 0$  in the formula for  $h$ , we get division by zero. What happens as  $\sigma \rightarrow 0$ , so that shares become certainties?

$$\begin{aligned} \text{Now } h &= \exp(nv) \cdot \{(a-1) \cdot N\{-\sqrt{n}(v/\sigma + \frac{1}{2}\sigma)\} + 1\} \\ &\quad - \{(a-1) \cdot N\{-\sqrt{n}(v/\sigma - \frac{1}{2}\sigma)\} + 1\} \end{aligned}$$

As  $\sigma \rightarrow 0$ ,  $N\{-\sqrt{n}(\nu/\sigma + \frac{1}{2}\sigma)\} \rightarrow N(-\infty) = 0$  if  $\nu > 0$ , and  $\rightarrow N(+\infty) = 1$  if  $\nu < 0$ .

$N\{-\sqrt{n}(\nu/\sigma - \frac{1}{2}\sigma)\}$  does the same.

Hence  $h \rightarrow \exp(n\nu) - 1 > 0$  if  $\nu > 0$

and  $h \rightarrow a(\exp(n\nu) - 1) < 0$  if  $\nu < 0$ .

Hence if  $\nu > 0$  shares are attractive, and if  $\nu < 0$  they are unattractive, as we might have expected.

### 9. Numerical results: generally

I have calculated  $h$  and all the relevant derivatives for all combinations of  $a$ ,  $\nu$ ,  $\sigma$  and  $n$  in the ranges:

$a = 1$  (0.2) 3 [i.e. from 1 to 3 in steps of 0.2], 11 values

$\nu = -0.01$  (0.005) 0.04, 11 values

$\sigma = 0$  (0.05) 0.3, 7 values

$n = 0$  (1) 10 (5) 30 (10) 60 (20) 100, 20 values

This gives 16,940 cases. [Note that there are occasional problems with the calculation of certain values, because of inaccuracy in the calculation of the normal distribution function,  $N(x)$ , for extreme values of  $x$ , say  $x > 8$ .]

There are some special cases which I discuss below, i.e.  $a = 1$ ,  $\nu \leq 0$ ,  $\sigma = 0$  and  $n = 0$ . Apart from these we have the following results.

The value of  $h$  is sometimes negative, sometimes positive. It is negative for high values of  $a$ , high values of  $\sigma$ , low values of  $\nu$  and low values of  $n$ . If it is negative for low values of  $n$ , it may become positive for larger values, keeping the other parameters constant. Thus shares are sometimes attractive, sometimes not, relative to cash, in this model.

The attractiveness of shares increases as the expected return on shares, parameterised by  $\nu$ , increases, confirmed by the fact that  $dh/d\nu > 0$  everywhere, as expected.

The attractiveness of shares decreases as the risk aversion factor,  $a$ , increases, confirmed by the fact that  $dh/da < 0$  everywhere, as expected.

The attractiveness of shares decreases as the standard deviation of the return, parameterised by  $\sigma$ , increases, confirmed by the fact that  $dh/d\sigma < 0$  everywhere, as expected.

If  $h$  is negative for low values of  $n$ ,  $dh/dn$  may be negative for low values of  $n$ , so  $h$  is getting more negative as  $n$  increases. As  $n$  continues to increase, there is a value of  $n$  at which  $dh/dn = 0$ , and above that value of  $n$  we find that  $dh/dn > 0$ , so  $h$  starts getting less negative. As  $n$  increases further,  $h$  crosses zero, and then becomes positive. In these circumstances, shares start out unattractive at low terms, become less attractive as term increases, for a bit; then it turns round and with long enough terms shares are attractive. The relationship of attractiveness of shares and term is complicated.

Some examples: for  $a = 1.2$ ,  $\sigma = 0.1$ ,  $\nu = 0.005$ , so quite low risk aversion and standard deviation, but low expected return, we have: for  $n = 1$ ,  $h = -0.00249$ ; for  $n = 2$ ,  $h = -0.00030$ ; for  $n = 3$ ,  $h = +0.00267$ . Thus shares are unattractive for the first two years, attractive thereafter.

At the other extreme: for  $a = 3.0$ ,  $\sigma = 0.3$ ,  $\nu = 0.005$ , so high risk aversion and high standard deviation, but low expected return, we have  $h$  negative for all values of  $n$  up to and including  $n = 100$ . I have not investigated when it turns positive. However, for  $\nu = 0.01$ ,  $h$  is negative up to term 80, positive at term 100.

There is probably no "typical" result, but a plausible middle of the road case is with  $a = 2.0$ ,  $\sigma = 0.2$ ,  $\nu = 0.02$ ; we have  $h$  negative up to  $n = 7$ , and positive thereafter. One does not have to hold shares for an exceptionally long period before they become sufficiently less risky to become attractive.

#### 10. *Numerical results: special cases*

Now for the special cases not considered in the general case above.

When  $a = 1$  we have a linear utility function.  $h > 0$  so shares are attractive whenever  $\nu > 0$ .

When  $\nu < 0$  shares should be unattractive, and  $h < 0$ .

When  $\nu = 0$  shares and cash give equal returns, and shares are unattractive with  $h < 0$ , except when  $a = 1$  or  $\sigma = 0$ .

When  $\sigma = 0$ , shares are more attractive than cash if  $\nu > 0$ , and then  $h > 0$ ; they are less attractive if  $\nu < 0$ , and then  $h < 0$ ; we are indifferent if  $\nu = 0$ , and then  $h = 0$ .

When  $n = 0$ ,  $h = 0$ ; we have no preferences over such a short interval.

## 11. Parameterisation

I initially parameterised the distribution of shares by putting the expected value of  $\ln S_n = n\nu$ , instead of  $n(\nu - \frac{1}{2}\sigma^2)$ . The expected return on shares is then  $\exp\{n(\nu + \frac{1}{2}\sigma^2)\}$ . This suits the algebra just as well until we come to consider variation of  $h$  with  $\sigma$ . As  $\sigma$  increases, the spread of returns from shares increases, but so also does the expected return. The derivative  $dh/d\sigma$  is not necessarily negative. Thus the parameterisation I have used is preferable.

## 12. Exponential utility function

Another approach is to use an exponential utility function,  $u(w) = -e^{-aw}$ . This is scaled so that  $u(0) = -1$  and  $u(w) \rightarrow 1$  as  $w \rightarrow \infty$ . It is defined for all values of  $w$ , and has constant absolute risk aversion  $= -u''(w)/u'(w) = a$ . It is not scale-invariant. If we replace  $w$  by  $kw$  we need to divide  $a$  by  $k$  to get the same effect.

A problem about an exponential utility function is that it is bounded upwards. It tends to a finite limit as  $w$  increases. This is both a disadvantage and an advantage.

If we use the same statistical model as previously for the returns on shares, assuming that they are lognormally distributed, we cannot proceed algebraically so easily (though numerical results could be calculated). I therefore change the assumptions about share returns. Assume that  $R_n = r^n S_n$ , as before, but assume that  $S_n$  is distributed normally, with mean  $E[S_n] = 1 + n\nu$  and variance  $\text{Var}[S_n] = n\sigma^2$ . Thus cash returns are compounded correctly, but share returns, relative to cash, increase only in a "simple interest" manner.

Now we return to formula (1). The equivalent is:

$$EU_n = E[U_n] = \int_{-\infty}^{\infty} -\exp\{-a(vr)^n\{1 + p.(x - 1)\}\} f_n(x). dx \quad (2)$$

As before, we put  $vr = 1$ , and get

$$\begin{aligned} EU_n &= \int_{-\infty}^{\infty} -\exp\{-a\{1 + p.(x - 1)\}\} f_n(x). dx \\ &= \int_{-\infty}^{\infty} -\exp\{-a + ap\} . \exp\{-apx\} f_n(x). dx \end{aligned}$$

We then substitute  $f_n(x) = 1/\sqrt{(2\pi)} . \exp\{-(x - 1 - n\nu)^2/(2n\sigma^2)\}$  to get

$$EU_n = \int_{-\infty}^{\infty} -\exp\{-a + ap\} . \exp\{-apx\} . 1/\sqrt{(2\pi)} . \exp\{-(x - 1 - n\nu)^2/(2n\sigma^2)\} . dx$$

A lot of rearranging (see Endnote) gives us:

$$\begin{aligned}
 EU_n &= -\exp\{-a + ap\} \cdot \exp\{-ap(1 + nv) + \frac{1}{2}a^2p^2n\sigma^2\} \\
 &\quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{(x - 1 - nv - apn\sigma^2)^2}{2n\sigma^2}\right\} \cdot dx \\
 &= -\exp\{-a - apnv + \frac{1}{2}a^2p^2n\sigma^2\}
 \end{aligned}$$

since the integral equals unity.

We wish to find the optimum value of  $p$ , that which maximises  $EU_n$ . To do this we differentiate  $EU_n$  with respect to  $p$  and set the derivative to zero. Differentiating gives us:

$$dEU_n/dp = -\exp\{-a - apnv + \frac{1}{2}a^2p^2n\sigma^2\} \cdot \{-anv + a^2pn\sigma^2\}$$

which equals zero when  $anv = a^2pn\sigma^2$ , or  $p = v/a\sigma^2$ .

But this is independent of  $n$ , so the optimum proportion of shares, *in this special case*, does not depend on the term. This may have misled others into quoting as a general rule what applies to a particular utility function, and a particular (not very realistic — a lognormal distribution for returns would be better) distribution for share returns.

[This result is unlikely to be new, but I have not spotted any previous reference in which it has appeared.]

### 13. Continuity

I observed at the end of Section 4 that the derivative of the utility function,  $u'(w)$ , was not defined at  $w = w_0$ . If this is felt to be a problem, one can rearrange the utility function so that instead of taking the lower branch of a pair of straight lines,  $u = a(w - w_0)$  and  $u = w - w_0$ , it lies on the lower branch of the hyperbola  $(u - a(w - w_0))(u - (w - w_0)) = c$ , where  $c$  can be as small as one likes. Then  $u'(w)$  is defined everywhere, is continuous, and changes reasonably (but quickly) in the neighbourhood of  $w = w_0$ . The second derivative,  $u''(w)$  is also defined everywhere, but changes very sharply in the neighbourhood of  $w = w_0$ . The coefficient of absolute risk aversion is very large in this neighbourhood, and approaches zero everywhere else. Is this reasonable behaviour for a utility function?

#### 14. *Concluding observations*

This paper was inspired by that of Thorley (1995), who uses a somewhat similar argument, but without explicitly bringing in utility theory. He shows that the downside risk gets smaller the longer the period for which shares are held.

Bodie (1995) takes the opposite approach, arguing that if the investor buys options to cover the downside risk of his investment in shares he is no better off than if he invested in cash. In my view Bodie's paper contains a fallacy: if the investor views shares as giving a better median return than cash then put options, valued using option pricing methodology, are more highly priced than the expected value of the risk they are covering, so only a sufficiently risk-averse investor would choose to buy them. Such a risk-averse investor would choose cash anyway, in my model. The investor who chooses shares in my model would not choose to buy options.

The theoretical assumptions on which options are priced are often described as being "risk-neutral". It would be better to call them "risk-aborrent", because, if the writer of options hedges the risks precisely, he presumably so abhors risk that he is not prepared to take any. The source option buyer, on the other hand, presumably takes a view that options are good value for money, for his circumstances and under his assumptions. Whether he chooses to reduce his other risks by buying options or chooses to use options to express his opinions about the market depends on his utility function, which is seldom either risk-neutral or risk-aborrent.

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ENDNOTE

In the exponential/normal case we have;

$$EU_n = \int_{-\infty}^{\infty} -\exp\{-a + ap\} \cdot \exp\{-apx\} \cdot 1/\sqrt{(2\pi)} \cdot \exp\{-(x-1-n\nu)^2/(2n\sigma^2)\} \cdot dx$$

$$= -\exp\{-a + ap\} \cdot \int_{-\infty}^{\infty} 1/\sqrt{(2\pi)} \cdot \exp\{-(x-1-n\nu)^2/(2n\sigma^2) - apx\} \cdot dx \quad \text{F1}$$

The bit inside the second exp{..} is  $-(x-1-n\nu)^2/(2n\sigma^2) - apx$

$$= -\{(x-1-n\nu)^2 + apx \cdot 2n\sigma^2\}/(2n\sigma^2) \quad \text{F2}$$

The bit inside {..} is  $(x-1-n\nu)^2 + apx \cdot 2n\sigma^2$

$$= x^2 - 2x(1+n\nu) + (1+n\nu)^2 + apx \cdot 2n\sigma^2$$

$$= x^2 - 2x(1+n\nu - apn\sigma^2) + (1+n\nu)^2$$

$$= x^2 - 2x(1+n\nu - apn\sigma^2) + (1+n\nu)^2 - 2(1+n\nu)(apn\sigma^2) + (apn\sigma^2)^2$$

$$\quad + 2(1+n\nu)(apn\sigma^2) - (apn\sigma^2)^2$$

$$= x^2 - 2x(1+n\nu - apn\sigma^2) + (1+n\nu - apn\sigma^2)^2$$

$$\quad + 2(1+n\nu)(apn\sigma^2) - (apn\sigma^2)^2$$

$$= (x - (1+n\nu - apn\sigma^2))^2 + 2n\sigma^2\{(1+n\nu)ap - \frac{1}{2}(ap\sigma)^2n\}$$

Substitute back into (F2), so the bit inside exp{..}

$$= -\{(x - (1+n\nu - apn\sigma^2))^2 + 2n\sigma^2\{(1+n\nu)ap - \frac{1}{2}(ap\sigma)^2n\}\}/(2n\sigma^2)$$

$$= -\{(x - (1+n\nu - apn\sigma^2))^2/(2n\sigma^2) + (1+n\nu)ap - \frac{1}{2}(ap\sigma)^2n\}$$

Substitute back into (F1), to give:

$$EU_n = -\exp\{-a + ap\} \cdot \int_{-\infty}^{\infty} 1/\sqrt{(2\pi)} \cdot \exp\{-(x - (1+n\nu - apn\sigma^2))^2/(2n\sigma^2) - (1+n\nu)ap + \frac{1}{2}(ap\sigma)^2n\} \cdot dx$$

$$= -\exp\{-a + ap\} \cdot \exp\{-(1+n\nu)ap + \frac{1}{2}(ap\sigma)^2n\} \cdot \int_{-\infty}^{\infty} 1/\sqrt{(2\pi)} \cdot \exp\{-\{(x - (1+n\nu - apn\sigma^2))^2/(2n\sigma^2)\}\} \cdot dx$$

$$= -\exp\{-a + ap\} \cdot \exp\{-ap(1+n\nu) + \frac{1}{2}(ap\sigma)^2n\} \cdot \int_{-\infty}^{\infty} 1/\sqrt{(2\pi)} \cdot \exp\{-\{(x - (1+n\nu - apn\sigma^2))^2/(2n\sigma^2)\}\} \cdot dx$$