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INVESTMENT CHANNEL CHOICE IN DEFINED CONTRIBUTION RETIREMENT FUNDS : THE USE OF UTILITY FUNCTIONS

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In this paper a method of recommending investment channel choices to members of defined contribution retirement funds is suggested. The method is based on an interactive system, which uses the member's answers to a series of questions to derive a utility function. That function, together with a stochastic investment model, is then used to recommend an optimum apportionment of the member's investment between the respective channels. Allowance is made for future service as well as past service. The sensitivity of the recommendations to the parameters of the utility function and the stochastic model is discussed.

1. INTRODUCTION

1.1 In defined contribution retirement funds, members may have the right to choose the investment channels in which their contributions are to be invested, and to apportion their fund investments between those channels. In order to help a member decide on such choices and apportionments, it is suggested in this paper that, by obtaining the member's answers to a series of questions, his or her utility function may be estimated, and that this function, combined with a stochastic model of returns on the alternative channels, may be used to recommend a combination of channels that will maximise the member's expected utility.

1.2 The member's utility function, and the probability density function of the amount of the benefit, must be determined as at the date on which the member intends to draw his or her benefit (the 'exit date'). The probability density function will depend on the output of the stochastic model, the rules of the fund and the circumstances of the member (i.e. whether he or she qualifies for a retirement or resignation benefit) at that date.

1.3 Both the utility function and the probability density function will be specified in real terms. The benefit under consideration is the amount payable as at the exit date in respect of service up to that date.

1.4 It may be noted from the above that no allowance is made for the different benefits that may become payable on death in service or on ill-health retirement. This is deliberate. The only modes of exit allowed for are resignation and early, normal or late retirement. The reason for this is that it is assumed that appropriate risk benefits are provided.

1.5 In a recent paper the author (Thomson, 1998) developed the arguments presented in this paper for benefits in respect of service up to the date of calculation. This paper extends that argument to benefits in respect of service up to the date of exit and presents the results of sensitivity tests.

1.6 Sections 2 to 4 set out the information required with regard to benefit provisions and the member's particulars, the utility function, and the stochastic model. This information is used to find the investment channel apportionment that maximises the expected utility of the benefit at exit as explained in section 5. Section 6 discusses the sensitivity tests and section 7 concludes.

2. BENEFIT PROVISIONS AND MEMBER'S PARTICULARS

2.1 For the purposes of this paper it is assumed that the benefit at the exit date will be the total of the member's accumulated contributions and a proportion of the employer's accumulated contributions depending on the member's age and service at that date. Let:

$M(s)$ denote the member's accumulated contributions at time s;

$E(s)$ denote the employer's accumulated contributions in respect of the member at that time;

$g(x,n)$ denote the proportion of the employer's accumulated contributions payable to a member on voluntary exit at age x with n years' service;

x_0 denote the member's age as at time 0 (i.e. the date of calculation); and

n_0 denote the member's service at time 0.

Then the real benefit payable at time t (i.e. the exit date) will be:

$$B = M(t) + g(x_0+t, n_0+t)E(t).$$

2.2 For each fund, the function $g(x,n)$ must be specified. If full accumulated contributions are payable regardless of age at exit or length of service, then $g(x,n) = 1$ for all values of x and n.

2.3 The member must either specify the values of $M(0)$, $E(0)$, x_0 and n_0 (or particulars from which those values may be calculated), or enter an identity code that will enable the system to obtain those particulars from the fund's data base. The latter is preferable, as it avoids errors that might otherwise be made by the member and serves as a member-audit to identify errors that may have been made by the administrators of the fund.

2.4 In order to determine the values of $M(t)$ and $E(t)$ it will also be necessary to obtain the following information:

in respect of the fund:

c_m = the member's contribution rate; and

c_e = the employer's contribution rate;

in respect of the member:

S = the member's salary at the date of calculation.

For the purposes of specification of the employer's contribution rate, only contributions towards members' retirement benefits should be included. It is assumed that the contribution rates are constant. Values of B will then be determined from the stochastic model as explained in section 5.

3. THE UTILITY FUNCTION

3.1 For the purposes of this paper the axioms of Von Neumann and Morgenstern (1947) are assumed to hold. In order to determine a member's utility function at a particular time, it is necessary to obtain the member's answers to certain questions. The following questions, which have been adapted from sources such as Bowers et al (1986), may be posed by means of an interactive program.

3.2 The first question is of the following form:

"If you could choose your channels:
(1) so that there is a 50-50 chance of getting:
(a) a benefit of X; or
(b) a benefit of Y; or
(2) so that you will get Z with certainty;
which of (1) and (2) would you choose?
Enter 1 or 2, or if you are indifferent, leave blank: ________"

In this question, X and Y could be the upper and lower 90% confidence limits of the benefit, and Z could be their geometric mean. For the purpose of determining the confidence limits, a typical investment channel apportionment may be used, and the stochastic model applied as explained in section 4. If the member is indifferent between the alternatives (1) and (2), the use of the geometric mean suggests a logarithmic utility function.

3.3 The second question is asked if and only if the member has entered 1 or 2 in response to the first question. If the member has entered 1, the question is:

"To what value would the amount in (2) have to be increased in order to change your answer?
________"

If the member has entered 2, the question is:

"To what value would the amount in (2) have to be decreased in order to change your answer? ________"

3.4 We now provisionally fix three points on the member's utility curve. For X and Y we may set arbitrary values:

$$u(X)=0 \text{ and } u(Y)=1.$$

These values may be set arbitrarily because utility curves may be linearly transformed without affecting the maximisation of expected utility. In other words, the problem

$$\text{maximise } \mathrm{E}\{Au(x)+B\}$$

gives the same result as

$$\text{maximise } \mathrm{E}\{u(x)\}.$$

3.5 Because of the specification of the 50-50 chance, the value of the utility function for a benefit of Z (after any increase or decrease indicated by the member in the second question) is 0.5; i.e.:

$$u(Z) = \tfrac{1}{2}\{u(X)+u(Y)\} = 0.5.$$

3.6 The range (X,Y) may now be split into two ranges (X,Z) and (Z,Y). For each of these ranges we ask the same pair of questions, again with geometric means in alternative (2). The answers to these questions give us five points on the provisional utility curve, two of which are arbitrary and three of which provide meaningful information on the shape of the member's utility curve.

3.7 It would be theoretically possible to continue asking similar questions indefinitely, thus ostensibly obtaining more and more information on the shape of the member's utility curve. However, not only is the member's patience likely to wear thin, but we are not likely to obtain much more information than we could get by fitting a suitable function to the five observed values. Furthermore, because the responses are subjective, some allowance must be made for error. In any event, at some stage it will be necessary to find a method of determining values for benefit levels between those specified. For the sake of simplicity, we may as well fit a curve to the five points we have.

3.8 Let us denote the five benefit amounts as x_0, .., x_4 in increasing order of magnitude, and the corresponding observed utility values as:

$$\overset{o}{u}(x_i) = \tfrac{i}{4}. \tag{1}$$

3.9 For the purposes of this paper we use a utility function of the form:

$$u(x) = a_1 x + a_2 \ln(x) - \frac{a_3}{x} + a_4 \tag{2}$$

where a_j is the jth component of a vector $\boldsymbol{a}$ of parameters to be determined in respect of the member such that:

$$u(x_0) = 0; \text{ and}$$
$$u(x_4) = 1.$$

This function is a weighted aggregate of the three component utility functions shown in Figure 1. In that figure the three component functions have been shifted and scaled against the vertical axis for comparative purposes. Observed values have been shown for illustrative purposes. The three component functions allow for a fairly broad range of utility functions. As shown in the appendix, if $a_1 \geq 0$, $a_2 \geq 0$ and $a_3 \geq 0$, a utility function of this form has a number of characteristics that are appropriate for the recommendation of an investment channel.

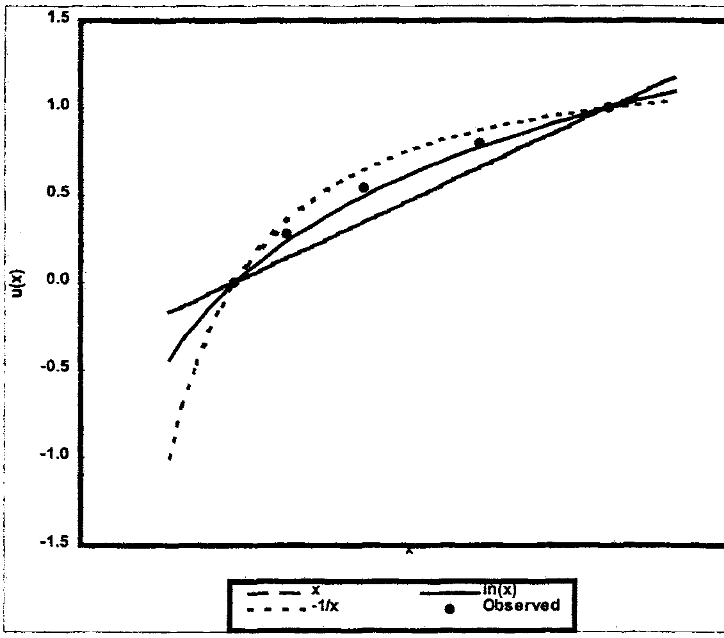

Figure 1. Component utility functions

3.10 Suppose that the observed values are:

$$\overset{\circ}{u}(x_i) = a_1 x_i + a_2 \ln(x_i) - \frac{a_3}{x_i} + a_4 + \varepsilon_i \text{ for } i = 1,2,3; \qquad (3)$$

where $\varepsilon_2 = \eta_2$ (4)

$\varepsilon_1 = \frac{1}{2}(\eta_2 + \eta_1)$ (5)

$\varepsilon_3 = \frac{1}{2}(\eta_2 + \eta_3)$ (6)

and η_i is an error term.

The specification of ε_i may be explained with reference to Figure 2.

3.11 In Figure 2, suppose the true utility of x_2 is u_2, so that:

$$u(x_2) = u_2.$$

From equation (1) the observed utility is:

$$\overset{\circ}{u}(x_2) = 0.50,$$

so that the error term is:

$$\eta_2 = \varepsilon_2 = 0.50 - u_2.$$

Suppose the true utility of x_1 is u_1, so that:

$$u(x_1) = u_1 .$$

The observed utility is:

$$\mathring{u}(x_1) = 0.25,$$

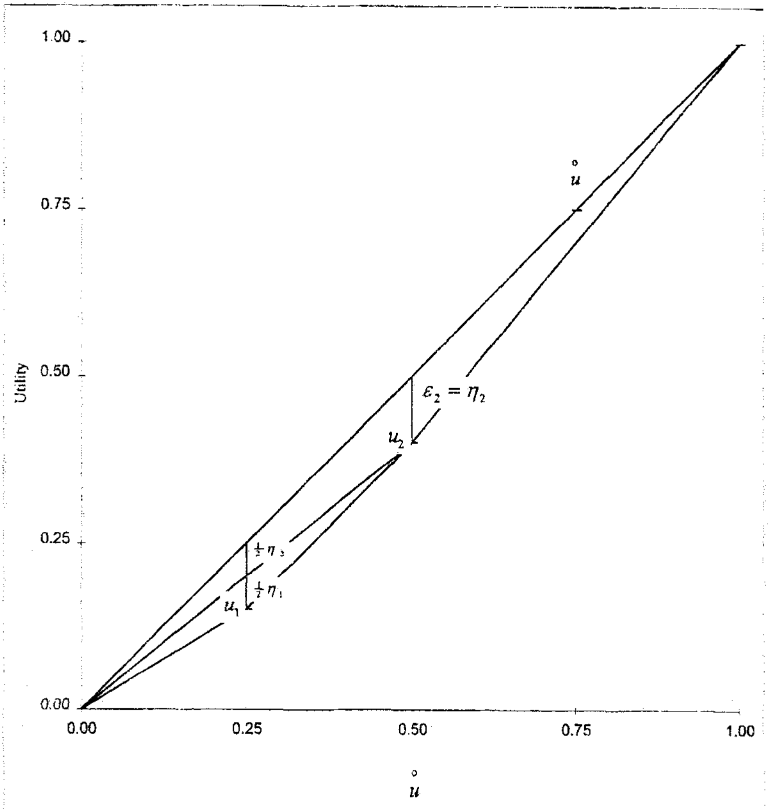

Figure 2. Error in the observed utility

so that the error term is:

$$\varepsilon_1 = 0.25 - u_1 .$$

This term may be partly explained by the error in $\mathring{u}(x_2)$. As shown in the figure, the portion of ε_1 that may be so explained is $\frac{1}{2}\eta_2$. The remaining error is likely to be smaller than η_2 because the range of possible error is (0, 0.5) instead of (0,1).

Assuming proportionality we may take the remaining error as $\frac{1}{2}\eta_1$, so that η_1 and η_2 may be assumed to have similar ranges. Equation (5) follows. The rationale for equation (6) is similar.

3.12 We now determine $\boldsymbol{a}$ so as to minimise

$$S = \sum_{i=1}^{3} \eta_i^2 \tag{7}$$

where, from equations (4), (5) and (6):

$$\eta_i = 2\varepsilon_i - \varepsilon_2,$$

and, from equation (3):

$$\varepsilon_i = \overset{\circ}{u}(x_i) - \{a_1 x_i + a_2 \ln(x_i) - \frac{a_3}{x_i} + a_4\};$$

subject to the constraints:

$$a_1 x_0 + a_2 \ln(x_0) - \frac{a_3}{x_0} + a_4 = 0;$$

$$a_1 x_4 + a_2 \ln(x_4) - \frac{a_3}{x_4} + a_4 = 1;$$

$$a_1 \geq 0;$$

$$a_2 \geq 0;$$

$$a_3 \geq 0.$$

This is a quadratic programming problem, which may be solved as such (e.g. Walsh, 1975).

3.13 It may be noted that, because the form of the utility function as specified in (2) enables us to use quadratic programming to find the utility function, the result gives a global minimum of (7), which might not otherwise be the case.

4. THE STOCHASTIC MODEL

4.1 It is assumed that a stochastic model of investment returns on the investment channels available to members of the fund, and of inflation, has been developed. The model is assumed to be a state space model specified as:

$$\boldsymbol{y}_u = \boldsymbol{H}\boldsymbol{z}_u + \boldsymbol{d} \tag{8}$$

where:

$$\boldsymbol{z}_u = \boldsymbol{F}\boldsymbol{z}_{u-1} + \boldsymbol{G}\boldsymbol{e}_u. \tag{9}$$

In this model:

$\boldsymbol{y}_u$ is a K-component vector comprising the real average forces of return on the respective channels during year u;

$\boldsymbol{z}_u$ is a vector with at least K components representing the state space;

$\boldsymbol{F}$, $\boldsymbol{G}$ and $\boldsymbol{H}$ are matrices of appropriate dimensions;

$\boldsymbol{d}$ is a K-component vector representing the means of the real returns on the respective channels; and

$\boldsymbol{e}_u$ is a vector of serially uncorrelated disturbances with a multivariate normal distribution with mean zero and covariance matrix Σ.

4.2 The value of z_0 may be defined as at an initial date, which will generally be earlier than the date of calculation as defined in ¶1.3. Furthermore, the date of exit will not necessarily be an integral number of years after the initial date. In applying the stochastic model to the determination of the distribution of the benefit these discrepancies need to be allowed for. This matter is dealt with in section 5.

4.3 From equations (8) and (9) it may be seen that $\boldsymbol{y}_u$, being a linear function of $\boldsymbol{e}_u$, which is a multivariate normal, is itself a multivariate normal. Its mean may be shown to be:

$$\mathrm{E}(\boldsymbol{y}_u) = \boldsymbol{H}\boldsymbol{F}^u z_0 + \boldsymbol{d} \qquad (10)$$

and its covariance matrix:

$$\mathrm{Cov}(\boldsymbol{y}_u, \boldsymbol{y}_v) = \boldsymbol{H}\left(\sum_{w=v-u}^{v-1} \boldsymbol{F}^w \boldsymbol{G}\boldsymbol{G}'\boldsymbol{F}'^w\right)\boldsymbol{H}' \text{ for } u \le v; \qquad (11)$$

in particular:

$$\mathrm{Cov}(\boldsymbol{y}_u) = \boldsymbol{H}\left(\sum_{w=0}^{u-1} \boldsymbol{F}^w \boldsymbol{G}\boldsymbol{G}'\boldsymbol{F}'^w\right)\boldsymbol{H}'.$$

5. The Problem and its Solution

5.1 In defining the problem we must bear in mind that the member is able to review his or her choices from time to time, and that those choices can be based on revised probability distributions of the benefit at exit, allowing for the returns earned to the date of the review.

5.2 We therefore divide the term from the initial date (which is assumed without loss of generality to coincide with the commencement of a particular year) to the end of the year of exit into N+1 periods, each comprising an integral number of years, $t(m)$ for m = 0 to N, so that:

$t(0)$ is the period from the initial date to the start of the year of calculation;

$t(m) = t^*$, a specified minimum period, for $1<m<N$;

$t(N) = t^*$ if the exit date coincides with the end of a year;
t^*+1 otherwise; and

$t(1)$ comprises the balance of the term.

The value of N is maximised, so that:

$t^* \leq t(1) < 2t^*$.
The division of the term into periods is illustrated in Figure 3.

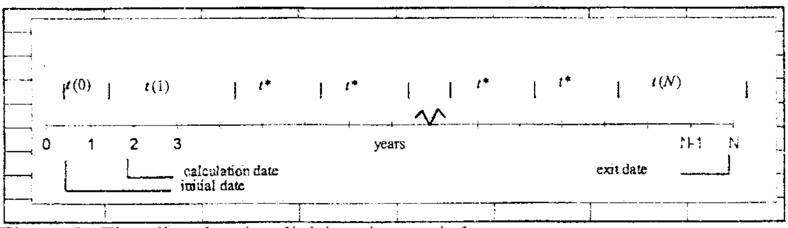

Figure 3. Time-line showing divisions into periods

5.3 Let $T(m)$ denote the period from the initial date to the end of the mth period, so that:

$$T(m) = \sum_{w=0}^{m} t(m).$$

Let $r_k(m)$ be a random variable denoting the total real force of return on the kth channel during the mth period, so that:

$$r_k(m) = \sum_{u=T(m-1)}^{T(m)} q(u,m) y_{k,u};$$

where:

$q(u,m)$ is the portion of year u to be included in the member's future service during period m; and

$y_{k,u}$ is the kth component of the vector y_u defined by equation (8).

In general:

$q(u,m) =$ 1 if year u falls in period m;
0 otherwise.

However:

for $m = 0$, we have $q(u,0) = 0$;

for $m = 1$ and u equal to the year of calculation, we have $q(u,1)$ equal to the interval from the date of calculation to the end of that year; and

for $m = N$ and u equal to the year of exit, we have $q(u,N)$ equal to the interval from the commencement of the year of exit to the date of exit.

5.4 From ¶¶4.3 and 5.3 it may be noted that, since y_u is a multivariate normal, $r_k(m)$ is normally distributed with mean:

$$\mu_k(m) = \sum_{u=T(m-1)}^{T(m)} q(u,m) \mu_k(u)$$

and covariance:

$$\sigma_{kj}(m) = \sum_{u,v=T(m-1)}^{T(m)} q(u,m)q(v,m)\mathrm{Cov}(y_{k,u}, y_{j,v}).$$

where $\mu_k(u)$ is the kth component of $E(y_u)$ obtained from equation (10) and $\mathrm{Cov}(y_{k,u}, y_{j,v})$ is obtained from equation (11).

5.5 For the sake of simplicity (as explained below) it is assumed that half the contributions payable during each period (in real terms) are paid at the beginning of that period and the other half at the end.

5.6 Let $B(m)$ denote the benefit accumulated at the end of the mth period before the contribution then due. Let $C(m)$ denote the contributions due during the mth period and accruing towards the benefit at the exit date. And let $p_k(m)$ denote the proportion invested in channel k during the mth period. Then, for $m = 1,...,N$:

$$B(m) = [B(m-1) + \tfrac{1}{2}\{C(m-1) + C(m)\}]\exp\{\sum_{k=1}^{K} p_k(m)r_k(m)\} \qquad (12)$$

where:

$B(0) = M(0) + g(x_0+t, n_0+t)E(0)$; and

$C(m) = t(m)S\{c_m + g(x_0+t, n_0+t)c_e\}$ for $1 \le m \le N$;

$\qquad\quad 0$ for $m = 0$.

5.7 For the purposes of subdivision into periods as explained in ¶5.2, it should be noted that t^* should be sufficiently large so that, for all values of m, j and k (including j=k):

$$\mathrm{Cov}\{r_k(m), r_j(m-1)\} \approx 0.$$

On the other hand, subject to that requirement, it should not be made so great that the simplified assumption regarding the payment of contributions might result in excessive errors of approximation.

5.8 First we consider the effect of the final contribution of $\tfrac{1}{2}C(N)$ assumed to be made on the date of exit. The benefit at exit may be defined as:

$$B = B(N) + \tfrac{1}{2}C(N).$$

For a given value of $B(N)$, say $B(N) = b^*$, it follows from equation (2) that the utility function of B will be:

$$U(B) = a_1(b^*+c) + a_2\ln(b^*+c) - \frac{a_3}{b^*+c} + a_4; \qquad (13)$$

where:

$$c = \tfrac{1}{2}C(N) \text{ and } B = b^*+c.$$

5.9 For the sake of simplicity (as explained below) we convert (13) into the approximate form:

$$U(B) \approx \tilde{u}(b^*) \equiv a_{N1}b^* + a_{N2}\ln b^* - \frac{a_{N3}}{b^*} + a_{N4} \tag{14}$$

where a_{Ni} is the ith component of a new vector $\boldsymbol{a_N}$ of parameters. For this purpose we select four values of b^* within the range of $B(N)$, and denote them $b_1{}^*$ to $b_4{}^*$ in ascending order. The values of $b_1{}^*$ and $b_4{}^*$ may be taken as the 90% confidence limits of $B(N)$ for typical values of $p_k(m)$. The other values may be determined as:

$$b_2^* = \sqrt[3]{b_1^{*2} b_4^*} \text{ and } b_3^* = \sqrt[3]{b_1^* b_4^{*2}} \text{ .}$$

5.10 Let

$$\boldsymbol{B^*} = \begin{Vmatrix} b_1^* & \ln b_1^* & -\frac{1}{b_1^*} & 1 \\ b_2^* & \ln b_2^* & -\frac{1}{b_2^*} & 1 \\ b_3^* & \ln b_3^* & -\frac{1}{b_3^*} & 1 \\ b_4^* & \ln b_4^* & -\frac{1}{b_4^*} & 1 \end{Vmatrix} .$$

From equation (13) we then determine:

$$\boldsymbol{u^*} = \begin{Vmatrix} U(b_1^* + c) \\ U(b_2^* + c) \\ U(b_3^* + c) \\ U(b_4^* + c) \end{Vmatrix} .$$

In order to determine $\boldsymbol{a_N}$ so as to satisfy equation (14), we let:

$$\boldsymbol{B^* a_N} = \boldsymbol{u^*} .$$

Thus:

$$\boldsymbol{a_N} = \boldsymbol{B^{*-1} u^*} . \tag{15}$$

5.11 Now for $m = N,...,1$ we proceed as follows. From ¶5.4 the conditional probability density function of Y=ln$B(m)$ for $B(m\text{-}1) = b^*$ is:

$$f(y|b^*) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\tfrac{1}{2}\left(\tfrac{y-\mu}{\sigma}\right)^2\right\} \tag{16}$$

where, from equation (12):

$$\mu = \ln(b^* + c) + \sum_{k=1}^{K} p_k(m)\mu_k(m) ; \tag{17}$$

$$\sigma^2 = \sum_{k,j=1}^{K} p_k(m)p_j(m)\sigma_{kj}(m) ; \text{ and} \tag{18}$$

$$c = \tfrac{1}{2}\{C(m-1) + C(m)\}. \qquad (19)$$

Thus the expected utility of $B(m)$ for $B(m\text{-}1) = b^*$ is:

$$E\{U(B(m))|b^*\} \approx \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (a_{m1}e^{y} + a_{m2}y - a_{m3}e^{-y} + a_{m4}) \exp\left\{-\tfrac{1}{2}\left(\tfrac{y-\mu}{\sigma}\right)^2\right\} dy$$

$$= a_{m1}\exp(\mu + \tfrac{1}{2}\sigma^2) + a_{m2}\mu - a_{m3}\exp(-\mu + \tfrac{1}{2}\sigma^2) + a_{m4}. \qquad (20)$$

5.12 The maximisation of (20), subject to:

$p_k(m) \geq 0$ for all k; and

$$\sum_{k=1}^{K} p_k(m) = 1 ;$$

is a nonlinear programming problem, which may be solved for any value of b^*. For the sake of simplicity (as explained below) we convert the maximum value of (20) into the approximate form:

$$\max E\{U(B(m)) \mid b^*\} \approx \tilde{u}(b^*) \equiv a_{m1}b^* + a_{m2}\ln b^* - \frac{a_{m3}}{b^*} + a_{m4}, \qquad (21)$$

using the same method as in ¶5.9. Using the same method as in ¶5.10 we then determine $\boldsymbol{a}_m$. Equation (21) defines an 'indirect' or 'derived' utility function (Mossin, 1968) for each period, which is used successively to define similarly derived utility functions for previous periods by means of equations (15) to (20).

5.13 The process described in ¶¶5.11 and 5.12 is continued until we reach the first period. For that period, instead of finding the maximum value of (20) for a series of values of $B(0)$, we find the values of $p_k(1) \geq 0$ that maximise (20) for

$$b^* + c = B(0) + \tfrac{1}{2}C(1).$$

From the results given by Mossin (*op. cit.*) it follows that those values maximise the expected utility of the member's benefit at the exit date. They may thus be presented to the member as the recommended apportionment.

It may be noted that (20) is expressed in closed form, thus obviating the need for approximate integration, which would necessitate considerably more computer time. It is for this reason that the simplifications referred to in ¶¶5.5, 5.9 and 5.12 are introduced.

6. Sensitivity

6.1 From equation (20) it may be seen that, if:

$$\left|\frac{d\sigma^2}{dp_i}\right| < 2\left|\frac{d\mu}{dp_i}\right| \text{ for all } I; \text{ and}$$

$$a_{mj} \geq 0 \text{ for } j=1,2,3,$$

then $p_i = 1$ for $\mu_i = \max(\mu_i)$;
in other words, under these conditions, during period m, the full investment should be placed in the channel with the greatest expected return.

6.2 For simplicity we consider the case of two investment channels with:

$$\mu_1(m) > \mu_2(m).$$

Now, from equation (17):

$$\frac{d\mu}{dp_1(m)} = \mu_1(m) + \frac{dp_2(m)}{dp_1(m)}\mu_2(m).$$

Since $p_1(m) + p_2(m) = 1$,

$$\frac{dp_2(m)}{dp_1(m)} = -1.$$

Thus

$$\frac{d\mu}{dp_1(m)} = \mu_1(m) - \mu_2(m).$$

And from equation (18):

$$\frac{d\sigma^2}{dp_1(m)} = 2p_1(m)\sigma_{11}(m) + 2\left\{p_2(m) + p_1(m)\frac{dp_2(m)}{dp_1(m)}\right\}\sigma_{12}(m) + 2p_2(m)\frac{dp_2(m)}{dp_1(m)}\sigma_{22}(m)$$

$$= 2\left[p_1(m)\sigma_{11}(m) + \{p_2(m) - p_1(m)\}\sigma_{12}(m) - p_2(m)\sigma_{22}(m)\right].$$

Therefore:

$$\left|p_1(m)\sigma_{11}(m) + \{p_2(m) - p_1(m)\}\sigma_{12}(m) - p_2(m)\sigma_{22}(m)\right| < \left|\mu_1(m) - \mu_2(m)\right| \tag{22}$$

$$\Leftrightarrow \left|\frac{d\sigma^2}{dp_1(m)}\right| < 2\left|\frac{d\mu}{dp_1(m)}\right| \text{ and, by symmetry, } \left|\frac{d\sigma^2}{dp_2(m)}\right| < 2\left|\frac{d\mu}{dp_2(m)}\right|.$$

6.3 Now the left-hand side of (22) reaches its maximum when $p_1(m)$=1 and $p_2(m)$=0 or vice versa. Thus for (22) to apply for all values of $p_1(m)$ it is necessary and sufficient that:

$$\left|\sigma_{11}(m) - \sigma_{12}(m)\right| < \mu_1(m) - \mu_2(m) \tag{23}$$

and

$$\left|\sigma_{22}(m) - \sigma_{12}(m)\right| < \mu_1(m) - \mu_2(m) \tag{24}$$

6.4 The method used for the approximation of the derived utility functions does not necessarily ensure that the coefficients a_{m1}, a_{m2} and a_{m3} will be non-negative. If μ and σ^2 were constant for all periods and if no future contributions were payable, the multi-period approach would be unnecessary. The fitted utility function could then be used and subject to conditions (23) and (24) the full investment should be placed in channel 1. From this it may be inferred that, if the variation in If μ and σ^2 is low and the annual contributions are low relative to accumulated contributions and if:

$$\left|\sigma_{ii}(m) - \sigma_{12}(m)\right| < \mu_1(m) - \mu_2(m) \tag{25}$$

then the full investment should be placed in channel 1.

6.5 In view of the fuzziness of the above results, it is helpful to consider numerical examples. For this purpose it was again assumed that there are two investment channels. For each combination of a set of utility functions, member particulars and terms to exit, and for a specified stochastic investment model, the proportion recommended for investment channel 1 was calculated.

6.6 The utility functions used were as follows:

$$u(x) = x;$$

$$u(x) = \ln x;$$

$$u(x) = -\frac{1}{x}; \text{ and}$$

$$u(x) = 10^{-5}x - \frac{10^5}{x}.$$

It may be noted that these are particular cases of equation (2).

6.7 Under normal circumstances, for a given level of accumulated contributions at the calculation date, the longer the term to exit, the greater will be the benefit. Unless some care is taken to separate the effects of the magnitude of the benefit from those of the term to exit, it will not be possible to distinguish them. For this reason it was assumed that the total period from entry to exit was 30 years. The accumulated contributions at the calculation date were equal to the contributions accumulated from the entry date to the calculation date with expected investment returns based on the model and the investment channel apportionments that would have been recommended from time to time over that period. For this purpose a salary of \$20,000, \$50,000 or \$100,000 was assumed. (The '\$' sign represents an unspecified currency.) A fully vesting contribution rate of 15% of salary was assumed.

6.8 The parameters assumed for the model were as follows:

$$F = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix};$$

$$G = \begin{Vmatrix} .02 & 0 \\ 0 & .01 \end{Vmatrix};$$

$$H = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix};$$

$$d = \begin{Vmatrix} .04 \\ .03 \end{Vmatrix}; \text{ and}$$

$$z_0 = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix}.$$

It may be noted that these parameters imply that:

$$\mathrm{Cov}(y_n, y_{n-1}) = 0;$$

and for a one-year period:

$$E(y_1) = .04; E(y_2) = .03;$$
$$\mathrm{Var}(y_1) = .0004; \mathrm{Var}(y_2) = .0001;$$
$$\mathrm{Cov}(y_1, y_2) = 0.$$

6.9 Other assumptions made for the purposes of the sensitivity tests were that:

(1) for the purposes of ¶5.9, 'typical' values of $p_k(m)$ were taken as .5; and

(2) the minimum period t^* referred to in ¶5.2 was taken as 1 year.

6.10 Over the entire range of the tests it was found that $p_1(1)=1$. This substantiates the findings in ¶6.4 for the parameters selected.

6.11 It is instructive to consider the point at which $\mathrm{Var}(y_1)$ becomes sufficiently great to warrant diversification into channel 2. This was done for the same combinations of parameters by increasing G_{11} (which, with H and F as specified, represents the standard deviation of y_1) to the critical value at which $p_1(1)$ starts to decline. As expected (for the reasons explained in ¶6.13), it was found that, for $u(x) = x$ or $\ln(x)$, an increase in G_{11} did not warrant diversification. For $u(x) = -\frac{1}{x}$ the results, which are shown in Table 1, were the same for all levels of salary. The reason for this is that the relative risk aversion is constant, as shown in the appendix. For

$$u(x) = 10^{-5}x - \frac{10^5}{x}$$

an increase in G_{11} did not warrant diversification.

6.12 It is of interest that, despite the fact that the utility function is isoelastic, the critical value of G_{11} differs by term to exit. This is because future contributions disturb the effects of isoelasticity. The greater the term to exit, the greater the standard deviation tolerated before diversification out of the channel with the higher expected yield becomes advisable.

Table 1. Critical values of G_{11} for $u(x) = -\,^1/_x$

Term to exit (years)	Critical value of G_{11}
1	.101
5	.110
10	.134
30	.259

6.13 From equation (20) it may be noted that, for a risk-neutral investor, for whom:

$$u(x) = a_1 x + a_4,$$

expected utility is maximised by maximising:

$$\mu + \tfrac{1}{2}\sigma^2.$$

For such an investor the desirability of an investment channel is enhanced by an increase in the variance of its return. This runs counter to the concept of mean-variance analysis, where the object is to obtain the highest expected return for a given variance, or the lowest variance for a given expected return. The paradox is resolved by the fact that the approach used in this paper, by assuming lognormal returns, allows for the effects of compounding. The utility function is expressed in terms of the benefit after compounding, not in terms of the annual return. Thus a risk-neutral investor is indifferent as between a 50-50 chance of a benefit of .5 or 1.5 and a benefit of 1 with certainty. On the other hand, after compounding, a lognormally distributed benefit with an expected value of 1 is equally likely to amount to 2 as to .5. The enhanced upside potential is attractive to a risk-neutral investor, and the greater the variance, the greater the enhancement. It may also be noted from equation (20) that, for an investor with logarithmic utility, the variance of the return is immaterial. Such an investor is indifferent as between a 50-50 chance of a benefit of .5 or 2 and a benefit of 1 with certainty.

6.14 Whether the function specified by equation (2) (with non-negative coefficients) is adequate to explain the risk aversion of any member is a matter for further research. It may be noted that the first three terms of that function are the start of a series whose derivatives are:

$$a_{j+1} x^{-j} \text{ for } j = 0,1,\ldots$$

To allow for greater risk aversion the function could be modified by selecting higher negative powers. Conversely, if the levels of risk aversion accommodated by equation (2) are unnecessarily high, fractional powers could be used. In either case the methods of section 5 could be adapted. The arguments of ¶6.1 to 6.4 would also have to be modified.

7. CONCLUSION

7.1 This paper sets out methods that may be used to estimate the utility function of the benefits payable to a member on exit from a defined contribution retirement fund and to use that function, combined with a stochastic model of returns on the alternative channels, to recommend a combination of channels that will maximise the member's expected utility.

7.2 It would be of interest to test the ranges of the parameters of the utility function over a sample of members of funds and to explore possible methods of estimating those parameters by means of questions that members may find easier to answer. Another matter of interest would be to test the sensitivity of the results to the errors of approximation introduced as a result of the simplifications adopted in this paper. Finally, a number of authors—both actuarial (notably Ramsay, 1993) and other (notably Allais, 1953)—have drawn attention to suggested weaknesses in utility theory. Consideration should be given to the possible defence of utility theory against their arguments in the context of the recommendation of appropriate investment channels for defined contribution schemes.

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APPENDIX

CHARACTERISTICS OF THE UTILITY FUNCTION

A.1 The utility function used in this paper is of the form:

$$u(x) = \sum_{j=1}^{4} a_j g_j(x) \qquad \text{(equation (5))}.$$

A.2 From (5) we have:

$$u'(x) = a_1 + \frac{a_2}{x} + \frac{a_3}{x^2};$$

$$u''(x) = -\frac{a_2}{x^2} - \frac{2a_3}{x^3}; \text{ and}$$

$$u'''(x) = \frac{2a_2}{x^3} + \frac{6a_3}{x^4}.$$

A.3 In order to ensure that

$$\forall x > 0: u'(x) > 0 \qquad \text{(A1)}$$

it is necessary and sufficient that:

$$a_1 \geq 0 \text{ and } a_2 \geq -2\sqrt{a_1 a_3} \text{ and } a_3 \geq 0$$

with at least one of the inequalities sharp.

A.4 The member's absolute risk aversion function is defined (Pratt, 1964) as:

$$r(x) = -\frac{u''(x)}{u'(x)}.$$

In order to ensure that (A1) applies and that

$$\forall x > 0: r(x) > 0$$

it is necessary and sufficient that:

$$a_1 \geq 0 \text{ and } a_2 \geq 0 \text{ and } a_3 \geq 0 \qquad \text{(A2)}$$

with at least one of the inequalities sharp.

A.5 It may be noted that if, as would normatively be the case, the member has decreasing risk aversion, then:

$$r'(x) = \frac{\{u''(x)\}^2 - u'(x)u'''(x)}{\{u'(x)\}^2} < 0. \qquad \text{(A3)}$$

If $a_1 \geq 0$ and $a_2 \geq 0$ and $a_3 \geq 0$ with one of the inequalities sharp then (A3) will in fact hold for the utility function (5).

A.6 The relative risk aversion function is defined (*ibid.*) as:

$$R(x) = xr(x).$$

If

$$R'(x) = 0 \tag{A4}$$

then $u(x)$ is said to be 'iso-elastic' and it is possible to make 'myopic' decisions (i.e. decisions for one period at a time without consideration of subsequent periods) which will nevertheless be optimal (Mossin, 1968). This simplifies considerably the decision-making process.

A.7 If only one of the inequalities in (A2) is sharp, (A4) will hold. However, it is not considered appropriate to presuppose that members' utility functions will be iso-elastic, and multiperiod optimisation methods must therefore be used.