

## GUARANTEED EQUITY-LINKED PRODUCTS

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### Abstract

Equity-linked products with an asset value guarantee are now becoming very popular in Germany both as pure investment contracts and in the context of life insurance policies. All of these products have a common feature: at maturity, the contract pays off a preagreed (deterministic) amount. In addition, the contract's payoff is linked - via the so-called index participation rate - to an underlying equity index (or basket). Hence, if the index performs well over the lifetime of the contract, a considerably higher amount might be paid back.

This paper shows how the fair rate of index participation  $x$  can be calculated within a deterministic interest rate economy. In particular, we discuss two different, popular products, derive explicit pricing formulas and discuss, how  $x$  depends on the costs, the preagreed amount, the level of interest rates and the volatility of the index. Indeed,  $x$  depends in a continuously differentiable way on these parameters, which is important when marketing the product. We also discuss, how  $x$  depends on the maturity of the contract and analyse the products' historical performance.

Building on the ideas of an earlier version of this paper, our group did extensive research in the field of equity-linked life insurance policies in Germany which is of considerable interest for the financial community.

# 1 Introduction

In America as well as in Europe, products, where the investor participates in the development of a stock market but is also offered a certain guaranteed (prefixed and deterministic) minimum payoff at maturity, are sold successfully by investment firms as well as life insurance companies.<sup>1</sup> In Germany, such contracts are very popular by now, as well. Within only a few months after their introduction, about 4 bn DEM were invested in such contracts. Further interest is in place due to an ongoing discussion on, e.g., pension funds.

The basic idea of such products is simple. The investor buys a security (paying the face value of, e.g., 100), that guarantees to pay back at maturity the invested money (or some other deterministic sum, e.g., a certain guaranteed rate of interest can be earned on the invested capital). Furthermore, the investor gets back an additional sum depending on the performance of the underlying index. This additional sum is determined via the so-called rate of index participation. This is the percentage rate by which the investor participates in index gains. The rate of index participation depends on the current market conditions (term structure of interest rates and index volatility), the fees and the payoff-function of the contract. It is fixed before the contract is marketed and guaranteed for the term of the contract.<sup>2</sup>

In the present paper, we introduce a model for the economy and, within this model, we explain how the fair rate of index participation is calculated. For two specific products, we analyse the dependence of the participation rate on costs and the market conditions, which is important to "adjust" the participation rate from a marketing point of view. Finally, we look at historical returns of the two products.

Our paper is organised as follows: In Section 2, we introduce our model for the economy and in Section 3, we discuss two interesting products, being already sold. We furthermore give explicit pricing formulas for these securities.<sup>3</sup> The proofs for these formulas are given in Appendix A. In Section 4, we discuss the conditions under which the rate of index participation exists and how it depends on the different parameters. Mathematical details are given in Appendix B. Section 5

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<sup>1</sup>In the U.K., e.g., guaranteed equity-linked insurance policies in 1996 had a market share of 20% with respect to new business. This means an amount of 2bn GBP in premium. In Germany, the first guaranteed equity-linked insurance policy was sold in August 1996 by a foreign provider. Up to now, there are six german insurance companies offering such policies. An introductory discussion of guaranteed equity-linked life insurance in Germany can be found in [Hi 96] and [No/Sc 97].

<sup>2</sup>As the rate of index participation depends heavily on the market parameters, the conditions of the contract are fixed and the security is then marketed and sold over a certain period of time, in general six to eight weeks.

<sup>3</sup>The importance of explicit pricing formulas for an efficient evaluation of the rate of index participation is stressed in Section 4.

shows the theoretical historical returns of our products and in Appendix C and D, we analyse the impact of a time dependent term structure of interest rates and of the term of the contract, respectively.

## 2 The model for the economy

We assume the underlying index to be a performance index and to follow a geometric Brownian motion in the time interval  $[0, T]$ , i.e.

$$\frac{dS_t}{S_t} = \mu(t)dt + \sigma(t)dW_t \quad (1)$$

with time dependent drift and volatility  $\mu(t)$  and  $\sigma(t)$  ( $> 0$ ), respectively. Here,  $W_t$  denotes a Wiener process. We furthermore assume the short rate process  $r(t)$  to be deterministic and to fit the current riskless term structure of interest rates, i.e.

$$\int_{t_1}^{t_2} r(t)dt = (t_2 - t_1) \cdot f_{t_1, t_2},$$

where  $f_{t_1, t_2}$  denotes today's continuous, annualized observable forward rate for the period of time  $0 \leq t_1 < t_2 \leq T$ .<sup>4</sup>

According to [Ha/Pl 81], there exists a unique probability measure  $Q$ <sup>5</sup> such that the price  $A_t$  at time  $t$  of any index-linked security, that pays off  $A_T$  at time  $T$  and does not generate any other payoffs can be calculated by

$$A_t = E_Q \left[ e^{-\int_t^T r(s)ds} A_T \middle| \mathcal{I}_t \right], \quad 0 \leq t \leq T. \quad (2)$$

Here,  $E_Q \{ \cdot | \mathcal{I}_t \}$  denotes the expected value under the (so-called equivalent martingale) measure  $Q$  given the information available at time  $t$ . Applying Girsanov's

<sup>4</sup>It is possible to allow for stochastic processes  $\mu$ ,  $r$  and  $\sigma$ , as well, which significantly increases the complexity of the problem. In what follows, we mainly discuss the case of a constant short rate ( $r(t) \equiv r$ ) and volatility ( $\sigma(t) \equiv \sigma$ ). Time dependent parameters are analysed in Appendix C.

<sup>5</sup>The unique existence of such a measure  $Q$  is essentially equivalent to the assumption of a complete, arbitrage-free market.

theorem, we get the dynamics of the process (1) under  $Q$ :

$$\frac{dS_t}{S_t} = r(t)dt + \sigma(t)dW_t^Q, \quad (1')$$

where  $W_t^Q$  denotes a Wiener process under  $Q$ . Given  $S_0 (> 0)$ , the solution of (1') (cf. [Ka/Sh 88] for the conditions that  $r(t)$  and  $\sigma(t)$  have to fulfill) is given by

$$S_t = S_0 \exp \left\{ f(t) + W_{g(t)}^Q \right\} \quad \text{with} \quad f(t) = \int_0^t r(s) - \frac{\sigma^2(s)}{2} ds \quad (1'')$$

$$\text{and} \quad g(t) = \int_0^t \sigma^2(s) ds.$$

In particular,  $\ln(S_t/S_0)$  has a Gaussian distribution with expectation  $f(t)$  and variance  $g(t)$ . The explicit calculation of (2) is often difficult or impossible. Then, numerical methods like Monte Carlo simulations are required.

### 3. Two different payoff patterns and pricing formulas

In what follows, we examine two interesting index-linked contracts with a term of  $T = 5$  years,<sup>6</sup> that are bought today ( $t = 0$ ) for 1 DEM and generate a payoff at maturity of

$$A_T^1 = \prod_{k=1}^T \left\{ 1 + \max \left[ i, \frac{S_k - S_{k-1}}{S_{k-1}} \cdot x \right] \right\}$$

$$A_T^2 = (1 + i)^T + \max \left[ 0, \frac{S_T - S_0}{S_0} \cdot x \right],$$

where  $i$  denotes the guaranteed rate of interest (in general  $i \geq 0$ ) and  $x (> 0)$  denotes the rate of index participation. In  $A_T^1$ , compound interest is earned on the invested capital. The interest earned in a specific year is  $i$  or  $x$  times the return of the index in that year. Product  $A_T^2$  has no lock-in feature. Here, additionally to the guaranteed sum  $G = (1 + i)^T$ ,  $x$  times the return of the index during the term

<sup>6</sup>Other values of  $T$  are examined in Appendix D.

of the contract is paid (if positive). The minimum sum payable at expiration  $T$  is obviously the same for both products:  $G = (1 + i)^T$ .

In Germany, products with a payoff function similar to  $A_T^2$  and  $i = 0\%$  and a term of 3 to 5 years are very popular. For many investors, however, a product with a lock-in-feature like product  $A_T^1$  might be more interesting, as such a product is not as sensitive with respect to a possible crash slightly before expiration. Hence, payoff patterns similar to  $A_T^1$  are often used in guaranteed equity-linked life insurance products.<sup>7</sup>

For these payoff functions, today's value can be calculated according to (2). Under the additional assumption  $r(t) \equiv r$  (= continuous, annualized, riskless short rate) and  $\sigma(t) \equiv \sigma$  (= annualized index volatility), we get

$$A_0^1 = e^{-Tr} \left\{ 1 + i + (x + i)e^r c\left(\frac{x}{x+i}, 1, 1\right) \right\}^T \quad (2^1)$$

$$A_0^2 = e^{-Tr} \left\{ (1 + i)^T + xe^{Tr} c(1, 1, T) \right\} , \quad (2^2)$$

where  $c(\alpha, \beta, \gamma)$  denotes today's value of a European call-option on the index with today's index value  $\alpha$ , strike  $\beta$  and time to maturity  $\gamma$  according to the well-known Black/Scholes formula. A proof for these formulas (for general  $r(t)$  and  $\sigma(t)$ ) is given in Appendix A.

Figure 1 shows the prices  $A_0$  as functions of  $x$  and  $i$ , given  $r = 4.9\%$  and  $\sigma = 12.98\%$ . Figure 2 shows the prices as functions of  $r$  and  $\sigma$ , given  $x = 50\%$  and  $i = 2\%$ .

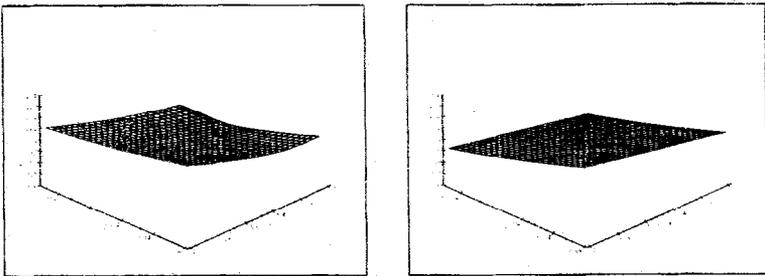


Figure 1:  $A_0^1(x, i)$  (left) and  $A_0^2(x, i)$  (right)

<sup>7</sup>Cf. e.g. [No/Sc 97], [No/Ru 97a].

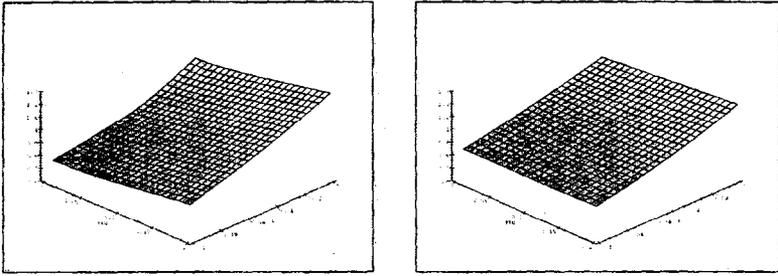


Figure 2:  $A_0^1(r, \sigma)$  (left) and  $A_0^2(r, \sigma)$  (right)

#### 4 Calculation of the fair rate of index participation in a model allowing for costs

As we assume today's value of the securities to be 1 DEM, the fair rate of index participation is calculated (given costs of  $a$  DEM that are deducted up front) as a solution  $x > 0$  of the implicit equation

$$A_0^j = A_0^j(x^j, i, r, \sigma) = 1 - a. \quad (3)$$

In the case  $j = 1$ ,<sup>8</sup> it is furthermore necessary that  $x + i > 0$ , such that (2<sup>1</sup>) and hence (3) are well defined. We therefore look for solutions  $x > \max\{0, -i\}$  and the question is, for which parameters  $i > -100\%$ ,  $r > 0$ ,  $\sigma > 0$ ,  $a$  arbitrary such a solution exists.<sup>9</sup> In Appendix B, we calculate the set of parameters, for which such a solution exists and show, that  $x$  is always unique and furthermore a continuously differentiable function of the parameters. Furthermore,  $x$  is decreasing in  $i$ ,  $\sigma$  and  $a$  and increasing in  $r$  as expected.

Figure 3 shows  $x = x(a, i)$  for given  $r = 4.9\%$  and  $\sigma = 12.98\%$ . Figure 4 shows  $x = x(r, \sigma)$  for given  $a = 5\%$  and  $i = 2\%$ , which approximately equals the current rate of inflation in Germany.

<sup>8</sup>As  $A_0^2$  is linear in  $x$ , the following is only of interest for the case  $j = 1$ .

<sup>9</sup>This choice of the parameters is no restriction, since for  $i \leq -100\%$  the whole capital might be destroyed or, even worse, the investor might have to make an additional payment at  $t = T$ .

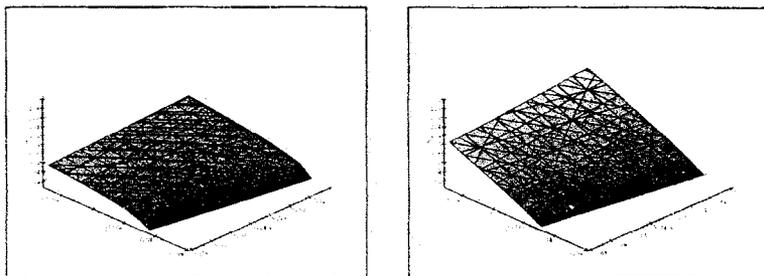


Figure 3:  $x^1(a, i)$  (left) and  $x^2(a, i)$  (right)

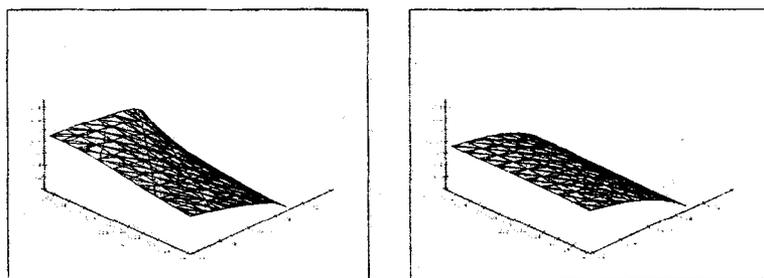


Figure 4:  $x^1(r, \sigma)$  (left) and  $x^2(r, \sigma)$  (right)

The time required to calculate solutions of (3), especially to create the above figures is neglectable, if the formulas (2') are used. If, however, no explicit pricing formulas are available and (2) has to be calculated by numerical methods like Monte Carlo Simulation techniques, this takes a considerable amount of time.<sup>10</sup> Explicit pricing formulas are even more important, if the distribution of the price of the security over the term of the contract has to be analysed, which is relevant in the context of guaranteed equity-linked life insurance and the problem of calculating the so-called additional policy reserves (required under German legislation) cf., e.g., [No/Sc 97], [No/Ru 97a] and [No/Ru 97b].

<sup>10</sup>All figures and calculations were generated using MAPLE V3.

## 5 Historical return of the products

In this Section, we look at the historical performance of our products. We assume a guaranteed rate of interest of  $i = 2\%$  and costs of  $a = 0\%$ ,  $3\%$  and  $5\%$ , respectively. We then derive for January 1, 1982-1991 the fair rates of index participation  $x^j$  for our products, given the historical values of  $r$  and  $\sigma$  and then calculate the annualized 5 year return of our products.<sup>11</sup>

In table 1, we give these results as well as the return of the DAX30 and the REXP (a bond-market performance index).<sup>12</sup>

|    | Product 1 |           |           | Product 2 |           |           | DAX30  | REXP  |
|----|-----------|-----------|-----------|-----------|-----------|-----------|--------|-------|
|    | $i = 2\%$ |           |           | $i = 2\%$ |           |           |        |       |
|    | $a = 5\%$ | $a = 3\%$ | $a = 0\%$ | $a = 5\%$ | $a = 3\%$ | $a = 0\%$ |        |       |
| 82 | 17.7%     | 18.6%     | 19.9%     | 19.6%     | 20.6%     | 21.9%     | 23.9%  | 11.0% |
| 83 | 13.8%     | 14.8%     | 16.1%     | 9.7%      | 10.4%     | 11.4%     | 12.6%  | 8.7%  |
| 84 | 12.8%     | 13.6%     | 14.7%     | 9.1%      | 9.7%      | 10.5%     | 11.4%  | 8.7%  |
| 85 | 15.3%     | 16.5%     | 18.3%     | 11.9%     | 12.9%     | 14.3%     | 16.9%  | 6.4%  |
| 86 | 7.1%      | 7.7%      | 8.6%      | 2.2%      | 2.2%      | 2.3%      | 0.5%   | 4.6%  |
| 87 | 5.9%      | 6.5%      | 7.4%      | 2.7%      | 2.8%      | 3.0%      | 2.0%   | 5.1%  |
| 88 | 4.5%      | 5.0%      | 5.6%      | 4.9%      | 5.4%      | 6.1%      | 9.1%   | 6.4%  |
| 89 | 7.8%      | 8.5%      | 9.5%      | 7.2%      | 7.9%      | 8.8%      | 11.3%  | 8.3%  |
| 90 | 6.4%      | 6.7%      | 7.3%      | 3.7%      | 3.9%      | 4.1%      | 3.3%   | 7.4%  |
| 91 | 6.9%      | 7.2%      | 7.7%      | 8.0%      | 8.5%      | 9.1%      | 10.0%  | 10.5% |
| ⊙  | 9.82%     | 10.51%    | 11.51%    | 7.90%     | 8.43%     | 9.15%     | 10.10% | 7.71% |

Table 1: historical, annualized 5 year return

The results show the difference between the payoff-patterns  $A_1^j$  and  $A_2^j$ : Only in three years (1982/88/91), the return of product 2 exceeds product 1. In all other years, product 1 outperformed product 2 by up to 6% (in 1986). The average return of product 1 is 2% higher than that of product 2, although the returns are less volatile. From a risk/return point of view, historically, product 1 was preferable. If we look at the returns in case  $a = 5\%$  (which is a usual up-front fee for investment funds), product 2 in average only returned the (riskless!) REXP-return and performed significantly worse than the DAX30. Product 1, however, in average performed as well as the DAX30.

<sup>11</sup>The riskless rate of interest for 5 years was calculated from the REX1 up to REX5 values (coupon 7.5%) by bootstrapping, the DAX30 volatility was estimated from the logarithm of the overnight returns of the preceding 250 trading days. The annualized 5 year return for product  $j$  was then calculated as a solution  $R^j$  of the equation  $(1 + R^j)^5 = 1 + (A_{t+5}^j - A_t^j)/A_t^j$ .

<sup>12</sup>When calculating the DAX30- and REXP-returns, we did not consider any involved costs.

Of course, table 1 only gives an (arbitrary) sample of historical returns, but it shows, how sensitive these returns are with respect to changes in the product design. Hence, the decision of an investor to buy a specific product will depend heavily on his expectation of future market scenarios. Whether or not such guaranteed products are attractive for an investor depends on his risk aversion. Of course, imposing a guarantee will (in average) lower the return.<sup>13</sup>

## A Derivation of the pricing formulas

We first determine today's value  $P_t$  of a performance option paying back

$$P_T = \max \left[ i, \frac{S_{t_2} - S_{t_1}}{S_{t_1}} \cdot x \right] ; \quad i \geq 0, x > 0, 0 \leq t_1 < t_2 \leq T$$

at time  $T$ .

For  $t_1 \leq t \leq t_2$ , we get from (2):

$$\begin{aligned} P_t &= E_Q \left[ e^{-\int_t^T r(s) ds} P_T \Big| \mathcal{F}_t \right] \\ &= e^{-\int_t^T r(s) ds} \left\{ i + \frac{x}{S_{t_1}} e^{\int_t^{t_2} r(s) ds} E_Q \left[ e^{-\int_t^{t_2} r(s) ds} \max \left[ 0, S_{t_2} - S_{t_1} \left( 1 + \frac{i}{x} \right) \right] \Big| \mathcal{F}_t \right] \right\} \\ &= e^{-\int_t^T r(s) ds} \left\{ i + \frac{x}{S_{t_1}} e^{\int_t^{t_2} r(s) ds} c_i \left( S_{t_1}, S_{t_1} \left( 1 + \frac{i}{x} \right), t_2 - t \right) \right\} \\ &= e^{-\int_t^T r(s) ds} \left\{ i + (x+i) e^{\int_t^{t_2} r(s) ds} c_t \left( \frac{S_t}{S_{t_1}} \cdot \frac{x}{x+i}, 1, t_2 - t \right) \right\} \end{aligned}$$

$$\text{Here, } c_t(\alpha, \beta, \gamma) = \alpha N(d_1) - \beta e^{-\int_t^{t+\gamma} r(s) ds} N(d_2)$$

<sup>13</sup>In particular, one could ask whether the desired guarantee can be achieved cheaper by a good diversification strategy. Furthermore, we would like to remark, that volatility might not be the "right" risk measure for an individual investor, since this does not take into account his specific situation (e.g. the need for a certain minimum return to match some future liabilities). Hence, it might be worthwhile to consider so-called downside-risk measures like lower partial moments, cf., e.g., [Ha 91].

with 
$$d_1 = \frac{\ln \frac{\alpha}{\beta} + \int_t^{t+\gamma} r(s) + \frac{\sigma^2(s)}{2} ds}{\left( \int_t^{t+\gamma} \sigma^2(s) ds \right)^{\frac{1}{2}}}, \quad d_2 = d_1 - \left( \int_t^{t+\gamma} \sigma^2(s) ds \right)^{\frac{1}{2}}$$

and 
$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{s^2}{2}} ds$$

denotes today's value of a European call-option on the index with current index value  $\alpha$ , strike  $\beta$  and term to maturity  $\gamma$  according to the Black/Scholes formula.

Hence, we get for  $t < t_1$ :

$$\begin{aligned} P_t &= E_Q \left[ e^{-\int_t^T r(s) ds} P_T \Big| \mathcal{F}_t \right] = E_Q \left[ e^{-\int_t^{t_1} r(s) ds} E_Q \left[ e^{-\int_{t_1}^T r(s) ds} P_T \Big| \mathcal{F}_{t_1} \right] \Big| \mathcal{F}_t \right] \\ &= E_Q \left[ e^{-\int_t^{t_1} r(s) ds} e^{-\int_{t_1}^T r(s) ds} \left\{ i + (x+i) e^{\int_{t_1}^{t_2} r(s) ds} c_{t_1} \left( \frac{x}{x+i}, 1, t_2 - t_1 \right) \right\} \Big| \mathcal{F}_t \right] \\ &= e^{-\int_t^T r(s) ds} \left\{ i + (x+i) e^{\int_t^{t_2} r(s) ds} c_{t_1} \left( \frac{x}{x+i}, 1, t_2 - t_1 \right) \right\}. \end{aligned}$$

Taking also into account the trivial case  $t_2 \leq t \leq T$ , we get

$$P_t = \begin{cases} e^{-\int_t^T r(s) ds} \left\{ i + (x+i) e^{\int_t^{t_2} r(s) ds} c_{t_1} \left( \frac{x}{x+i}, 1, t_2 - t_1 \right) \right\} & ; t \leq t_1 \\ e^{-\int_t^T r(s) ds} \left\{ i + (x+i) e^{\int_t^{t_2} r(s) ds} c_t \left( \frac{S_t}{S_{t_1}} \cdot \frac{x}{x+i}, 1, t_2 - t \right) \right\} & ; t_1 \leq t \leq t_2 \\ e^{-\int_t^T r(s) ds} \max \left[ i, \frac{S_{t_2} - S_{t_1}}{S_{t_1}} \cdot x \right] & ; t_2 \leq t \leq T \end{cases}$$

As the returns  $(S_k - S_{k-1})/S_{k-1}$  are independent, we get with  $t = 0$ ,  $t_k = i$  and  $T = \bar{5}$ :

$$A_0^t = E_Q \left[ e^{-\int_0^T r(s) ds} A_T^t \Big|_{t=0} \right]$$

$$= \begin{cases} e^{-\int_0^T r(s) ds} \prod_{k=1}^T \left\{ 1 + i + (x+i)e^{i-k} c_{k-1} \left( \frac{x+i}{x+i}, 1, 1 \right) \right\} & ; j = 1 \\ e^{-\int_0^T r(s) ds} \left\{ (1+i)^T + x e^{\int_0^T r(s) ds} c_0(1, 1, T) \right\} & ; j = 2, \end{cases}$$

and letting  $r(t) \equiv r$ ,  $\sigma(t) \equiv \sigma$  we get (2<sup>1,2</sup>).<sup>14</sup>

## B Dependence of the rate of index participation on the parameters

We here only investigate the equation  $A_0^1(x, i, r, \sigma) = 1 - a$ , as  $A_0^2$  is linear in  $x$ . We denote by  $\mathbb{R}$  and  $\mathbb{R}_+$  the real numbers and the positive real numbers, respectively. We first recall the partial derivatives (risk parameters) of the Black/Scholes formula  $c = c(S, X, T - t, r, \sigma)$  for today's value of an European call-option on the index with today's index value  $S$ , strike  $X$ , (constant) short rate  $r$ , index volatility  $\sigma$  and time to maturity  $T - t$ . cf. e.g. [Hu 93].

$$\frac{\partial c}{\partial S} = N(d_1), \quad \frac{\partial c}{\partial r} = X(T-t)e^{-r(T-t)}N(d_2), \quad \frac{\partial c}{\partial \sigma} = S\sqrt{T-t}N'(d_1),$$

where 
$$d_1 = \frac{\ln \frac{S}{X} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

<sup>14</sup>If similar options are traded, one might prefer to use implied instead of historical volatilities in the pricing and derivation of the index participation rate.

On the open set  $O = \{(x, i, r, \sigma) \in \mathbb{R}^4 : x > \max[0, -i], i > -1, r > 0, \sigma > 0\}$  we get

$$\frac{\partial A_0^1}{\partial x} = \{e^r N(d_1) - N(d_2)\} \cdot B > 0, \quad \frac{\partial A_0^1}{\partial i} = \{1 - N(d_2)\} \cdot B > 0$$

$$\frac{\partial A_0^1}{\partial \sigma} = x e^r N'(d_1) \cdot B > 0, \quad \frac{\partial A_0^1}{\partial r} = -\{1 + i - (x + i)N(d_2)\} \cdot B,$$

where  $B = T e^{-Tr} \left\{ 1 + i + (x + i)e^r c \left( \frac{x}{x+i}, 1, 1 \right) \right\}^{T-1} > 0$ .

In particular,  $A_0^1$  is strictly increasing in  $x$ ,  $i$ ,  $\sigma$  and obviously unbounded in  $x$ . In general (especially if  $x < 1$ ),  $A_0^1$  is decreasing in  $r$ .

We now define on the open set  $O \times \mathbb{R} (\subseteq \mathbb{R}^5)$  the continuously differentiable function  $f = f(x, i, r, \sigma, a) = A_0^1(x, i, r, \sigma) + a - 1$  and for  $i > -1$ ,  $r > 0$ ,  $\sigma > 0$ ,  $a \in \mathbb{R}$  the continuous function  $g(i, r, \sigma, a) = e^{-Tr} \{1 + i + e^r \max[0, -i]\}^T + a - 1$ . Obviously, it holds, that  $f = f(x, i, r, \sigma, a) \rightarrow g(i, r, \sigma, a)$  for  $x \rightarrow \max[0, -i]$ . This convergence is strictly increasing in  $x$ . Hence, for given parameters  $i > -1$ ,  $r > 0$ ,  $\sigma > 0$ ,  $a \in \mathbb{R}$ , there exists a unique  $x > \max[0, -i]$  with  $f(x, i, r, \sigma, a) = 0^{15}$  if and only if  $g(i, r, \sigma, a) < 0$ . Hence, on the open set  $G = \{(i, r, \sigma, a) \in \mathbb{R}^4 : i > -1, r > 0, \sigma > 0, a \in \mathbb{R}, g(i, r, \sigma, a) < 0\}$  we can define a unique function  $x = x(i, r, \sigma, a)$  with  $f(x(i, r, \sigma, a), i, r, \sigma, a) = 0$ . As  $\partial f / \partial x > 0$  on  $G$ , it follows from the theorem of implicate functions that  $x$  is continuously differentiable on  $G$  (cf. [Ru 82]). In particular, we get

$$x' = \left( \frac{\partial x}{\partial i}, \frac{\partial x}{\partial r}, \frac{\partial x}{\partial \sigma}, \frac{\partial x}{\partial a} \right) = - \left( \frac{\partial A_0^1}{\partial x} \right)^{-1} \left( \frac{\partial A_0^1}{\partial i}, \frac{\partial A_0^1}{\partial r}, \frac{\partial A_0^1}{\partial \sigma}, 1 \right).$$

## C The rate of index participation when the short rate is time dependent

In Appendix A, we derived the pricing formulas  $A_0^j$  allowing for a time dependent term structure of index rates as well as a time dependent index volatility. We now still assume the index volatility to be constant, i.e.  $\sigma(t) = 12.98\%$ , but we use a short rate  $r(t)$  that fits the current term structure of interest rates.

<sup>15</sup>Note that  $f$  is unbounded in  $x$ .

The 3-dimensional plots corresponding to Figures 3 and 4 we get in the case of time dependent  $r(t)$ , can not be distinguished from Figures 3 and 4. Figure 5 shows for product 1,  $T = 5$  years and  $a = 5\%$  a sectional drawing through the surface  $x = x(a, i)$ . The lower graph was created using time dependent  $r(t)$ , the higher one assuming  $r(t) \equiv 4.9\%$ .

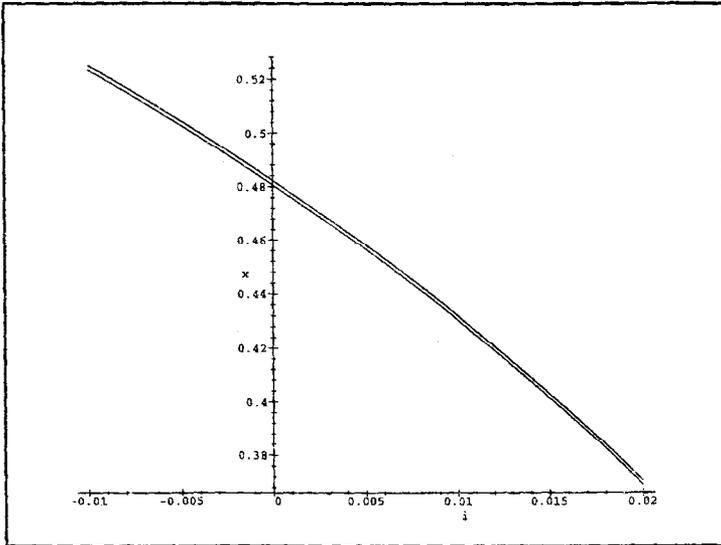


Figure 5:  $x^1(5\%, i)$

The difference between the case of a time dependent  $r(t)$  and a constant short rate is smaller than one might have expected.<sup>16</sup>

<sup>16</sup>Using techniques like the univariate GARCH(1,1)-method (cf., e.g., [Bo 86], [En 82]) one could also investigate the influence of a time dependent volatility function  $\sigma(t)$ .

## D Dependence of the rate of index participation on the term of the contract

The following figures show how the rates of index participation depend on the term  $T$  of the contract and the guaranteed rate of interest  $i$ . To make the products more comparable, we let  $a = 0\%$ . We furthermore let  $r(t) \equiv \hat{r}(T)$  for  $t \leq T$  for each maturity  $T$ , where  $\hat{r}(T)$  denotes the observed annualized rate of interest for the interval  $[0, T]$ , and  $\sigma = 12.98\%$ . We then see from Figure 6, that the rates of index participation  $x^j$  are generally increasing in  $T$ .

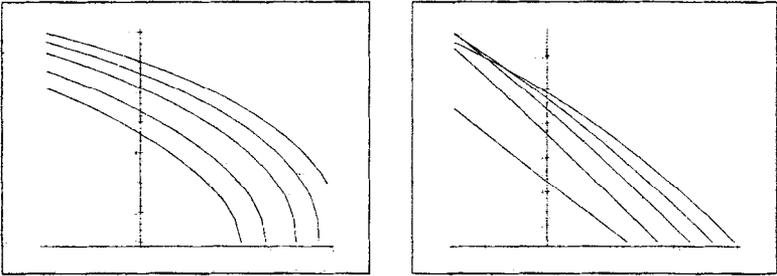


Figure 6:  $x^1(i, T)$  (left) and  $x^2(i, T)$  (right) for  $T = 1, 3, 5, 7, 10$  (increasing order)

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