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Risk/Arbitrage Strategies: A New Concept for Asset/Liability Management, Optimal Fund Design and Optimal Portfolio Selection in a Dynamic, Continuous-Time Framework
Part IV: An Impulse Control Approach to Limited Risk Arbitrage

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Abstract. Asset/Liability management, optimal fund design and optimal portfolio selection have been key issues of interest to the (re)insurance and investment banking communities, respectively, for some years - especially in the design of advanced risk-transfer solutions for clients in the Fortune 500 group of companies. The new concept of ***limited risk arbitrage investment management*** in a diffusion type securities and derivatives market introduced in our papers *Risk/Arbitrage Strategies: A New Concept for Asset/Liability Management, Optimal Fund Design and Optimal Portfolio Selection in a Dynamic, Continuous-Time Framework Part I: Securities Markets* and *Part II: Securities and Derivatives Markets*, AFIR 1997, Vol. II, p. 543, is immediately applicable to ALM, optimal fund design and portfolio selection problems in the investment banking and life insurance areas. However, in order to adequately model the (RCLL) risk portfolio dynamics of a large, internationally operating (re)insurer with considerable ("catastrophic") non-life exposures, significant model extensions are necessary (see also the paper *Baseline for Exchange Rate - Risks of an International Reinsurer*, AFIR 1996, Vol. I, p. 395). To this end, we examine here an ***alternative risk/arbitrage investment management methodology in which given an arbitrary trading or portfolio management policy the limited risk arbitrage objectives are periodically enforced by (impulsive) corrective actions at a certain cost.*** The mathematical framework used is that related to the optimal singular control of Markov jump diffusion processes in R^n with dynamic programming and continuous-time martingale representation techniques.

Key Words and Phrases. Risk/Arbitrage tolerance band, risk exposure control costs, impulsive risk exposure control strategies.

Contents (all five parts of the publication series).

1. Introduction

Risk/Arbitrage strategies [see ***Part I: Securities Markets and Part II: Securities and Derivatives Markets***] are trading or portfolio management strategies in the securities and derivatives markets that guarantee (with probability one) a limited risk exposure over the entire investment horizon and at the same time achieve a maximum (with guaranteed floor) rate of portfolio value appreciation over each individual trading period. They ensure an efficient allocation of investment risk in these integrated financial markets and are the solutions of the general investment management and asset allocation problem

$$\max_{(c,\theta)\in A(v)} E[\int_0^H U^c(t,c(t))dt + U^V(V_v^{c\theta}(H))]$$
$$E[\int_0^H \varsigma(t)c(t)dt + \varsigma(H)V_v^{c\theta}(H)] \le v \qquad (1.1)$$

with drawdown control

$$D(t)V_v^{c\theta}(t) > \alpha M_v^{c\theta}(t)\,,\; 0 \le t \le H \qquad (1.2)$$

$[M_v^{c\theta}(t) = \max_{0\le s\le t} D(s)V_v^{c\theta}(s)]$, limited risk arbitrage objectives

$$\left|\nu(t)^T \Delta(t)\right| \le \delta(t)\,,\; 0 \le t \le H \quad \text{(instantaneous investment risk)} \qquad (1.3a)$$
$$\left|\nu(t)^T \Gamma(t)\right| \le \gamma(t)\,,\; 0 \le t \le H \quad \text{(future portfolio risk dynamics)} \qquad (1.3b)$$
$$\nu(t)^T \Theta(t) \ge \vartheta(t)\,,\; 0 \le t \le H \quad \text{(portfolio time decay dynamics)} \qquad (1.3c)$$
$$\nu(t)^T \Lambda(t) \ge \lambda(t)\,,\; 0 \le t \le H \quad \text{(portfolio value appreciation dynamics)} \qquad (1.3d)$$

$[\theta(t) = I_X(t)\nu(t)]$ and additional inequality and equality constraints

$$g(t, X(t), D(t), \varsigma(t), \nu(t)) \le 0\,,\; 0 \le t \le H \qquad (1.4a)$$
$$h(t, X(t), D(t), \varsigma(t), \nu(t)) = 0\,,\; 0 \le t \le H \qquad (1.4b)$$

(e.g., market frictions, etc.) in a securities and derivatives market

$$dX(t) = I_X(t)[M(t)dt + \Sigma(t)dW(t)]$$
$$dD(t) = -D(t)r(t)dt \qquad d\varsigma(t) = -\varsigma(t)[r(t)dt + A(t)^T dW(t)] \qquad (1.5)$$
$$M(t) = \begin{bmatrix} M_1(t) \\ M \\ M_L(t) \end{bmatrix} \quad \Sigma(t) = \begin{bmatrix} \Sigma_{11}(t) & \Lambda & \Sigma_{1N}(t) \\ M & & M \\ \Sigma_{L1}(t) & \Lambda & \Sigma_{LN}(t) \end{bmatrix}$$

with associated [expressed in terms of an underlying Markov risk exposure assessment and control model $(t, S(t))$ in which $S(t)$ is any N-vector of state variables that completely characterize the investor's intertemporal exposure to adverse market effects] instantaneous investment risk, future derivatives risk dynamics, options time decay dynamics and asset value appreciation dynamics

$$\Delta(t) = \begin{bmatrix} \Delta_1(t) \\ M \\ \Delta_L(t) \end{bmatrix} \quad \Gamma(t) = \begin{bmatrix} \Gamma_1(t) \\ M \\ \Gamma_L(t) \end{bmatrix} \quad \Theta(t) = \begin{bmatrix} \Theta_1(t) \\ M \\ \Theta_L(t) \end{bmatrix} \quad \Lambda(t) = \begin{bmatrix} \Lambda_1(t) \\ M \\ \Lambda_L(t) \end{bmatrix} \qquad (1.6)$$

[where $\Delta_i(t) = \nabla_S X_i(t,S(t))$ is the delta (N-vector), $\Gamma_i(t) = \nabla_S^2 X_i(t,S(t))$ the gamma ($N \times N$-matrix), etc. of traded asset $X_i(t,S(t))$ in the market, $1 \le i \le L$, and the market prices of risk associated with the exogenous sources $W(t)$ of market uncertainty are $A(t) = \Sigma(t)^T K(t)^{-1}[M(t) - r(t)1_L]$ with the asset price covariance matrix $K(t) = \Sigma(t)\Sigma(t)^T$]. If this financial economy is dynamically complete, then (in a Markovian framework) the value function

$$V_v^{c_{y_v},\theta_{y_v}}(t) = V_{\omega_{y_v} v}^{c_{y_v},\theta_{y_v}}(t) = V(t,X(t),Z(t)) \qquad (1.7)$$

of the limited risk arbitrage investment management and asset allocation portfolio satisfies the linear partial differential equation

$$\frac{\partial V}{\partial t} + A^T \nabla V + \frac{1}{2}\operatorname{tr}(BB^T \nabla^2 V) - \nabla V^T B \alpha_{\omega_{y_v}} - V r_{\omega_{y_v}} + I^c(t,Z) = 0 \qquad (1.8)$$

with boundary conditions $V(0,X,Z) = v$ and $V(H,X,Z) = I^V(Z)$ where

$$A = \begin{bmatrix} X_1 M_1^{\omega_{y_v}} \\ M \\ X_N M_N^{\omega_{y_v}} \\ -Z r_{\omega_{y_v}} \end{bmatrix} \qquad B = \begin{bmatrix} X_1 \Sigma_{11} & \Lambda & X_1 \Sigma_{1N} \\ M & & M \\ X_N \Sigma_{N1} & \Lambda & X_N \Sigma_{NN} \\ -Z\alpha_1^{\omega_{y_v}} & \Lambda & -Z\alpha_N^{\omega_{y_v}} \end{bmatrix} \qquad \nabla = \begin{bmatrix} \frac{\partial}{\partial X_1} \\ M \\ \frac{\partial}{\partial X_N} \\ \frac{\partial}{\partial Z} \end{bmatrix} = \begin{bmatrix} \nabla_X \\ \nabla_Z \end{bmatrix} \qquad (1.9)$$

[and $\alpha_\omega(t) = \Sigma(t)^T K(t)^{-1}[M_\omega(t) - r_\omega(t)1_N]$, $M_\omega(t) = M(t) + \omega(t) + \delta(\omega(t)|K_t^\alpha)1_N$ and $r_\omega(t) = r(t) + \delta(\omega(t)|K_t^\alpha)$ holds]. The optimal trading strategy is

$$\theta_{y_v} = [B\Sigma^{-1}]^T \nabla V = I_X \nabla_X V - K^{-1}[M_{\omega_{y_v}} - r_{\omega_{y_v}} 1_N] Z \nabla_Z V. \qquad (1.10)$$

In the incomplete case we have the quasi-linear partial differential equation

$$\frac{\partial V}{\partial t} + A^T \nabla V + \frac{1}{2}\operatorname{tr}(BB^T \nabla^2 V) - \nabla V^T B \hat{\alpha}_{\hat{a}_{z_v}}^{\omega_{y_v}} - V r_{\omega_{y_v}} + I^c(t,Z) = 0 \qquad (1.11)$$

for the portfolio value function

$$V_v^{c_{y_v},\theta_{y_v}}(t) = V_{\omega_{y_v} v}^{c_{y_v},\theta_{y_v}}(t) = V(t,X(t),Y(t),Z(t)) \qquad (1.12)$$

with boundary conditions $V(0,X,Y,Z) = v$ and $V(H,X,Y,Z) = I^V(Z)$ where

$$A = \begin{bmatrix} X_1 M_1^{\omega_{y_v}} \\ M \\ X_L M_L^{\omega_{y_v}} \\ Y_1 \hat{a}_1^{z_v} \\ M \\ Y_{N-L} \hat{a}_{N-L}^{z_v} \\ -Z r_{\omega_{y_v}} \end{bmatrix} \qquad B = \begin{bmatrix} X_1 \Sigma_{11} & \Lambda & X_1 \Sigma_{1N} \\ M & & M \\ X_L \Sigma_{L1} & \Lambda & X_L \Sigma_{LN} \\ Y_1 b_{11} & \Lambda & Y_1 b_{1N} \\ M & & M \\ Y_{N-L} b_{N-L1} & \Lambda & Y_{N-L} b_{N-LN} \\ -Z\hat{\alpha}_1^{\omega_{y_v}\hat{a}_{z_v}} & \Lambda & -Z\hat{\alpha}_N^{\omega_{y_v}\hat{a}_{z_v}} \end{bmatrix} \qquad \nabla = \begin{bmatrix} \nabla_X \\ \nabla_Y \\ \nabla_Z \end{bmatrix} \qquad (1.13)$$

[and $\alpha_{\omega}(t) = \Sigma(t)^T K(t)^{-1}[M_{\omega}(t) - r_{\omega}(t)1_L]$, $M_{\omega}(t) = M(t) + \omega(t) + \delta(\omega(t)|K_t^{\alpha})1_L$ and $r_{\omega}(t) = r(t) + \delta(\omega(t)|K_t^{\alpha})$ holds and moreover

$$\nu\lambda_{z_v} = \frac{B_Y{}^T \nabla_Y V}{Z \nabla_Z V} = b^T \frac{I_Y \nabla_Y V}{Z \nabla_Z V} \quad (1.14)$$

for the completion premium $\nu\lambda_{z_v} \in K(\Sigma)$, $\alpha\lambda_{z_v}^{\omega_{y_v}} = \alpha_{\omega_{y_v}} + \nu\lambda_{z_v}$, associated with the market prices of risk $\alpha_{\omega_{y_v}} \in K^{\perp}(\Sigma)$]. The optimal asset allocation is

$$\theta_{y_v} = [B\Sigma^T K^{-1}]^T \nabla V = I_X \nabla_X V - K^{-1}[M_{\omega_{y_v}} - r_{\omega_{y_v}} 1_L] Z \nabla_Z V. \quad (1.15)$$

During the construction process that led to these optimal solutions of the above stochastic control problem for strictly limited risk investments in (highly geared) derivative financial products several assumptions about an investor's utility functions $U^c(t,c)$ and $U^V(V)$ had to be made, especially

$$R^c(t,c) = -\frac{c\dfrac{\partial^2 U^c(t,c)}{\partial c^2}}{\dfrac{\partial U^c(t,c)}{\partial c}} \leq 1 \text{ and } R^V(V) = -\frac{V\dfrac{d^2 U^V(V)}{dV^2}}{\dfrac{dU^V(V)}{dV}} \leq 1 \quad (1.16)$$

for the associated coefficients of relative risk aversion. In a general dynamic programming framework [see ***Part III: A Risk/Arbitrage Pricing Theory***] all these restrictions (beyond the standard differentiability and boundedness assumptions) on the investor's overall risk management objectives can be removed and furthermore efficient alternative numerical solution methods derived.

2. Dynamic Programming

Risk/Arbitrage Controls. In order to apply standard HJB solution techniques we have to make the additional assumption that $u(t) \in U$ where $U \subseteq R^{L+1}$ is a compact set holds for the progressively measurable controls $u(t) = (c(t), \theta(t))$. The diffusion type controlled state variable is $x(t) = (V(t), X(t))$ and $\overline{A}_{tx}$ denotes the set of all feasible controls $u(s)$ on the time interval $[t, H]$ when the time t state is x. The state space characteristics are then

$$dx(t) = a(t, x(t), u(t))dt + b(t, x(t), u(t))dW(t)$$

$$a(t) = \begin{bmatrix} \theta(t)^T [M(t) - r(t)1_L] \\ + V(t)r(t) - c(t) \\ X_1(t)M_1(t) \\ M \\ X_L(t)M_L(t) \end{bmatrix} \quad b(t) = \begin{bmatrix} & \theta(t)^T \Sigma(t) & \\ X_1(t)\Sigma_{11}(t) & \Lambda & X_1(t)\Sigma_{1N}(t) \\ M & & M \\ X_L(t)\Sigma_{L1}(t) & \Lambda & X_L(t)\Sigma_{LN}(t) \end{bmatrix} \quad (2.1)$$

[where the coefficients $a(t,x,u) \in R^{L+1}$ and $b(t,x,u) \in R^{(L+1)\times N}$ satisfy the usual conditions that guarantee a unique strong (continuous) solution of the associated evolution equation with bounded absolute moments] and the utility functions

$$L(t,x(t),u(t)) = U^c(t,c(t)) \text{ and } \psi(x(H)) = U^V(V(H)) \quad (2.2)$$

in the maximization criterion

$$\begin{aligned} J(t,x,u) &= E_{tx}[\int_t^H L(s,x(s),u(s))ds + \psi(x(H))] \\ V(t,x) &= \sup_{u \in \bar{A}_{tx}} J(t,x,u) \end{aligned} \quad (2.3)$$

[we are only interested in the case where for the value function

$$V_{AS}(t,x) = V_{PM}(t,x) = V_{(\Omega,\Phi,\pi,F,W)}(t,x) \quad (2.4)$$

holds] are assumed to be continuous and to satisfy a polynomial growth condition in both the state

$$x = [x_0 \quad \Lambda \quad x_L]^T \qquad \nabla_x = \begin{bmatrix} \frac{\partial}{\partial x_0} & \Lambda & \frac{\partial}{\partial x_L} \end{bmatrix}^T = \begin{bmatrix} \nabla_V \\ \nabla_X \end{bmatrix}$$

$$H(t,x,p,A) = \sup_{u \in \bar{A}_{tx}} \begin{bmatrix} a(t,x,u(t))^T p \\ +\frac{1}{2} tr[b(t,x,u(t))b(t,x,u(t))^T A] \\ + L(t,x,u(t)) \end{bmatrix} \quad (2.5)$$

and the control

$$u = [u_0 \quad \Lambda \quad u_L]^T \qquad \nabla_u = \begin{bmatrix} \frac{\partial}{\partial u_0} & \Lambda & \frac{\partial}{\partial u_L} \end{bmatrix}^T = \begin{bmatrix} \nabla_c \\ \nabla_k \end{bmatrix} \quad (2.6)$$

variables [which we have mapped into $\bar{\theta}(t)$ - strategy space for convenience: note that the associated constraint sets $K_t^\alpha = T_t^\alpha(K_t)$ (drawdown control) in $\bar{\theta}(t)$ - strategy space are compact if and only if the originally given constraint sets N_t (limited risk arbitrage objectives) in $v(t)$ - strategy space are compact whereas in general the risk/arbitrage constraint transforms $K_t = Y_{v,c}\bar{\theta}_t^{vc}(N_t)$ in $\bar{\theta}(t)$ - strategy space are (infinite) convex cones generated by $I_X(t)(N_t)$-rays emanating from the origin]. Key to the dynamic programming approach is the second order, non-linear Hamilton-Jacobi-Bellman (HJB) partial differential equation

$$\frac{\partial V}{\partial t} + H(t,x,\nabla_x V, \nabla_x^2 V) = 0 \quad (2.7)$$

with boundary data $V(H,x) = \psi(x)$. We first assume that this boundary value problem is uniformly parabolic, i.e., that there exists an $\varepsilon > 0$ such that for all $\xi \in R^{L+1}$,

$$\xi^T[b(t,x,u)b(t,x,u)^T]\xi \geq \varepsilon\|\xi\|^2. \quad (2.8)$$

Then under the standard differentiability and boundedness assumptions that have to be imposed on the coefficients $a(t,x,u)$ and $b(t,x,u)$ determining the state dynamics and the utility functions $L(t,x,u)$ and $\psi(x)$ the above Cauchy problem has a unique $C^{1,2}$ solution $W(t,x)$ which is bounded together with its partial derivatives. With this candidate for the optimal value function of the risk/arbitrage control problem we consider the maximization program

$$\max_{u \in \bar{A}_{\alpha}} F(t,x,u(t))$$

$$F(t,x,u) = a(t,x,u)^T \nabla_x W(t,x) + \frac{1}{2} tr[b(t,x,u)b(t,x,u)^T \nabla_x^2 W(t,x)] + L(t,x,u) \qquad (2.9)$$

in control space U and denote with $U_{t\alpha}$ the set of corresponding solutions [which are the time t values of feasible controls $u(s)$ on $[t,H]$, i.e., of the form $u(t)$ with $u \in \overline{A}_{\alpha}$]. By measurable selection we can then determine a bounded and Borel measurable function $\bar{u}(t,x)$ with the property $\bar{u}(t,x) \in U_{t\alpha}$ (almost everywhere t,x). If an application of this optimal Markov control policy to the above state dynamics satisfies

$$\int_t^H \pi_{t\alpha}^{\bar{u}}[(s, x_{\bar{u}}(s)) \in N] ds = 0 \qquad (2.10)$$

for every Lebesgue null set $N \subseteq R^{L+2}$, then

$$W(t,x) = V_{PM}(t,x) = V_{(\Omega,\Phi,\pi,F,W)}(t,x) = J_{(\Omega^{\bar{u}},\Phi^{\bar{u}},\pi^{\bar{u}},F^{\bar{u}},W^{\bar{u}})}(t,x,\bar{u}) \qquad (2.11)$$

and

$$W(t,x) = E_{t\alpha}^{\bar{u}}[\int_t^{\tau} L(s,x_{\bar{u}}(s),\bar{u}(s,x_{\bar{u}}(s)))ds + W(\tau, x_{\bar{u}}(\tau))] \qquad (2.12)$$

for any stopping time $\tau \in [t,H]$ (dynamic programming principle). This is the case if [after completion with additional state variables $x_{L+1}(t),..,x_{N-1}(t)$] the $N \times N$-matrix $b(t)$ satisfies

$$\xi^T b(t,x,u)\xi \geq \varepsilon \|\xi\|^2, \ \varepsilon > 0 \text{ and } \xi \in R^N, \qquad (2.13)$$

a property that implies uniform parabolicity of the associated HJB boundary value problem.

Viscosity Solutions. In the degenerate parabolic case we retain the above standard differentiability and boundedness conditions on the coefficients $a(t,x,u)$ and $b(t,x,u)$ determining the state dynamics and the utility functions $L(t,x,u)$ and $\psi(x)$. Then the value function

$$\begin{aligned} V_{PM}(t,x) &= \sup_{(\Omega,\Phi,\pi,F,W)} V_{(\Omega,\Phi,\pi,F,W)}(t,x) \\ &= \sup_{(\Omega,\Phi,\pi,F,W)} \sup_{u \in \bar{A}_{\alpha}} E_{t\alpha}[\int_t^H L(s,x(s),u(s))ds + \psi(x(H))] \end{aligned} \qquad (2.14)$$

associated with limited risk arbitrage control is continuous in time and state and semiconvex in the state variable x. Furthermore, we have

$$V_{PM}(t,x) \geq E_{t\alpha}[\int_t^{\tau} L(s,x(s),u(s))ds + V_{PM}(\tau, x(\tau))] \qquad (2.15)$$

for every reference probability system (Ω,Φ,π,F,W), every feasible control $u\in\overline{A}_{tx}$ and any stopping time $\tau\in[t,H]$. Also, if $\varepsilon>0$ is given, then there exists a reference probability system (Ω,Φ,π,F,W) and a feasible control process $u\in\overline{A}_{tx}$ such that

$$V_{PM}(t,x)-\varepsilon\le E_{tx}[\int_t^{\tau} L(s,x(s),u(s))ds+V_{PM}(\tau,x(\tau))] \quad (2.16)$$

for any stopping time $\tau\in[t,H]$ (dynamic programming principle). Moreover, the equality

$$V_{PM}(t,x)=V_{(\Omega,\Phi,\pi,F,W)}(t,x) \quad (2.17)$$

holds for every reference probability system (Ω,Φ,π,F,W) and if in addition $W(t,x)$ is a classical solution of the above HJB boundary value problem, then we have

$$V_{PM}(t,x)=V_{(\Omega,\Phi,\pi,F,W)}(t,x)=W(t,x). \quad (2.18)$$

The dynamic programming principle can therefore also be written in the (generic) form

$$V(t,x)=\sup_{u\in\overline{A}_{tx}} E_{tx}[\int_t^{t+\Delta t} L(s,x(s),u(s))ds+V(t+\Delta t,x(t+\Delta t))]. \quad (2.19)$$

With the two parameter family of non-linear operators

$$[T_{tt+\Delta t}\phi](x)=\sup_{u\in\overline{A}_{tx}} E_{tx}[\int_t^{t+\Delta t} L(s,x(s),u(s))ds+\phi(x(t+\Delta t))] \quad (2.20)$$

on the class of continuous state functions $\phi(x)$ and the family of non-linear, elliptic, second order partial differential operators

$$[G_t\phi](x)=H(t,x,\nabla_x\phi,\nabla_x^2\phi) \quad (2.21)$$

for at least twice continuously differentiable state functions $\phi(x)$ we have then

$$\lim_{\Delta t\downarrow 0}\frac{[T_{tt+\Delta t}\varphi(t+\Delta t,\cdot)](x)-\varphi(t,x)}{\Delta t}=\frac{\partial\varphi}{\partial t}(t,x)+[G_t\varphi(t,\cdot)](x) \quad (2.22)$$

for every $C^{1,2}$ test function $\varphi(t,x)$ [i.e., $\{G_t\}$ is the infinitesimal generator of the operator semigroup $\{T_{tt+\Delta t}\}$ on $C(R^{L+1})$] as well as

$$V(t,x)=[T_{tT}\psi](x)=[T_{tt+\Delta t}V(t+\Delta t,\cdot)](x) \quad (2.23)$$

(abstract dynamic programming principle) and consequently $V(t,x)$ is a uniformly continuous viscosity solution of the (abstract) HJB dynamic programming equation

$$\frac{\partial V}{\partial t}(t,x)+[G_tV(t,\cdot)](x)=0 \quad (2.24)$$

which satisfies the boundary condition $V(H,x)=\psi(x)$. If on the other hand $V_1(t,x)$ is a corresponding continuous and bounded viscosity supersolution and $V_2(t,x)$ a continuous and bounded viscosity subsolution, then

$$\sup_{t,x}[V_2(t,x)-V_1(t,x)]=\sup_x[V_2(H,x)-V_1(H,x)] \quad (2.25)$$

holds and therefore $V(t,x)$ is uniquely determined by the Cauchy data

$$V(H,x)=\psi(x). \quad (2.26)$$

Finite Difference Approximation. A discrete approximation $V_h(t,x)$ of the value function $V(t,x)$ associated with limited risk arbitrage investment management and a corresponding optimal Markov control policy $\bar{u}_h(t,x)$ can be determined numerically by considering a time discretization

$$t = ih,\ 0 \le i \le m \qquad H = mh \qquad (2.27)$$

and a lattice structure

$$x = \delta \begin{bmatrix} j_0 \\ M \\ j_L \end{bmatrix} \qquad (2.28)$$

in state space where $j_0,..,j_L$ are integers and the two relevant discretization parameters h and δ satisfy

$$c(t,x,u) = b(t,x,u)b(t,x,u)^T$$

$$c_{ll}(t,x,u) - \sum_{\substack{k=0 \\ k \ne l}}^{L} |c_{kl}(t,x,u)| \ge 0 \qquad (2.29)$$

$$h\sum_{l=0}^{L}\left[c_{ll}(t,x,u) - \frac{1}{2}\sum_{\substack{k=0 \\ k \ne l}}^{L} |c_{kl}(t,x,u)| + \delta |a_l(t,x,u)| \right] \le \delta^2.$$

We first approximate the controlled continuous-time diffusion process $x(t)$ by a controlled discrete-time Markov chain $x_h(t)$ that evolves on this lattice with one step transition probabilities

$$\pi_t^u(x, x \pm \delta e_l) = \frac{h}{2\delta^2}\left[c_{ll}(t,x,u) - \sum_{\substack{k=0 \\ k \ne l}}^{L} |c_{kl}(t,x,u)| + 2\delta a_l^{\pm}(t,x,u) \right] \qquad (2.30a)$$

$$\left.\begin{aligned} \pi_t^u(x, x + \delta e_k \pm \delta e_l) &= \frac{h}{2\delta^2} c_{kl}^{\pm}(t,x,u) \\ \pi_t^u(x, x - \delta e_k \pm \delta e_l) &= \frac{h}{2\delta^2} c_{kl}^{\mu}(t,x,u) \end{aligned}\right\} \quad [k \ne l] \qquad (2.30b)$$

$$\pi_t^u(x,x) = 1 - \frac{h}{\delta^2}\sum_{l=0}^{L}\left[c_{ll}(t,x,u) - \frac{1}{2}\sum_{\substack{k=0 \\ k \ne l}}^{L} |c_{kl}(t,x,u)| + \delta |a_l(t,x,u)| \right] \qquad (2.30c)$$

[$e_0,..,e_L$ is the standard basis in R^{L+1} and $\pi_t^u(x,y) = 0$ for all other grid points y on the above lattice $X_h \subseteq R^{L+1}$]. The corresponding dynamic programming equation is

$$V_h(t,x) = \max_{u \in A_m}\left[\sum_{y \in X_h} \pi_t^{u(t)}(x,y)V_h(t+h,y) + hL(t,x,u(t)) \right] \qquad (2.31)$$

with boundary condition $V_h(H,x) = \psi(x)$ and an associated optimal Markov control policy $\bar{u}_h(t,x)$ maximizes the expression

$$\sum_{y \in X_h} \pi_t^{u(t)}(x,y)V_h(t+h,y) + hL(t,x,u(t)) \qquad (2.32)$$

in $\overline{A}_\alpha$ [backwards in time from $H-h$ to 0]. With the finite differences

$$\Delta_t^- V = \frac{V(t,x)-V(t-h,x)}{h} \qquad \Delta_{x_l}^{\pm} V = \pm\frac{V(t,x\pm\delta e_l)-V(t,x)}{\delta}$$
$$\Delta_{x_l}^2 V = \frac{V(t,x+\delta e_l)+V(t,x-\delta e_l)-2V(t,x)}{\delta^2} \qquad (2.33a)$$

$$\Delta_{x_k x_l}^+ V = \frac{2V(t,x)+V(t,x+\delta e_k+\delta e_l)+V(t,x-\delta e_k-\delta e_l)}{2\delta^2} - \frac{V(t,x+\delta e_k)+V(t,x-\delta e_k)+V(t,x+\delta e_l)+V(t,x-\delta e_l)}{2\delta^2} \qquad (2.33b)$$

$$\Delta_{x_k x_l}^- V = \frac{V(t,x+\delta e_k)+V(t,x-\delta e_k)+V(t,x+\delta e_l)+V(t,x-\delta e_l)}{2\delta^2} - \frac{2V(t,x)+V(t,x+\delta e_k-\delta e_l)+V(t,x-\delta e_k+\delta e_l)}{2\delta^2} \qquad (2.33c)$$

we then also discretize the continuous-time HJB equation

$$\frac{\partial V}{\partial t}+H(t,x,\nabla_x V,\nabla_x^2 V)=0 \qquad (2.34)$$

and with

$$H_h(t,x,p_l^{\pm},A_{ll},A_{kl}^{\pm}) = \max_{u\in A_\alpha}\left[\sum_{l=0}^{L}\left(\begin{array}{c} a_l^+(t,x,u(t))p_l^+ \\ -a_l^-(t,x,u(t))p_l^- \\ +\dfrac{c_{ll}(t,x,u(t))}{2}A_{ll} \\ +\sum\limits_{\substack{k=0\\k\neq l}}^{L}\left[\begin{array}{c}\dfrac{c_{kl}^+(t,x,u(t))}{2}A_{kl}^+ \\ -\dfrac{c_{kl}^-(t,x,u(t))}{2}A_{kl}^-\end{array}\right]\end{array}\right)+L(t,x,u(t))\right] \qquad (2.35)$$

find that

$$V_h(t-h,x)=V_h(t,x)+hH_h(t,x,\Delta_{x_l}^{\pm}V_h,\Delta_{x_l}^2 V_h,\Delta_{x_k x_l}^{\pm}V_h) \qquad (2.36)$$

holds for the value function of the discrete-time Markov chain control problem. This form of the associated dynamic programming equation can be rewritten as

$$V_h(t,x)=[T_{\pi+h}^h V_h(t+h,\cdot)](x) \qquad (2.37)$$

with the family of discrete-time operators

$$[T_{\pi+h}^h\phi](x)=\phi(x)+hH_h(t+h,x,\Delta_{x_l}^{\pm}\phi,\Delta_{x_l}^2\phi,\Delta_{x_k x_l}^{\pm}\phi) \qquad (2.38)$$

for bounded state functions $\phi(x)$ on the lattice $X_h \subseteq R^{L+1}$ which satisfies

$$\lim_{\substack{(\tau,\xi)\to(t,x)\\ h\downarrow 0}} \frac{[T_{\pi+h}^h\varphi(\tau+h,\cdot)](\xi)-\varphi(\tau,\xi)}{h} = \frac{\partial\varphi}{\partial t}(t,x)+H(t,x,\nabla_x\varphi,\nabla_x^2\varphi) \qquad (2.39)$$

for every $C^{1,2}$ test function $\varphi(t,x)$ (consistency) and consequently we have uniform convergence

$$\lim_{\substack{(\tau,\xi)\to(t,x)\\ h\downarrow 0}} V_h(\tau,\xi) = V(t,x) \qquad (2.40)$$

of the discrete-time Markov chain control problem to the continuous-time diffusion process control problem. The same is true on compact sets if instead of the full infinite lattice $X_h \subseteq R^{L+1}$ only a bounded sublattice (with arbitrary definition of the transition probabilities at the boundary) is considered in actual numerical calculations.

If we now instead of requiring our securities and derivatives investment management strategies $\theta(t)$ to be of the risk/arbitrage (without drawdown control) type, i.e.,

$$\bar{\theta}(t) \in K_t,\ 0 \le t \le H, \qquad (2.41)$$

for an arbitrarily given trading strategy $\theta(t)$ (reference allocation) consider the stochastic evolution of the corresponding portfolio value and sensitivities

$$\left(V_v^\theta(t), \Delta_v^\theta(t), \Gamma_v^\theta(t), \Theta_v^\theta(t), \Lambda_v^\theta(t)\right) \qquad (2.42)$$

over the investment horizon $[0,H]$, then the limited risk arbitrage objectives can periodically be enforced at a certain cost by using impulsive controls that keep the portfolio value and sensitivities within a specified tolerance band

$$(0,\delta,\gamma,\vartheta,\lambda). \qquad (2.43)$$

The (jump diffusion type) state variable of such an alternative (singular or impulse control) approach to limited risk arbitrage investment management is thus

$$x(t) = \left(X(t), V_v^\theta(t), \Delta_v^\theta(t), \Gamma_v^\theta(t), \Theta_v^\theta(t), \Lambda_v^\theta(t)\right) \qquad (2.44)$$

[where $X(t)$ is the price process of the traded assets - bonds, stocks and options - spanning the securities and derivatives market] and impulsive control occurs whenever the state variable comes close to the risk/arbitrage tolerance band in which case the state evolution is reflected back into its interior.

3. Impulse Control Approach

Jump Diffusion State Variables. The uncontrolled RCLL state dynamics in our impulse control model for strictly limited risk investments in securities and derivative financial products are determined by the stochastic differential equation

$$\begin{aligned} dx(t) &= a(t,x(t))dt + b(t,x(t))dW(t) + dJ(t) \\ J(t) &= \int_0^t \int_{R^n} q(s,x(s-),y)N(dsdy) \end{aligned} \qquad (3.1)$$

where the coefficients $a(t,x) \in R^n$ and $b(t,x) \in R^{n\times N}$ of the diffusion part satisfy the usual conditions that guarantee a corresponding unique strong solution with bounded absolute moments. The additional (Poisson) jump process $J(t)$ is characterized by the bounded and measurable parameter $q(t,x,y) \in R^n$ which is continuous in time t and state x and a Poisson random measure $N(dtdy)$ on the Borel σ-algebra $B([0,\infty)\times R^n)$ with intensity

$$E[N(dtdy)] = \lambda dt \Pi(dy) \qquad (3.2)$$

[where the associated probability measure $\Pi(dy)$ on the Borel sets $B(R^n)$ has compact support $\Gamma \subseteq R^n$] and therefore has the continuous jump rate

$$\lambda(t,x) = \lambda \int_{\{y:q(t,x,y)\neq 0\}} \Pi(dy) \qquad (3.3)$$

and corresponding continuous (in time t and state x) jump distribution

$$\overline{\Pi}(t,x,Q) = \int_Q \overline{\Pi}(t,x,dy) = \int_{\{y:q(t,x,y)\in Q, q(t,x,y)\neq 0\}} \Pi(dy). \qquad (3.4)$$

Under these assumptions the above evolution equation for the state variable has a unique strong solution $x(t)$ [with at most finitely many jumps in the time interval $[0,H]$ representing the relevant investment horizon] for each initial condition $x(0)=x$ with $x \in G$ where $G \subseteq R^n$ is the interior of the investor's risk/arbitrage tolerance band $\overline{G} = (0,\delta,\gamma,\vartheta,\lambda)$. Furthermore, the Ito formula

$$f(t,x(t)) = f(0,x(0)) + \begin{bmatrix} \int_0^t [Af](s,x(s))ds + \\ \int_0^t [\nabla_x f^T b](s,x(s))dW(s) \end{bmatrix} + J_f(t) \qquad (3.5)$$

holds for $C^{1,2}$ functionals $f(t,x)$ of the jump diffusion state variable $x(t)$ with the associated integro-differential operator

$$[Af](t,x) = \begin{matrix} [\frac{\partial f}{\partial t} + a^T \nabla_x f + \frac{1}{2} tr(bb^T \nabla_x^2 f)](t,x) + \\ \lambda(t,x) \int_\Gamma [f(t,x+y) - f(t,x)]\overline{\Pi}(t,x,dy) \end{matrix} \qquad (3.6)$$

and the martingale

$$J_f(t) = \begin{matrix} \sum_{s\leq t} [f(s,x(s)) - f(s,x(s-))] - \\ \int_0^t \lambda(s,x(s)) \left[\int_\Gamma [f(s,x(s)+y) - f(s,x(s))]\overline{\Pi}(s,x(s),dy) \right] ds. \end{matrix} \qquad (3.7)$$

Singular Controls. Started at an admissible point $x \in G$ the state variable $x(t)$ evolves in time until it comes close to the boundary ∂G of the risk/arbitrage tolerance band. At each boundary point $y \in \partial G$ a set $R(y)$ of admissible reflection directions is assumed to be given [e.g., the interior normals $n(y) \perp \partial G$ on the hyperplanes $(0,\delta,\gamma,\vartheta,\lambda)$ at all points $y \in \partial G$ where they exist] and the state evolution is then reflected back into G in one of these admissible directions. We also allow (relaxed) intertemporal control of the state variable while it meets the investor's limited risk arbitrage objectives, i.e., resides in $\overline{G}$; and therefore consider the general singular (reflected jump diffusion) control model

$$x(t) = x + \begin{bmatrix} \int_0^t \int_U \overline{a}(s,x(s),u)m_s(du)ds + \\ \int_0^t b(s,x(s))dW(s) \end{bmatrix} + J(t) + F(t) + z(t) \qquad (3.8a)$$

$$J(t)=\int_0^t\int_{R^*}q(s,x(s-),y)N(dsdy) \quad x+q(t,x,y)\in\overline{G},\ x\in\overline{G} \quad (3.8b)$$

$$F(t)=\sum_{i=1}^{l}r_i f_i(t) \quad r_i\in R(y_i),\ y_i\in\partial G \quad f_i(t)\geq 0,\ df_i(t)\geq 0 \quad (3.8c)$$

$$z(t)=\int_0^t r(s)d|z|(s),\ r(s)\in R(x(s))\ [\mu_z(ds)\text{ a.e.}] \quad |z|(t)=\int_0^t 1_{x(s)\in\partial G}d|z|(s) \quad (3.8d)$$

which is based on a Lipschitz continuous solution mapping in the Skorokhod problem for (G,R) and under our above assumptions [and the usual compact control space $U\subseteq R^m$] has a unique strong solution $x(t)\in\overline{G}$, $0\leq t\leq H$, for every $x\in G$. Note that any (conventional) progressively measurable control process $u(t)\in U$, $0\leq t\leq H$, has a relaxed control representation $m_t^{u(\cdot)}(du)$ [by an adapted random measure on the Borel sets $B(U)$] such that

$$\bar{a}(t,x(t),u(t))=\int_U \bar{a}(t,x(t),u)m_t^{u(\cdot)}(du) \quad (3.9)$$

holds. The value function associated with impulsive limited risk arbitrage investment management is then

$$J(t,x,m,f)=E_{tx}\begin{bmatrix}\int_t^H\int_U L(s,x(s),u)m_s(du)ds+\\ \int_t^H M(s,x(s))^T df(s)+\\ \int_t^H N(s,x(s))^T dz(s)\end{bmatrix} \qquad V(t,x)=\inf_{AS} J(t,x,m,f) \quad (3.10)$$

where the continuous reflection part of the bounded and continuous total risk exposure control costs (L,M,N) satisfies $N(t,y)^T r\geq 0$, $r\in R(y)$, on ∂G and the infimum is taken over all admissible (relaxed/singular) control systems. The corresponding (formal) dynamic programming equation is of the form

$$\min\left\{\inf_{m_{tx}(du)}\begin{bmatrix}[A^{m_{tx}(du)}V](t,x)\\ +\int_U L(t,x,u)m_{tx}(du)\end{bmatrix},\min_{1\leq i\leq l}\begin{bmatrix}\frac{\partial V}{\partial t}(t,x)\\ +\nabla_x V(t,x)^T r_i\\ +M_i(t,x)\end{bmatrix}\right\}=0 \quad (3.11)$$

$$\frac{\partial V}{\partial t}(t,x(t))+[\nabla_x V(t,x(t))+N(t,x(t))]^T r(t)=0,\ x(t)\in\partial G \qquad V(H,x)=0$$

where the parabolic integro-differential operator $[A^{m_{tx}(du)}V](t,x)$ (in relaxed control notation) is defined with the controlled drift term $\int_U \bar{a}(t,x,u)m_{tx}(du)$, i.e.,

$$[A^{m_{tx}(du)}V](t,x)=\begin{bmatrix}\frac{\partial V}{\partial t}+\int_U[\bar{a}(t,x,u)^T\nabla_x V]m_{tx}(du)+\frac{1}{2}tr(bb^T\nabla_x^2 V)\end{bmatrix}(t,x) \\ +\lambda(t,x)\int_\Gamma[V(t,x+y)-V(t,x)]\overline{\Pi}(t,x,dy). \quad (3.12)$$

Markov Chain Approximation. A discrete approximation $V_h(t,x)$ of the value function $V(t,x)$ in the above impulsive risk exposure control model and a corresponding optimal control policy consisting of an ordinary Markov component $\bar{u}_h(t,x)$, a singular control component $\bar{f}_h(t)$ and an associated reflection component $\bar{z}_h(t)$ can be determined numerically by considering a controlled discrete-time Markov chain $x_h(t)$ with interpolation interval Δt^h_{txu} and one step transition probabilities $\pi^h_{txu}(y)$ that is locally consistent with the singular control reflected jump diffusion state dynamics $x(t)$ and evolves on a lattice structure

$$x = h\begin{bmatrix} j_1 \\ M \\ j_n \end{bmatrix} \quad (3.13)$$

where $j_1,\ldots,j_n$ are integers and h is the relevant approximation parameter. Let furthermore G_h and ∂G_h be the corresponding discretizations of the interior G and the boundary ∂G of the risk/arbitrage tolerance band. With the discrete time parameter $k = 0,1,2,\ldots,K_h$ enumerating the interpolation steps

$$\begin{aligned} &\Delta t^h_k = \Delta t^h(t^h_k, x^h_k, u^h_k),\ x^h_k = x_h(t^h_k),\ u^h_k = u_h(t^h_k) \\ &t^h_0 = 0,\ t^h_{k+1} = t^h_k + \Delta t^h_k,\ t^h_{K_h} = H \end{aligned} \quad (3.14)$$

in $[0,H]$, an admissible [i.e., the resulting discrete-time state dynamics $x_h(t)$ have the Markov property] discrete-time control process $u_h(t)$ and the conditional expectations

$$E^h_{kxu}[\cdot] = E[\cdot | x^h_i, u^h_i : i \le k, x^h_k = x, u^h_k = u] \quad (3.15)$$

local consistency with the diffusion part of the continuous-time state dynamics $x(t)$ means that

$$\begin{aligned} E^h_{kxu}[\Delta x^h_k] &= \bar{a}(t^h_k,x,u)\Delta t^h(t^h_k,x,u) + o(\Delta t^h(t^h_k,x,u)) \\ &= \bar{a}_h(t^h_k,x,u)\Delta t^h(t^h_k,x,u) \\ V^h_{kxu}[\Delta x^h_k] &= c(t^h_k,x)\Delta t^h(t^h_k,x,u) + o(\Delta t^h(t^h_k,x,u)) \\ &= c_h(t^h_k,x,u)\Delta t^h(t^h_k,x,u) \end{aligned} \quad (3.16)$$

and

$$\sup_{k,x,u} \Delta t^h(t^h_k,x,u) \xrightarrow{h\downarrow 0} 0 \qquad \max_{0\le k<K_h} \|\Delta x^h_k\| \xrightarrow{h\downarrow 0} 0 \quad (3.17)$$

holds where $\Delta x^h_k = x^h_{k+1} - x^h_k$ and $c(t,x) = b(t,x)b(t,x)^T$ is the diffusion process covariance matrix. A corresponding interpolation interval and one step transition probabilities for the controlled Markov chain $x_h(t)$ on the lattice $X_h \subseteq R^n$ can then as in the preceding paragraph be obtained by a finite difference approximation of the (formal) HJB dynamic programming equation or else by the following direct construction. Let

$$E(t,x) = \{v_i(t,x) : 1 \le i \le m(t,x)\} \quad (3.18)$$

with the cardinality $m(t,x) \in N$ uniformly bounded in time t and state x be any set of admissible evolution directions for the Markov chain approximation $x_h(t)$ [e.g., in the finite difference approach used in the preceding paragraph we chose $E(t,x) = \{\pm e_l, \pm e_k \pm e_l : 0 \le k, l \le L\}$ to define the discrete-time state evolution] and

$$\overline{X}_h(t,x) = \{x + hv_i(t,x) : 1 \le i \le m(t,x)\},\ \overline{X}_h(t,x) \subseteq X_h, \quad (3.19)$$

be the corresponding set of states reachable in a single associated transition from state x at time t. Local consistency of $x_h(t)$ with the diffusion part of $x(t)$ then implies the relationships

$$\begin{aligned} h\sum_{i=1}^{m(t,x)} \pi^h_{txu}(x + hv_i(t,x))v_i(t,x) &= \bar{a}(t,x,u)\Delta t^h_{txu} + o(\Delta t^h_{txu}) \\ h^2 \sum_{i=1}^{m(t,x)} \pi^h_{txu}(x + hv_i(t,x))v_i(t,x)v_i(t,x)^T &= c(t,x)\Delta t^h_{txu} + o(\Delta t^h_{txu}) \end{aligned} \quad (3.20)$$

for the unknown interpolation interval and state transition probabilities. On the other hand, with any given non-negative numbers $q_i^0(t,x,u)$ and $q_i^1(t,x)$, $1 \le i \le m(t,x)$, that satisfy

$$\begin{gathered} \sum_{i=1}^{m(t,x)} q_i^0(t,x,u)v_i(t,x) = \bar{a}(t,x,u) \\ \sum_{i=1}^{m(t,x)} q_i^1(t,x)v_i(t,x)v_i(t,x)^T = c(t,x) \qquad \sum_{i=1}^{m(t,x)} q_i^1(t,x)v_i(t,x) = 0 \end{gathered} \quad (3.21)$$

and the definitions

$$\begin{gathered} Q_h(t,x,u) = \sum_{i=1}^{m(t,x)} hq_i^0(t,x,u) + q_i^1(t,x) \\ \Delta t^h_{txu} = \frac{h^2}{Q_h(t,x,u)} \qquad \pi^h_{txu}(x + hv_i(t,x)) = \frac{hq_i^0(t,x,u) + q_i^1(t,x)}{Q_h(t,x,u)} \end{gathered} \quad (3.22)$$

if for the interpolation interval

$$\sup_{t,x,u} \Delta t^h_{txu} \xrightarrow{h \downarrow 0} 0 \quad (3.23)$$

holds, then we obtain a Markov chain $x_0^h,..,x_k^h,..,x_{K_h}^h$ that is locally consistent with the diffusion part of the continuous-time state dynamics $x(t)$ and with a piecewise constant interpolation

$$u_h(t) = u_k^h,\ \tau_k^h \le t < \tau_{k+1}^h, \quad (3.24)$$

[in which the time steps $\tau_0^h = 0, \tau_1^h,.., \tau_k^h,.., \tau_{K_h}^h = H$ are suitably chosen - in order to preserve the Markov property - alternatives to $t_0^h = 0, t_1^h,.., t_k^h,.., t_{K_h}^h = H$ and the intervals $\Delta\tau_k^h = \tau_{k+1}^h - \tau_k^h$ and $\Delta t_k^h = t_{k+1}^h - t_k^h$ are related to each other via $\Delta t_k^h = \Delta t^h(t_k^h, x_k^h, u_k^h)$ and $E^h_{kxu}[\Delta\tau_k^h] = \Delta t^h(\tau_k^h, x, u)$] of a corresponding admissible discrete-time control policy $u_0^h,.., u_k^h,.., u_{K_h-1}^h$ has a Markov process interpolation

$$x_h(t) = x + \int_0^t \bar{a}_h(s, x_h(s), u_h(s))ds + M_h(t) \quad (3.25)$$

in $[0, H]$ where the martingale $M_h(t)$ has quadratic variation

$$\langle M_h \rangle_t = \int_0^t c_h(s, x_h(s), u_h(s))ds \quad (3.26)$$

and approximates the stochastic integral

$$M(t) = \int_0^t b(s, x(s))dW(s) \quad (3.27)$$

with respect to the Wiener process as $h \downarrow 0$. Furthermore, the approximating drift rate vector and covariance matrix have the representations

$$\bar{a}_h(t,x,u) = \lim_{\Delta t \to 0} \frac{E^h_{txu}[\Delta x_h(t)]}{\Delta t} = \bar{a}(t,x,u) + \frac{o(\Delta t^h_{txu})}{\Delta t^h_{txu}} \quad (3.28a)$$

$$\begin{aligned} c_h(t,x,u) &= \lim_{\Delta t \to 0} \frac{V^h_{txu}[\Delta x_h(t)]}{\Delta t} = \lim_{\Delta t \to 0} \frac{V^h_{txu}[\Delta M_h(t)]}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{E^h_{txu}[\Delta M_h(t) \Delta M_h(t)^T]}{\Delta t} = c(t,x) + \frac{o(\Delta t^h_{txu})}{\Delta t^h_{txu}} \end{aligned} \quad (3.28b)$$

where $\Delta x_h(t) = x_h(t + \Delta t) - x_h(t)$ and $\Delta M_h(t) = M_h(t + \Delta t) - M_h(t)$. Note that the above conditional expectations are

$$E^h_{txu}[\cdot] = E[\cdot | x_h(s), u_h(s) : s \le t, \tau^h_k \le t, x_h(t) = x, u_h(t) = u] \quad (3.29)$$

with the exponentially distributed, i.e.,

$$\begin{aligned} &\pi(\Delta\tau^h_k \le t | x^h_i, u^h_i, \tau^h_i : i \le k, x^h_k = x, u^h_k = u) = 1 - e^{-\frac{t}{\Delta t^h_{txu}}} \\ &E^h_{kxu}[\cdot] = E[\cdot | x^h_i, u^h_i, \tau^h_i : i \le k, x^h_k = x, u^h_k = u] \qquad E^h_{kxu}[\Delta\tau^h_k] = \Delta t^h(\tau^h_k, x, u) \end{aligned} \quad (3.30)$$

[$\Delta\tau^h_k = \tau^h_{k+1} - \tau^h_k$], moments $\tau^h_0 = 0, \tau^h_1, .., \tau^h_k, .., \tau^h_{K_h} = H$ of change of the Markov process interpolation $x_h(t)$ of $x^h_0, .., x^h_k, .., x^h_{K_h}$ in $[0, H]$. In the jump diffusion case local consistency holds if the Poisson process coefficient $q(t, x, y)$ has an approximation

$$q_h(t,x,y) \xrightarrow{h \downarrow 0} q(t,x,y) \qquad x + q_h(t,x,y) \in \bar{G}_h, x \in \bar{G}_h \quad (3.31)$$

[where $q_h(t, x, y)$ is bounded - uniformly in the approximation parameter h - and measurable and convergence is uniform on compact sets in time t and state x for each $y \in \Gamma$] and there exists a parameter $\delta^h_{txu} = o(\Delta t^h_{txu})$ such that the jump diffusion transition probabilities are

$$\pi^h_{txu}(y) = (1 - \lambda\Delta t^h_{txu} - \delta^h_{txu})\pi^h_{txu}(y|D) + (\lambda\Delta t^h_{txu} + \delta^h_{txu})\Pi(q_h(t,x,\cdot) = y - x) \quad (3.32)$$

where Δt^h_{txu} and $\pi^h_{txu}(y|D)$ are a (locally consistent) interpolation interval and one step transition probabilities of the continuous diffusion part. Note that the Poisson process $J(t)$ has the representation

$$J(t) = \sum_{t_n \le t} q(t_n, x(t_n-), y_n) \qquad t_0 = 0, t_n \xrightarrow{n \to \infty} \infty \quad (3.33)$$

[where the random variables Δt_n $[\Delta t_n = t_{n+1} - t_n]$ and y_n characterizing the associated jumps are mutually independent with the time intervals Δt_n exponentially distributed with mean $1/\lambda$ and the locations y_n in state space having the common distribution $\Pi(\cdot)$ and in addition $\{\Delta t_i, y_i : i \geq n\}$ are independent of $\{\Delta t_i, y_i : i < n, x(s) : s < t_n\}$] and therefore (an interpolation with the Markov property of) a locally consistent discrete-time Markov chain approximation to the jump diffusion part of the state dynamics $x(t)$ can be written in the form

$$x_h(t) = x + \begin{bmatrix} \int_0^t \bar{a}(s, x_h(s), u_h(s))ds \\ + \\ M_h(t) + J_h(t) + \varepsilon_1^h(t) \end{bmatrix} \qquad E[\sup_{0 \leq t \leq H} \|\varepsilon_1^h(t)\|] \xrightarrow{h \downarrow 0} 0 \qquad (3.34)$$

where the martingale $M_h(t)$ has quadratic variation

$$\langle M_h \rangle_t = \int_0^t c(s, x_h(s))ds + \varepsilon_2^h(t) \qquad E[\sup_{0 \leq t \leq H} \|\varepsilon_2^h(t)\|] \xrightarrow{h \downarrow 0} 0 \qquad (3.35)$$

and the term

$$J_h(t) = \sum_{t_n^h \leq t} q_h(t_n^h, x_h(t_n^h -), y_n^h) \qquad (3.36)$$

[where the random variables y_n^h have the common distribution $\Pi(\cdot)$ and are independent of $\{x_k^h, u_k^h : k \leq n, y_m^h : m < n, m \leq n\}$] approximates the jump process $J(t)$ as $h \downarrow 0$. Furthermore, $t_n^h = \tau_k^h$ for some $0 \leq k \leq K_h$ holds for each jump time $0 \leq t_n^h \leq H$ of the Poisson process approximation $J_h(t)$. Locally consistent reflection of the controlled Markov chain $x_h(t)$ [approximating the jump diffusion part of the continuous-time state dynamics $x(t)$] at the boundary discretization ∂G_h requires the definition of a corresponding (uncontrolled) interpolation interval $\Delta t_{tx}^h = 0$ (because of the assumed instantaneous nature of the reflection steps) and one step transition probabilities $\pi_{tx}^h(y|R)$ for points $x \in \partial G_h$ with admissible discrete-time reflection directions $R_h(x)$ such that with $c_1 > 0$, $c_2(h) \xrightarrow{h \downarrow 0} 0$ and $c_3 > 0$ we have

$$\begin{aligned} E_{kx}^h[\Delta x_k^h | R] &= cr + o(h), c_1 h \leq c \leq c_2(h), r \in R_h(x) \\ V_{kx}^h[\Delta x_k^h | R] &= O(h^2) \end{aligned} \qquad \sum_{y \in G_h} \pi_{kx}^h(y|R) \geq c_3. \qquad (3.37)$$

At a reflection step (k, x) we then define

$$\Delta z_k^h = E_{kx}^h[\Delta x_k^h | R] = cr + o(h) \qquad \Delta \tilde{z}_k^h = \Delta x_k^h - E_{kx}^h[\Delta x_k^h | R] \qquad (3.38)$$

$[\Delta z_k^h = \Delta \tilde{z}_k^h = 0$ at all other steps] and the corresponding interpolations

$$z_h(t) = \sum_{\tau_{k+1}^h \leq t} \Delta z_k^h \qquad \tilde{z}_h(t) = \sum_{\tau_{k+1}^h \leq t} \Delta \tilde{z}_k^h \qquad (3.39)$$

with the moments $\tau_0^h = 0, \tau_1^h, .., \tau_k^h, .., \tau_{K_h}^h = H$ of change associated with $\Delta t_{tx}^h = 0$, $x \in \partial G_h$, and a locally consistent jump diffusion interpolation interval Δt_{txu}^h, $x \in G_h$, and find that for the above deviations from the conditional mean state displacements at the reflecting boundary

$$E[\sup_{0\le t\le H}\|\tilde{z}_h(t)\|^2]\xrightarrow{h\downarrow 0}0 \quad (3.40)$$

holds. A Markov process interpolation is now of the form

$$\begin{aligned} x_h(t) &= x+\int_0^t \bar{a}(s,x_h(s),u_h(s))ds+M_h(t)+J_h(t)+z_h(t)+\tilde{z}_h(t)+\varepsilon_1^h(t) \\ &E[\sup_{0\le t\le H}\|\varepsilon_1^h(t)\|]\xrightarrow{h\downarrow 0}0 \end{aligned} \quad (3.41)$$

and the term $z_h(t)$ approximates the continuous reflection process $z(t)$ as $h\downarrow 0$. The remaining singular control part $F(t)$ of the continuous-time state dynamics $x(t)$ can finally be approximated in the same way as the continuous reflection part above [the singular control directions are admissible reflection directions, i.e., in a corresponding Markov chain approximation $r_i^h \in R_h(y_i^h)$ holds with $y_i^h \in \partial G_h$, and at each control step (k,x) either an admissible regular control u_k^h or then an admissible singular control of the (conditionally expected) form $hr_{i_k}^h$ with associated transition probability $\pi_{kx}^h(y|r_{i_k}^h)$ is applied to the discrete-time state dynamics $x_h(t)$]. At a singular control step (k,x) we consequently define

$$\Delta F_k^h = E_{kx}^h[\Delta x_k^h|r_{i_k}^h]=hr_{i_k}^h \qquad \Delta\tilde{F}_k^h=\Delta x_k^h-E_{kx}^h[\Delta x_k^h|r_{i_k}^h] \quad (3.42)$$

[$\Delta F_k^h=\Delta\tilde{F}_k^h=0$ at regular control steps] and the corresponding interpolations

$$F_h(t)=\sum_{\tau_{k+1}^h\le t}\Delta F_k^h \qquad \tilde{F}_h(t)=\sum_{\tau_{k+1}^h\le t}\Delta\tilde{F}_k^h \quad (3.43)$$

and find that similar to the reflection case for the intertemporal deviations from the conditional mean state displacements under singular control

$$E[\sup_{0\le t\le H}\|\tilde{F}_h(t)\|^2]\xrightarrow{h\downarrow 0}0 \quad (3.44)$$

holds. A Markov process interpolation is therefore of the form

$$\begin{aligned} x_h(t) &= x+\left[\begin{matrix}\int_0^t \bar{a}(s,x_h(s),u_h(s))ds \\ + \\ M_h(t)+J_h(t)+F_h(t)+\tilde{F}_h(t)+z_h(t)+\tilde{z}_h(t)+\varepsilon_1^h(t)\end{matrix}\right] \\ &E[\sup_{0\le t\le H}\|\varepsilon_1^h(t)\|]\xrightarrow{h\downarrow 0}0 \end{aligned} \quad (3.45)$$

where the term $F_h(t)$ approximates the singular control process $F(t)$ as $h\downarrow 0$ and the associated discrete-time dynamic programming equation is

$$V_h(t,x)=\min\left\{\begin{array}{c}\min_{u\in U}\left[\begin{array}{c}(1-\lambda\Delta t^h_{txu}-\delta^h_{txu})\sum_{y\in X_h}\pi^h_{txu}(y|D)V_h(t+\Delta t^h_{txu},y)+\\(\lambda\Delta t^h_{txu}+\delta^h_{txu})\int V_h(t+\Delta t^h_{txu},x+q_h(t,x,y))\Pi(dy)+\\\Delta t^h_{txu}L(t,x,u)\end{array}\right],\\\min_{1\le i\le l}\left[\sum_{y\in X_h}\pi^h_{tx}(y|r_i^h)V_h(t,y)+hM_i(t,x)\right]\end{array}\right\}$$

$$V_h(t,x)=\sum_{y\in X_h}\pi^h_{tx}(y|R)V_h(t,y)+N(t,x)^T\Delta z_h(t,x),\ x\in\partial G_h \tag{3.46}$$

$$V_h(H,x)=0.$$

By solving this equation backwards in time from $t^h_{K_h-1}$ $[t^h_{K_h}=H]$ to 0 we can determine an optimal Markov control policy $\bar{u}_h(t,x)$ [which we denote by $\bar{m}^h_{tx}(du)$ in relaxed control notation], the corresponding (optimal) intertemporal singular control impulses $\Delta\bar{f}_h(t,x)$ and the necessary reflection impulses $\Delta\bar{z}_h(t,x)$ at the boundary ∂G_h, i.e., an optimal discrete-time impulsive risk exposure control strategy $(\bar{u}_h=\bar{m}_h,\bar{f}_h,\bar{z}_h)$ and associated state evolution $\bar{x}_h(t)\in\bar{G}_h$, $0\le t\le H$. As a first step to then also deriving an optimal solution to the initially given continuous-time impulsive limited risk arbitrage investment management problem with a weak convergence argument and the above Markov chain approximations and (interpolations of) their discrete-time optimal solutions we consider the stochastic processes

$$\begin{array}{c}\hat{A}_h(t)=\left(\hat{x}_h(t),\hat{m}_h(t),\hat{f}_h(t),\hat{z}_h(t),\hat{W}_h(t),\hat{J}_h(t),\hat{N}_h(t),\hat{T}_h(t)\right)\\B_h(t)=\left(\bar{m}_h(t),W_h(t),N_h(t)\right)\end{array} \tag{3.47}$$

where we have used discrete-time approximations $W_h(t)$ [defined with a diagonal decomposition of the diffusion process covariance matrix $c(t,x)$] and $N_h(t)$ [defined by counting the jumps of the Poisson process approximation $J_h(t)$] to the Wiener process $W(t)$ and Poisson measure $N(t)$ driving the continuous-time state evolution $x(t)$ [note here that

$$M_h(t)=\int_0^t b(s,x_h(s))dW_h(s)+\varepsilon_3^h(t)\qquad E[\sup_{0\le t\le H}\|\varepsilon_3^h(t)\|]\xrightarrow{h\downarrow 0}0 \tag{3.48}$$

holds for the above defined martingale term approximating the stochastic integral

$$M(t)=\int_0^t b(s,x(s))dW(s) \tag{3.49}$$

with respect to the Wiener process] as well as the adjusted time scale

$$\Delta\hat{\tau}^h_k=\begin{cases}\Delta\tau^h_k & \text{diffusion step}\\ \|\Delta\bar{F}^h_k\| & \text{singular control step}\\ \|\Delta\bar{z}^h_k\| & \text{reflection step}\end{cases}\qquad \hat{\tau}^h_0=0,\ \hat{\tau}^h_{k+1}=\hat{\tau}^h_k+\Delta\hat{\tau}^h_k \tag{3.50a}$$

$$\hat{T}_h(0)=0, \hat{T}_h(\hat{\tau}_k^h)=\tau_k^h, \hat{T}_h{}'(t)=\begin{cases}1 & \text{on } [\hat{\tau}_k^h,\hat{\tau}_{k+1}^h] \text{ for diffusion steps} \\ 0 & \text{on } [\hat{\tau}_k^h,\hat{\tau}_{k+1}^h] \text{ for all other steps}\end{cases} \quad (3.50b)$$

[which ensures tightness of the families $\{\hat{f}_h\}$ (and therefore also $\{\hat{F}_h\}$) and $\{\hat{z}_h\}$ of the singular control and boundary reflection parts of the approximating discrete-time state dynamics $\bar{x}_h(t)$] for all relevant stochastic processes in the impulsive securities and derivatives risk exposure control models considered here, i.e., $\hat{x}_h(t)=\bar{x}_h(\hat{T}_h(t))$, $\hat{m}_h(t)=\overline{m}_h(\hat{T}_h(t))$, $\hat{f}_h(t)=\bar{f}_h(\hat{T}_h(t))$, $\hat{z}_h(t)=\bar{z}_h(\hat{T}_h(t))$, $\hat{W}_h(t)=W_h(\hat{T}_h(t))$, $\hat{J}_h(t)=J_h(\hat{T}_h(t))$ and $\hat{N}_h(t)=N_h(\hat{T}_h(t))$. The families $\{\hat{A}_h\}$ and $\{B_h\}$ are tight and the limits

$$\hat{A}(t)=\left(\hat{x}(t),\hat{m}(t),\hat{f}(t),\hat{z}(t),\hat{W}(t),\hat{J}(t),\hat{N}(t),\hat{T}(t)\right) \quad B(t)=\left(\overline{m}(t),W(t),N(t)\right) \quad (3.51)$$

of two corresponding weakly convergent subsequences satisfy: (1) $W(t)$ and $N(t)$ are a standard Wiener process and Poisson random measure with respect to the natural filtration; (2) $\overline{m}(t)$ is an admissible relaxed control with respect to $W(t)$ and $N(t)$; (3) $\hat{m}(t)=\overline{m}(\hat{T}(t))$, $\hat{W}(t)=W(\hat{T}(t))$, $\hat{N}(t)=N(\hat{T}(t))$ as well as $\hat{J}(t)=\int_0^t\int_{R^n} q(s,\hat{x}(s-),y)\hat{N}(dsdy)$ holds and the stochastic process $\hat{W}(t)$ is an $\hat{F}_t$-martingale, $\hat{F}_t=\sigma(\hat{A}(s):s\le t)$, with quadratic variation $\hat{T}(t)I_N$; (4)

$$\hat{x}(t)=x+\begin{bmatrix}\int_0^t\int_U \bar{a}(s,\hat{x}(s),u)\hat{m}_s(du)ds+ \\ \int_0^t b(s,\hat{x}(s))d\hat{W}(s)\end{bmatrix}+\hat{J}(t)+\hat{F}(t)+\hat{z}(t) \quad \text{where } \hat{F}(t)=\sum_{i=1}^{I} r_i\hat{f}_i(t) \text{ and}$$

$\hat{x}(t)\in\overline{G}$, $0\le t\le H$; (5) the process $\hat{z}(t)$ is differentiable and only changes when $\hat{x}(t)\in\partial G$ at which time $\frac{d\hat{z}(t)}{dt}\in R(\hat{x}(t))$ holds. If we now in a second step towards establishing a continuous-time optimal impulsive risk exposure control strategy $(\overline{m},\bar{f},\bar{z})$ and associated state evolution $\bar{x}(t)\in\overline{G}$, $0\le t\le H$, define the continuous inverse

$$T(t)=\inf\left\{s:\hat{T}(s)>t\right\} \quad (3.52)$$

of the above time scale adjustment and apply it to the obtained weak limit $\hat{A}(t)$, i.e., consider the stochastic process $A(t)=\hat{A}(T(t))$ with the components $\bar{x}(t)=\hat{x}(T(t))$, $\overline{m}(t)=\hat{m}(T(t))$, $\bar{f}(t)=\hat{f}(T(t))$, $\bar{z}(t)=\hat{z}(T(t))$, $W(t)=\hat{W}(T(t))$, $J(t)=\hat{J}(T(t))$ and $N(t)=\hat{N}(T(t))$, then under the additional assumption

$$\overline{\lim_{h\downarrow 0}}\max_{0\le k<K_h} E[\|\Delta\bar{F}_k^h\|^2]<\infty \quad (3.53)$$

$W(t)$ and $N(t)$ are a standard Wiener process and Poisson random measure with respect to the filtration $F_t=\sigma(A(s):s\le t)$, $\overline{m}(t)$ is an admissible relaxed control with respect to $W(t)$ and $N(t)$ and furthermore

$$\bar{x}(t) = x + \left[\begin{array}{c} \int_0^t \int_U \bar{a}(s,\bar{x}(s),u)\bar{m}_s(du)ds \\ + \int_0^t b(s,\bar{x}(s))d\bar{W}(s) \end{array} \right] + J(t) + \bar{F}(t) + \bar{z}(t) \quad \bar{x}(t) \in \bar{G}, 0 \le t \le H \quad (3.54a)$$

$$J(t) = \int_0^t \int_{R^n} q(s,\bar{x}(s-),y)N(dsdy) \quad x + q(t,x,y) \in \bar{G}, x \in \bar{G} \quad (3.54b)$$

$$\bar{F}(t) = \sum_{i=1}^{I} r_i \bar{f}_i(t) \quad r_i \in R(y_i), y_i \in \partial G \quad \bar{f}_i(t) \ge 0, d\bar{f}_i(t) \ge 0 \quad (3.54c)$$

$$\bar{z}(t) = \int_0^t \bar{r}(s)d|\bar{z}|(s), \bar{r}(s) \in R(\bar{x}(s))\,[\mu_{\bar{z}}(ds)\text{ a.e.}] \quad |\bar{z}|(t) = \int_0^t 1_{\bar{x}(s)\in\partial G}\, d|\bar{z}|(s) \quad (3.54d)$$

holds. Moreover, we have

$$\lim_{h \downarrow 0} V_h(t,x) = V(t,x)$$

$$= J(t,x,\bar{m},\bar{f}) = E_{tx}\left[\begin{array}{c} \int_t^H \int_U L(s,\bar{x}(s),u)\bar{m}_s(du)ds + \\ \int_t^H M(s,\bar{x}(s))^T d\bar{f}(s) + \\ \int_t^H N(s,\bar{x}(s))^T d\bar{z}(s) \end{array} \right]. \quad (3.55)$$

Note finally that the limiting relaxed control $\bar{m}_t(du)$ can given $\varepsilon > 0$ be approximated by a (conventional) piecewise constant [on time intervals $[k\delta,(k+1)\delta)$, $k = 0,1,2,..,K_\delta - 1$, $K_\delta \delta = H$], progressively measurable control process $u_\varepsilon(t)$ that takes its values in a finite subset $U_\varepsilon \subseteq U$ of the control space in the sense that the associated state dynamics $x_\varepsilon(t) \in \bar{G}$, $0 \le t \le H$, and cost functional

$$J(t,x,u_\varepsilon,\bar{f}) = E_{tx}\left[\begin{array}{c} \int_t^H L(s,x_\varepsilon(s),u_\varepsilon(s))ds + \\ \int_t^H M(s,x_\varepsilon(s))^T d\bar{f}(s) + \\ \int_t^H N(s,x_\varepsilon(s))^T d\bar{z}(s) \end{array} \right] \quad (3.56)$$

satisfy the inequalities

$$\pi_{tx}(\sup_{t \le s \le H} \|x_\varepsilon(s) - \bar{x}(s)\| > \varepsilon) \le \varepsilon \text{ and } |J(t,x,u_\varepsilon,\bar{f}) - J(t,x,\bar{m},\bar{f})| \le \varepsilon. \quad (3.57)$$

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