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Risk/Arbitrage Strategies: A New Concept for Asset/Liability Management, Optimal Fund Design and Optimal Portfolio Selection in a Dynamic, Continuous-Time Framework Part III: A Risk/Arbitrage Pricing Theory

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Abstract. Asset/Liability management, optimal fund design and optimal portfolio selection have been key issues of interest to the (re)insurance and investment banking communities, respectively, for some years - especially in the design of advanced risk-transfer solutions for clients in the Fortune 500 group of companies. Building on the new concept of ***limited risk arbitrage investment management*** in a diffusion type securities and derivatives market introduced in our papers *Risk/Arbitrage Strategies: A New Concept for Asset/Liability Management, Optimal Fund Design and Optimal Portfolio Selection in a Dynamic, Continuous-Time Framework Part I: Securities Markets* and *Part II: Securities and Derivatives Markets*, AFIR 1997, Vol. II, p. 543, we outline here a ***corresponding risk/arbitrage pricing theory that is consistent with an investor's overall risk management objectives and takes into account drawdown control and limited risk arbitrage constraints on admissible contingent claim replication/hedging strategies***. The mathematical framework used is that related to the optimal control of Markov diffusion processes in R^n with dynamic programming and continuous-time martingale representation techniques.

Key Words and Phrases. Risk/Arbitrage pricing theory (R/APT), risk/arbitrage contingent claim replication strategies, optimal financial instruments, LRA market indices, utility-based hedging, risk/arbitrage price, R/A-attainable, partial replication strategies.

Contents (all five parts of the publication series).

Part III: A Risk/Arbitrage Pricing Theory

Part IV: An Impulse Control Approach to Limited Risk Arbitrage (separate paper)

Part V: A Guide to Efficient Numerical Implementations (separate paper)

1. Introduction

Risk/Arbitrage strategies [see ***Part I: Securities Markets*** and ***Part II: Securities and Derivatives Markets***] are trading or portfolio management strategies in the securities and derivatives markets that guarantee (with probability one) a limited risk exposure over the entire investment horizon and at the same time achieve a maximum (with guaranteed floor) rate of portfolio value appreciation over each individual trading period. They ensure an efficient allocation of investment risk in these integrated financial markets and are the solutions of the general investment management and asset allocation problem

$$\max_{(c,\theta)\in A(v)} E[\int_0^H U^c(t,c(t))dt + U^V(V_v^{c\theta}(H))]$$
$$E[\int_0^H \varsigma(t)c(t)dt + \varsigma(H)V_v^{c\theta}(H)] \le v \qquad (1.1)$$

with drawdown control

$$D(t)V_v^{c\theta}(t) > \alpha M_v^{c\theta}(t)\,,\ 0 \le t \le H \qquad (1.2)$$

[$M_v^{c\theta}(t) = \max_{0\le s\le t} D(s)V_v^{c\theta}(s)$], limited risk arbitrage objectives

$$|v(t)^T\Delta(t)| \le \delta(t)\,,\ 0\le t\le H \quad \text{(instantaneous investment risk)} \qquad (1.3a)$$
$$|v(t)^T\Gamma(t)| \le \gamma(t)\,,\ 0\le t\le H \quad \text{(future portfolio risk dynamics)} \qquad (1.3b)$$
$$v(t)^T\Theta(t) \ge \vartheta(t)\,,\ 0\le t\le H \quad \text{(portfolio time decay dynamics)} \qquad (1.3c)$$
$$v(t)^T\Lambda(t) \ge \lambda(t)\,,\ 0\le t\le H \quad \text{(portfolio value appreciation dynamics)} \qquad (1.3d)$$

[$\theta(t) = I_X(t)v(t)$] and additional inequality and equality constraints

$$g(t,X(t),D(t),\varsigma(t),v(t)) \le 0\,,\ 0\le t\le H \qquad (1.4a)$$
$$h(t,X(t),D(t),\varsigma(t),v(t)) = 0\,,\ 0\le t\le H \qquad (1.4b)$$

(e.g., market frictions, etc.) in a securities and derivatives market

$$dX(t) = I_X(t)[M(t)dt + \Sigma(t)dW(t)]$$
$$dD(t) = -D(t)r(t)dt \qquad d\varsigma(t) = -\varsigma(t)[r(t)dt + A(t)^T dW(t)] \qquad (1.5)$$
$$M(t) = \begin{bmatrix} M_1(t) \\ M \\ M_L(t) \end{bmatrix} \quad \Sigma(t) = \begin{bmatrix} \Sigma_{11}(t) & \Lambda & \Sigma_{1N}(t) \\ M & & M \\ \Sigma_{L1}(t) & \Lambda & \Sigma_{LN}(t) \end{bmatrix}$$

with associated [expressed in terms of an underlying Markov risk exposure assessment and control model $(t,S(t))$] instantaneous investment risk, future derivatives risk dynamics, options time decay dynamics and asset value appreciation dynamics

$$\Delta(t) = \begin{bmatrix} \Delta_1(t) \\ M \\ \Delta_L(t) \end{bmatrix} \quad \Gamma(t) = \begin{bmatrix} \Gamma_1(t) \\ M \\ \Gamma_L(t) \end{bmatrix} \quad \Theta(t) = \begin{bmatrix} \Theta_1(t) \\ M \\ \Theta_L(t) \end{bmatrix} \quad \Lambda(t) = \begin{bmatrix} \Lambda_1(t) \\ M \\ \Lambda_L(t) \end{bmatrix} \qquad (1.6)$$

[where $\Delta_i(t) = \nabla_S X_i(t,S(t))$ is the delta (N-vector), $\Gamma_i(t) = \nabla_S^2 X_i(t,S(t))$ the gamma ($N \times N$-matrix), etc. of traded asset $X_i(t,S(t))$ in the market, $1 \le i \le L$, and the market prices of risk associated with the exogenous sources $W(t)$ of market uncertainty are $A(t) = \Sigma(t)^T K(t)^{-1}[M(t) - r(t)1_L]$ with the asset price covariance matrix $K(t) = \Sigma(t)\Sigma(t)^T$]. If this financial economy is dynamically complete, then (in a Markovian framework) the value function

$$V_v^{c_{y_v},\theta_{y_v}}(t) = V_{\omega_{y_v},v}^{c_{y_v},\vartheta_{y_v}}(t) = V(t,X(t),Z(t)) \qquad (1.7)$$

of the limited risk arbitrage investment management and asset allocation portfolio satisfies the linear partial differential equation

$$\frac{\partial V}{\partial t} + A^T \nabla V + \frac{1}{2}\mathrm{tr}(BB^T \nabla^2 V) - \nabla V^T B \alpha_{\omega_{y_v}} - V r_{\omega_{y_v}} + I^c(t,Z) = 0 \qquad (1.8)$$

with boundary conditions $V(0,X,Z) = v$ and $V(H,X,Z) = I^V(Z)$ where

$$A = \begin{bmatrix} X_1 M_1^{\omega_{y_v}} \\ M \\ X_N M_N^{\omega_{y_v}} \\ -Z r_{\omega_{y_v}} \end{bmatrix} \qquad B = \begin{bmatrix} X_1\Sigma_{11} & \Lambda & X_1\Sigma_{1N} \\ M & & M \\ X_N\Sigma_{N1} & \Lambda & X_N\Sigma_{NN} \\ -Z\alpha_1^{\omega_{y_v}} & \Lambda & -Z\alpha_N^{\omega_{y_v}} \end{bmatrix} \qquad \nabla = \begin{bmatrix} \dfrac{\partial}{\partial X_1} \\ M \\ \dfrac{\partial}{\partial X_N} \\ \dfrac{\partial}{\partial Z} \end{bmatrix} = \begin{bmatrix} \nabla_X \\ \nabla_Z \end{bmatrix} \qquad (1.9)$$

[and $\alpha_\omega(t) = \Sigma(t)^T K(t)^{-1}[M_\omega(t) - r_\omega(t)1_N]$, $M_\omega(t) = M(t) + \omega(t) + \delta(\omega(t)|K_t^\alpha)1_N$ and $r_\omega(t) = r(t) + \delta(\omega(t)|K_t^\alpha)$ holds]. The optimal trading strategy is

$$\theta_{y_v} = [B\Sigma^{-1}]^T \nabla V = I_X \nabla_X V - K^{-1}[M_{\omega_{y_v}} - r_{\omega_{y_v}} 1_N] Z \nabla_Z V. \qquad (1.10)$$

In the incomplete case we have the quasi-linear partial differential equation

$$\frac{\partial V}{\partial t} + A^T \nabla V + \frac{1}{2}\mathrm{tr}(BB^T \nabla^2 V) - \nabla V^T B \hat{\alpha}_{z_v}^{\omega_{y_v}} - V r_{\omega_{y_v}} + I^c(t,Z) = 0 \qquad (1.11)$$

for the portfolio value function

$$V_v^{c_{y_v},\theta_{y_v}}(t) = V_{\omega_{y_v},v}^{c_{y_v},\theta_{y_v}}(t) = V(t,X(t),Y(t),Z(t)) \qquad (1.12)$$

with boundary conditions $V(0,X,Y,Z) = v$ and $V(H,X,Y,Z) = I^V(Z)$ where

$$A = \begin{bmatrix} X_1 M_1^{\omega_{y_v}} \\ M \\ X_L M_L^{\omega_{y_v}} \\ Y_1 \hat{a}_1^{z_v} \\ M \\ Y_{N-L} \hat{a}_{N-L}^{z_v} \\ -Z r_{\omega_{y_v}} \end{bmatrix} \qquad B = \begin{bmatrix} X_1\Sigma_{11} & \Lambda & X_1\Sigma_{1N} \\ M & & M \\ X_L\Sigma_{L1} & \Lambda & X_L\Sigma_{LN} \\ Y_1 b_{11} & \Lambda & Y_1 b_{1N} \\ M & & M \\ Y_{N-L} b_{N-L1} & \Lambda & Y_{N-L} b_{N-LN} \\ -Z\hat{\alpha}_1^{\omega_{y_v},\hat{a}_{z_v}} & \Lambda & -Z\hat{\alpha}_N^{\omega_{y_v},\hat{a}_{z_v}} \end{bmatrix} \qquad \nabla = \begin{bmatrix} \nabla_X \\ \nabla_Y \\ \nabla_Z \end{bmatrix} \qquad (1.13)$$

[and $\alpha_\omega(t) = \Sigma(t)^T K(t)^{-1}[M_\omega(t) - r_\omega(t) 1_L]$, $M_\omega(t) = M(t) + \omega(t) + \delta(\omega(t)|K_t^\alpha) 1_L$ and $r_\omega(t) = r(t) + \delta(\omega(t)|K_t^\alpha)$ holds and moreover

$$v_{\lambda_{z_v}} = \frac{B_Y{}^T \nabla_Y V}{Z \nabla_Z V} = b^T \frac{I_Y \nabla_Y V}{Z \nabla_Z V} \quad (1.14)$$

for the completion premium $v_{\lambda_{z_v}} \in K(\Sigma)$, $\lambda_{\lambda_{z_v}}^{\omega_{y_v}} = \alpha_{\omega_{y_v}} + v_{\lambda_{z_v}}$, associated with the market prices of risk $\alpha_{\omega_{y_v}} \in K^{\perp}(\Sigma)$]. The optimal asset allocation is

$$\theta_{y_v} = [B \Sigma^T K^{-1}]^T \nabla V = I_X \nabla_X V - K^{-1}[M_{\omega_{y_v}} - r_{\omega_{y_v}} 1_L] Z \nabla_Z V. \quad (1.15)$$

The above idea of strictly limiting investment risk to values within a given tolerance band is also crucially important when the two problems (related to limited risk arbitrage investment management) of establishing the fair price of a claim contingent on the basic traded assets (bonds, stocks and options) in the securities and derivatives market and of hedging such a claim are considered. A risk/arbitrage contingent claim replication strategy furthermore ensures certain desirable properties (e.g., guaranteed floors on time decay and value appreciation as well as controlled drawdown) of the associated option value process.

2. Arbitrage Pricing Theory (APT)

Dynamically Complete Market. If the given securities and derivatives market is dynamically complete, then the arbitrage value of a contingent claim C written on the basic assets $X_1(t),..,X_N(t)$ available to investors in this market is

$$V_{p(C)}^{\theta_C}(t) = \frac{1}{D(t)} \widetilde{E}[D(T)C|F_t] \qquad \text{(European option)} \quad (2.1a)$$

$$V_{p(C)}^{\theta_C}(t) = \frac{1}{D(t)} \widetilde{E}[D(\tau_0)C(\tau_0)|F_t] \qquad \text{(American option)} \quad (2.1b)$$

(under the unique equivalent martingale measure) and we have its Markovian characterization

$$\frac{\partial V}{\partial t} + \frac{1}{2} \mathrm{tr}(BB^T \nabla^2 V) + r(X^T \nabla V - V) = 0 \quad (2.2)$$

with the boundary conditions $V(T, X) = h(X)$ for European options and $V(\tau_0, X) = h(\tau_0, X)$ for American options where

$$B = \begin{bmatrix} X_1 \Sigma_{11} & \Lambda & X_1 \Sigma_{1N} \\ M & & M \\ X_N \Sigma_{N1} & \Lambda & X_N \Sigma_{NN} \end{bmatrix} \qquad \nabla = \begin{bmatrix} \frac{\partial}{\partial X_1} \\ M \\ \frac{\partial}{\partial X_N} \end{bmatrix} \quad (2.3)$$

(Karatzas [1]). Furthermore, the associated trading strategy that hedges (or replicates) claim C is

$$\theta_C = [B\Sigma^{-1}]^T \nabla V = I_X \nabla V. \quad (2.4)$$

Incomplete Market. In the case of an incomplete securities and derivatives market the arbitrage value of a contingent claim C written on the basic assets $X_1(t),..,X_L(t)$ available to investors is

$$V_{p(C)}^{\theta_C}(t) = \frac{1}{D(t)} \tilde{E}_{\lambda_L}[D(T)C|F_t] \qquad \text{(European option)} \quad (2.5a)$$

$$V_{p(C)}^{\theta_C}(t) = \frac{1}{D(t)} \tilde{E}_{\lambda_L}[D(\tau_0)C(\tau_0)|F_t] \quad \text{(American option)} \quad (2.5b)$$

[under the minimax local martingale measure which is uniquely determined by the investor's overall risk management objectives $U^c(t,c)$ and $U^V(V)$] with associated risk minimizing replication costs

$$dC^{\theta_C}(t) = dL_C(t), \; C^{\theta_C}(0) = p(C), \quad (2.6)$$

and we have its Markovian characterization

$$\frac{\partial V}{\partial t} + \frac{1}{2}\mathrm{tr}(BB^T\nabla^2 V) + r(\begin{bmatrix} X \\ Y \end{bmatrix}^T \nabla V - V) = 0 \quad (2.7)$$

with [the additional market state variables $Y_1(t),..,Y_{N-L}(t)$ that are not prices of traded assets and] the boundary conditions $V(T,X,Y) = h(X,Y)$ for European options and $V(\tau_0,X,Y) = h(\tau_0,X,Y)$ for American options where

$$B = \begin{bmatrix} X_1\Sigma_{11} & \Lambda & X_1\Sigma_{1N} \\ M & & M \\ X_L\Sigma_{L1} & \Lambda & X_L\Sigma_{LN} \\ Y_1 b_{11} & \Lambda & Y_1 b_{1N} \\ M & & M \\ Y_{N-L} b_{N-L1} & \Lambda & Y_{N-L} b_{N-LN} \end{bmatrix} \qquad \nabla = \begin{bmatrix} \nabla_X \\ \nabla_Y \end{bmatrix} \quad (2.8)$$

(Schweizer [2]). Furthermore, the corresponding risk minimizing replication strategy for claim C is

$$\theta_C = [B\Sigma^T K^{-1}]^T \nabla V = I_X \nabla_X V. \quad (2.9)$$

Note that such an arbitrage pricing methodology is consistent with the investor's overall risk management objectives $U^c(t,c)$ and $U^V(V)$ but does not take into account drawdown control or limited risk arbitrage constraints on admissible trading strategies.

3. Risk/Arbitrage Pricing Theory (R/APT)

General Contingent Claims. A general claim contingent on the securities and derivatives $X_1(t),..,X_L(t)$ available to investors in the financial market under consideration is a pair (c, V_T) consisting of a non-negative, progressively measurable cashflow process $c(t)$ and a payoff V_T at maturity such that $E[(V_T + \int_0^T c(t)dt)^n] < \infty$ for some $n > 1$. In a complete securities and derivatives market setting [i.e., $L = N$] the arbitrage value of such a general contingent claim $C = (c, V_T)$ is

$$V_{p(C)}^{\theta_c}(t) = \frac{1}{D(t)} \tilde{E}[\int_t^T D(s)c(s)ds + D(T)V_T \Big| F_t] \quad (3.1)$$

and we have its Markovian characterization

$$\frac{\partial V}{\partial t} + \frac{1}{2}\mathrm{tr}(BB^T \nabla^2 V) + r(X^T \nabla V - V) + c = 0 \quad (3.2)$$

with boundary condition $V(T, X) = h(X)$ where

$$B = \begin{bmatrix} X_1\Sigma_{11} & \Lambda & X_1\Sigma_{1N} \\ M & & M \\ X_N\Sigma_{N1} & \Lambda & X_N\Sigma_{NN} \end{bmatrix} \text{ and } \nabla = \begin{bmatrix} \frac{\partial}{\partial X_1} \\ M \\ \frac{\partial}{\partial X_N} \end{bmatrix}. \quad (3.3)$$

Furthermore, the associated trading strategy that replicates/hedges this claim is

$$\theta_C = [B\Sigma^{-1}]^T \nabla V = I_X \nabla V. \quad (3.4)$$

In the case of an incomplete securities and derivatives market the arbitrage value of a general contingent claim $C = (c, V_T)$ is

$$V_{p(C)}^{\theta_c}(t) = \frac{1}{D(t)} \tilde{E}_{\lambda_*}[\int_t^T D(s)c(s)ds + D(T)V_T \Big| F_t] \quad (3.5)$$

with associated risk minimizing replication costs

$$dC^{\theta_c}(t) = dL_C(t),\ C^{\theta_c}(0) = p(C), \quad (3.6)$$

and we have its Markovian characterization

$$\frac{\partial V}{\partial t} + \frac{1}{2}\mathrm{tr}(BB^T \nabla^2 V) + r(\begin{bmatrix} X \\ Y \end{bmatrix}^T \nabla V - V) + c = 0 \quad (3.7)$$

with boundary condition $V(T, X, Y) = h(X, Y)$ where

$$B = \begin{bmatrix} X_1\Sigma_{11} & \Lambda & X_1\Sigma_{1N} \\ M & & M \\ X_L\Sigma_{L1} & \Lambda & X_L\Sigma_{LN} \\ Y_1 b_{11} & \Lambda & Y_1 b_{1N} \\ M & & M \\ Y_{N-L} b_{N-L1} & \Lambda & Y_{N-L} b_{N-LN} \end{bmatrix} \text{ and } \nabla = \begin{bmatrix} \nabla_X \\ \nabla_Y \end{bmatrix}. \quad (3.8)$$

Furthermore, the corresponding risk minimizing replication strategy is

$$\theta_C = [B\Sigma^T K^{-1}]^T \nabla V = I_X \nabla_X V. \quad (3.9)$$

Optimal Financial Instruments. An investor in the securities and derivatives market with utility functions $U^c(t,c)$ for intertemporal fund consumption and $U^V(V)$ for final wealth, acceptable drawdown control level $1-\alpha$, $0<\alpha<1$, and risk/arbitrage tolerances $(\delta,\gamma,\vartheta,\lambda)$ can given an initial wealth $v>0$ by trading in the available securities and derivatives $X_1(t),..,X_L(t)$ synthetically create a unique generalized contingent claim $(c_{y_v}, V_T^{y_v})$ with the properties

$$\max_{(c,V_T)\in\bar{A}(v)} E[\int_0^T U^c(t,c(t))dt + U^V(V_T)] = E[\int_0^T U^c(t,c_{y_v}(t))dt + U^V(V_T^{y_v})] \quad (3.10)$$

[$\bar{A}(v)$ here denotes the set of all general contingent claims (c,V_T) that given the investor's initial wealth $v>0$ can be created by continuous risk/arbitrage trading with drawdown control in the assets $X_1(t),..,X_L(t)$ (bonds, stocks and options) spanning the securities and derivatives market] and

$$D(t)V_v^{c_{y_v},\theta_{y_v}}(t) > \alpha M_v^{c_{y_v},\theta_{y_v}}(t)\,,\ 0\le t\le T \text{ (drawdown control)} \quad (3.11)$$

[$M_v^{c\theta}(t) = \max_{0\le s\le t} D(s)V_v^{c\theta}(s)$] as well as

$$\left|v_{y_v}(t)^T\Delta(t)\right| \le \delta\,,\ 0\le t\le T \quad \text{(instantaneous investment risk)} \quad (3.12a)$$

$$\left|v_{y_v}(t)^T\Gamma(t)\right| \le \gamma\,,\ 0\le t\le T \quad \text{(future derivatives risk dynamics)} \quad (3.12b)$$

$$v_{y_v}(t)^T\Theta(t) \ge \vartheta\,,\ 0\le t\le T \quad \text{(options time decay dynamics)} \quad (3.12c)$$

$$v_{y_v}(t)^T\Lambda(t) \ge \lambda\,,\ 0\le t\le T \quad \text{(asset value appreciation dynamics)} \quad (3.12d)$$

[$\theta(t) = I_X(t)v(t)$]. Such a generalized contingent claim then optimally satisfies all the investor's specifications for (derived) financial instruments admissible in a limited risk arbitrage transaction on a single-instrument basis (i.e., these synthetically created optimal financial instruments do not have to be hedged and can therefore be used as elementary hedges in a risk/arbitrage hedging concept) and its risk/arbitrage value

$$V_v^{c_{y_v},\theta_{y_v}}(t) = V_{\omega_{y_v},v}^{c_{y_v},\theta_{y_v}}(t) = \frac{1}{\varsigma_v^{\omega_{y_v}}(t)} E[\int_t^T \varsigma_v^{\omega_{y_v}}(s)c_{y_v}(s)ds + \varsigma_v^{\omega_{y_v}}(T)V_T^{y_v}\,\big|\,F_t] \quad (3.13)$$

properly reflects this fact [recall the identification

$$\bar{\theta}(t)\in K_t \iff v(t)\in N_t \leftrightarrow \left|v(t)^T\Delta(t)\right|\le\delta,\ \left|v(t)^T\Gamma(t)\right|\le\gamma,\ ..,\ v(t)^T\Lambda(t)\ge\lambda$$

that involves constraint set projection along $I_X(t)(N_t)$-rays emanating from the origin and generally applies in our discussions of limited risk arbitrage investment management in a martingale representation setting (Cvitanic and Karatzas [3]), with drawdown control $K_t^\alpha = T_t^\alpha(K_t)$ a simple (compact range) transformation in $\bar{\theta}(t)$-strategy space again along $I_X(t)(N_t)$-rays emanating from the origin].

LRA Market Indices. Furthermore, the asset allocation problem

$$J(i,\theta) = E[U^V(I_i^\theta(T))]$$
$$\bar{V}(i) = \sup_{\theta\in\bar{A}(i)} J(i,\theta) \quad (3.14)$$

with controlled state dynamics (market index)

$$dI(t) = [\theta(t)^T[M(t) - r(t)1_L] + I(t)r(t)]dt + \theta(t)^T \Sigma(t)dW(t),\ I(0) = i, \quad (3.15)$$

where the feasible controls $\theta \in \overline{A}(i)$ satisfy the risk/arbitrage objectives

$$|v(t)^T \Delta(t)| \leq \delta,\ 0 \leq t \leq T \quad \text{(instantaneous investment risk)} \quad (3.16a)$$

$$|v(t)^T \Gamma(t)| \leq \gamma,\ 0 \leq t \leq T \quad \text{(future risk dynamics)} \quad (3.16b)$$

$$v(t)^T \Theta(t) \geq \vartheta,\ 0 \leq t \leq T \quad \text{(time decay dynamics)} \quad (3.16c)$$

$$v(t)^T \Lambda(t) \geq \lambda,\ 0 \leq t \leq T \quad \text{(value appreciation dynamics)} \quad (3.16d)$$

[$\theta(t) = I_X(t)v(t)$] and additional inequality and equality constraints

$$g(t, I(t), v(t)) \leq 0,\ 0 \leq t \leq T \quad (3.17a)$$

$$h(t, I(t), v(t)) = 0,\ 0 \leq t \leq T \quad (3.17b)$$

can be solved by using an algebraic technique that involves KKT first order optimality conditions and Lagrange multipliers (Chow [4]). A corresponding optimal solution $\theta_{LRA}(t)$ defines the composition of a limited risk arbitrage index $I_{LRA}(t)$ in the securities and derivatives market $X_1(t),..,X_L(t)$.

Utility-Based Hedging. A first risk/arbitrage hedging concept can be based on the fact that for an investor with the overall risk management objectives $U^c(t,c)$ and $U^V(V)$ two general contingent claims (c_1, V_T^1) and (c_2, V_T^2) with the property

$$E[\int_0^T U^c(t, c_1(t))dt + U^V(V_T^1)] = E[\int_0^T U^c(t, c_2(t))dt + U^V(V_T^2)] \quad (3.18)$$

are equally acceptable. If therefore $(\overline{c}, \overline{V}_T)$ is a given general claim contingent on the basic traded assets $X_1(t),..,X_L(t)$ (bonds, stocks and options) spanning the securities and derivatives market, then we write

$$J(\overline{c}, \overline{V}_T) = E[\int_0^T U^c(t, \overline{c}(t))dt + U^V(\overline{V}_T)] \quad (3.19)$$

and consider the expected utility

$$J(v) = E[\int_0^T U^c(t, c_{y_v}(t))dt + U^V(V_T^{y_v})] = \max_{(c, V_T) \in \overline{A}(v)} E[\int_0^T U^c(t, c(t))dt + U^V(V_T)] \quad (3.20)$$

of optimal financial instruments as a function of initial wealth $v > 0$. Because this concave and strictly increasing function is continuous with

$$\lim_{v \uparrow \infty} J(v) = +\infty \quad (3.21)$$

there exists an initial wealth $v_{(\overline{c}, \overline{V}_T)} > 0$, the utility-based risk/arbitrage price of the given contingent claim $(\overline{c}, \overline{V}_T)$, such that

$$v_{(\overline{c}, \overline{V}_T)} = \min\{v > 0 : J(v) \geq J(\overline{c}, \overline{V}_T)\}. \quad (3.22)$$

The corresponding optimal financial instrument

$$(c_{(\overline{c}, \overline{V}_T)}, V_T^{(\overline{c}, \overline{V}_T)}) \in \overline{A}(v_{(\overline{c}, \overline{V}_T)}) \quad (3.23)$$

is then a utility-based hedge for contingent claim $(\overline{c}, \overline{V}_T)$.

Contingent Claim Replication. A risk/arbitrage replication strategy $\theta(t)$ for a given general claim $C = (c, V_T)$ contingent on the basic assets $X_1(t),..,X_L(t)$ traded in the securities and derivatives market satisfies the investor's drawdown control and limited risk arbitrage objectives, i.e.,

$$\bar{\theta}(t) \in K_t^{\alpha},\ 0 \le t \le T, \quad (3.24)$$

and the associated value process

$$V_v^{\theta}(t) = \frac{1}{D(t)}\left[v + \int_0^t D(s)[\theta(s)^T[M(s) - r(s)1_L] - c(s)]ds + \int_0^t D(s)\theta(s)^T\Sigma(s)dW(s)\right] \quad (3.25)$$

has the property

$$V_v^{\theta}(T) = V_T. \quad (3.26)$$

Using the maximum principle of limited risk arbitrage investment management such a replication strategy $\theta_C(t)$ with associated value process

$$V_{v_C}^{\theta_C}(t) = V_{\omega_C v_C}^{\theta_C}(t) = \frac{1}{\varsigma_{\omega_C}(t)} E[\int_t^T \varsigma_{\omega_C}(s)c(s)ds + \varsigma_{\omega_C}(T)V_T | F_t] \quad (3.27)$$

can be found for an initial wealth $v_C > 0$, the replication-based risk/arbitrage price of claim $C = (c, V_T)$, if

$$\max_{\bar{\omega} \in \Omega} E[\int_0^T \varsigma_{\bar{\omega}}(t)c(t)dt + \varsigma_{\bar{\omega}}(T)V_T] = E[\int_0^T \varsigma_{\omega_C}(t)c(t)dt + \varsigma_{\omega_C}(T)V_T] = v_C \quad (3.28)$$

holds for a securities and derivatives market variation parameter $\omega_C(t)$, i.e., if the given contingent claim is R/A-attainable. In a complete securities and derivatives market setting [i.e., $L = N$] the replication-based risk/arbitrage value of such a general claim is then

$$V_{p(C)}^{\theta_C}(t) = \frac{1}{D_{\omega_C}(t)} \tilde{E}_{\omega_C}[\int_t^T D_{\omega_C}(s)c(s)ds + D_{\omega_C}(T)V_T | F_t] \quad (3.29)$$

$[p(C) = v_C]$ and we have its Markovian characterization

$$\frac{\partial V}{\partial t} + \frac{1}{2}\mathrm{tr}(BB^T\nabla^2 V) + r_{\omega_C}(X^T\nabla V - V) + c = 0 \quad (3.30)$$

with boundary condition $V(T, X) = h(X)$ where

$$B = \begin{bmatrix} X_1\Sigma_{11} & \Lambda & X_1\Sigma_{1N} \\ M & & M \\ X_N\Sigma_{N1} & \Lambda & X_N\Sigma_{NN} \end{bmatrix} \text{ and } \nabla = \begin{bmatrix} \frac{\partial}{\partial X_1} \\ M \\ \frac{\partial}{\partial X_N} \end{bmatrix}. \quad (3.31)$$

Furthermore, the associated trading strategy that replicates/hedges this claim is

$$\theta_C = [B\Sigma^{-1}]^T\nabla V = I_X\nabla V. \quad (3.32)$$

If the securities and derivatives market under consideration is incomplete, then the replication-based risk/arbitrage value of claim $C = (c, V_T)$ is

$$V_{p(C)}^{\theta_C}(t) = \frac{1}{D_{\omega_C}(t)} \tilde{E}_{1_{z_v}}^{\varphi_C}[\int_t^T D_{\omega_C}(s)c(s)ds + D_{\omega_C}(T)V_T | F_t] \quad (3.33)$$

[$p(C) = v_C$] with associated risk minimizing replication costs

$$dC^{\theta_c}(t) = dL_C(t)\,,\; C^{\theta_c}(0) = p(C)\,, \quad (3.34)$$

and we have its Markovian characterization

$$\frac{\partial V}{\partial t} + \frac{1}{2}\mathrm{tr}(BB^T\nabla^2 V) + r_{\omega_c}(\begin{bmatrix} X \\ Y \end{bmatrix}^T \nabla V - V) + c = 0 \quad (3.35)$$

with boundary condition $V(T,X,Y) = h(X,Y)$ where

$$B = \begin{bmatrix} X_1\Sigma_{11} & \Lambda & X_1\Sigma_{1N} \\ M & & M \\ X_L\Sigma_{L1} & \Lambda & X_L\Sigma_{LN} \\ Y_1 b_{11} & \Lambda & Y_1 b_{1N} \\ M & & M \\ Y_{N-L}b_{N-L1} & \Lambda & Y_{N-L}b_{N-LN} \end{bmatrix} \text{ and } \nabla = \begin{bmatrix} \nabla_X \\ \nabla_Y \end{bmatrix}. \quad (3.36)$$

Furthermore, the corresponding risk minimizing replication strategy is

$$\theta_C = [B\Sigma^T K^{-1}]^T \nabla V = I_X \nabla_X V\,. \quad (3.37)$$

Note also that for such an R/A-attainable general contingent claim $C = (c, V_T)$ the replication methodology (with the associated concept of a parametrized fundamental partial differential equation in a Markovian setting) is equivalent to the above utility-based hedging technique [i.e., given the initial wealth $p(C) = v_C$ (replication-based risk/arbitrage price) the replication strategy $\theta_C(t)$ is optimal with respect to the investor's overall risk management objectives $U^c(t,c)$ and $U^V(V)$, that is,

$$\begin{aligned} J(v_C,c,\theta_C) &= E[\int_0^T U^c(t,c(t))dt + U^V(V_{v_C}^{c\theta_c}(T))] \\ &= \overline{V}(v_C) = \sup_{(c,\theta)\in \overline{A}(v_C)} J(v_C,c,\theta) \end{aligned} \quad (3.38)$$

and consequently also $p(C) = v_C = v_{(c,V_T)}$ (utility-based risk/arbitrage price) is satisfied] if

$$\begin{aligned} c(t) &= I^c(t, H_{\overline{\omega}}^{-1}(v_C)\varsigma_{\overline{\omega}}(t)) \\ V_T &= I^V(H_{\overline{\omega}}^{-1}(v_C)\varsigma_{\overline{\omega}}(T)) \end{aligned} \quad (3.39)$$

$$\left[H_\omega(y) = E[\int_0^T \varsigma_\omega(t) I^c(t, y\varsigma_\omega(t))dt + \varsigma_\omega(T) I^V(y\varsigma_\omega(T))] \right]$$

holds for some market parametrization $\overline{\omega}(t)$ [that could be different from $\omega_C(t)$]. After solving the non-linear optimization programs

$$\begin{aligned} &\min_{\overline{\omega}\in\Omega} E[\int_0^H U_*^c(t, y\varsigma_{\overline{\omega}}(t))dt + U_*^V(y\varsigma_{\overline{\omega}}(H))] \\ &\min_{a(t)} E[\int_0^H U_*^c(t, z\xi_a^\omega(t))dt + U_*^V(z\xi_a^\omega(H))] \end{aligned} \quad (3.40)$$

(LRA investment management) and

$$\max_{\varpi\in\Omega} E[\int_0^T \varsigma_{\varpi}(t)c(t)dt + \varsigma_{\varpi}(T)V_T]$$
$$\min_{a(t)} E[\int_0^H U^c_\bullet(t, z\zeta^{\varpi}_a(t))dt + U^V_\bullet(z\zeta^{\varpi}_a(H))] \qquad (3.41)$$

(LRA contingent claim analysis) which determine the appropriate securities and derivatives market parametrizations and completions the general limited risk arbitrage (LRA) investment management process therefore only requires solving a (quasi-) linear partial differential equation.

Partial Replication Strategies. Partial replication in a limited risk arbitrage investment management and asset allocation context is based on the notion of an investor's cost functions $C^c_{\bar{c}}(t,c)$ for not exactly matching intertemporal cashflows and $C^V_{\bar{V}}(V)$ for a mismatch in the payoffs at maturity associated with a general claim $(\bar{c}, \bar{V}_T)$ contingent on the basic assets $X_1(t),..,X_L(t)$ spanning the securities and derivatives market and a corresponding partial replication strategy $(c,\theta) \in A(v)$. The relevant stochastic control problem is

$$\min_{(c,\theta)\in A(v)} E[\int_0^T C^c_{\bar{c}}(t,c(t))dt + C^V_{\bar{V}}(V^{c\theta}_v(T))]$$
$$\bar{\theta}(t) \in K^a_t,\ 0 \le t \le T. \qquad (3.42)$$

In order to apply standard HJB (dynamic programming) solution techniques we have to make the additional assumption that $u(t) \in U$ where $U \subseteq R^{L+1}$ is a compact set (control space) holds for the progressively measurable controls $u(t) = (c(t), \theta(t))$. The diffusion type controlled state variable is $x(t) = (V(t), X(t))$ and $\bar{A}_{tx}$ denotes the set of all feasible controls $u(s)$ on the time interval $[t,H]$ when the time t state is x. The state space characteristics are then

$$dx(t) = a(t,x(t),u(t))dt + b(t,x(t),u(t))dW(t)$$

$$a(t) = \begin{bmatrix} \theta(t)^T[M(t) - r(t)1_L] \\ + V(t)r(t) - c(t) \\ X_1(t)M_1(t) \\ M \\ X_L(t)M_L(t) \end{bmatrix} \quad b(t) = \begin{bmatrix} & \theta(t)^T\Sigma(t) & \\ X_1(t)\Sigma_{11}(t) & \Lambda & X_1(t)\Sigma_{1N}(t) \\ M & & M \\ X_L(t)\Sigma_{L1}(t) & \Lambda & X_L(t)\Sigma_{LN}(t) \end{bmatrix} \qquad (3.43)$$

[where the coefficients $a(t,x,u) \in R^{L+1}$ and $b(t,x,u) \in R^{(L+1)\times N}$ satisfy the usual conditions that guarantee a unique strong solution of the associated evolution equation with bounded absolute moments] and the cost functions

$$L(t,x(t),u(t)) = C^c_{\bar{c}}(t,c(t)) \text{ and } \psi(x(T)) = C^V_{\bar{V}}(V(T)) \qquad (3.44)$$

in the minimization criterion

$$J(t,x,u) = E_{tx}[\int_t^T L(s,x(s),u(s))ds + \psi(x(T))]$$
$$V(t,x) = \inf_{u\in\bar{A}_{tx}} J(t,x,u) \qquad (3.45)$$

[we are only interested in the case where for the value function

$$V_{AS}(t,x) = V_{PM}(t,x) = V_{(\Omega,\Phi,\pi,F,W)}(t,x) \quad (3.46)$$

holds] are assumed to be continuous and to satisfy a polynomial growth condition in both the state

$$x = [x_0 \quad \Lambda \quad x_L]^T \qquad \nabla_x = \left[\frac{\partial}{\partial x_0} \quad \Lambda \quad \frac{\partial}{\partial x_L}\right]^T = \begin{bmatrix}\nabla_V \\ \nabla_X\end{bmatrix}$$

$$H(t,x,p,A) = \sup_{u \in \tilde{A}_{tx}} \begin{bmatrix} -a(t,x,u(t))^T p \\ -\frac{1}{2} tr[b(t,x,u(t))b(t,x,u(t))^T A] \\ -L(t,x,u(t)) \end{bmatrix} \quad (3.47)$$

and the control

$$u = [u_0 \quad \Lambda \quad u_L]^T \qquad \nabla_u = \left[\frac{\partial}{\partial u_0} \quad \Lambda \quad \frac{\partial}{\partial u_L}\right]^T = \begin{bmatrix}\nabla_c \\ \nabla_k\end{bmatrix} \quad (3.48)$$

variables [which we have mapped into $\overline{\theta}(t)$ - strategy space for convenience: note that the associated constraint sets $K_t^{\alpha} = T_t^{\alpha}(K_t)$ (drawdown control) in $\overline{\theta}(t)$ - strategy space are compact if and only if the originally given constraint sets N_t (limited risk arbitrage objectives) in $v(t)$ - strategy space are compact whereas in general the risk/arbitrage constraint transforms $K_t = Y_{v,c}\, \overline{\theta}_t^{vc}(N_t)$ in $\overline{\theta}(t)$ - strategy space are (infinite) convex cones generated by $I_X(t)(N_t)$ -rays emanating from the origin]. Key to the dynamic programming approach is the second order, non-linear Hamilton-Jacobi-Bellman (HJB) partial differential equation

$$-\frac{\partial V}{\partial t} + H(t,x,\nabla_x V, \nabla_x^2 V) = 0 \quad (3.49)$$

with boundary data $V(T,x) = \psi(x)$. We first assume that this boundary value problem is uniformly parabolic, i.e., that there exists an $\varepsilon > 0$ such that for all $\xi \in R^{L+1}$,

$$\xi^T[b(t,x,u)b(t,x,u)^T]\xi \ge \varepsilon\|\xi\|^2. \quad (3.50)$$

Then under the standard differentiability and boundedness assumptions that have to be imposed on the coefficients $a(t,x,u)$ and $b(t,x,u)$ determining the state dynamics and the cost functions $L(t,x,u)$ and $\psi(x)$ the above Cauchy problem has a unique $C^{1,2}$ solution $W(t,x)$ which is bounded together with its partial derivatives. With this candidate for the optimal value function of the partial replication problem we consider the minimization program

$$\min_{u \in \tilde{A}_{tx}} F(t,x,u(t))$$

$$F(t,x,u) = a(t,x,u)^T \nabla_x W(t,x) + \frac{1}{2} tr[b(t,x,u)b(t,x,u)^T \nabla_x^2 W(t,x)] + L(t,x,u) \quad (3.51)$$

in control space U and denote with U_{tx} the set of corresponding solutions [which are the time t values of feasible controls u(s) on [t,T], i.e., of the form u(t) with $u \in \overline{A}_{tx}$]. By measurable selection we can then determine a bounded and Borel measurable function $\overline{u}(t,x)$ with the property $\overline{u}(t,x) \in U_{tx}$ (almost everywhere t, x). If an application of this optimal Markov control policy to the above state dynamics satisfies

$$\int_t^T \pi_{tx}^{\overline{u}}[(s, x_{\overline{u}}(s)) \in N]ds = 0 \quad (3.52)$$

for every Lebesgue null set $N \subseteq R^{L+2}$, then

$$W(t,x) = V_{PM}(t,x) = V_{(\Omega,\Phi,\pi,F,W)}(t,x) = J_{(\Omega^{\overline{u}},\Phi^{\overline{u}},\pi^{\overline{u}},F^{\overline{u}},W^{\overline{u}})}(t,x,\overline{u}) \quad (3.53)$$

and

$$W(t,x) = E_{tx}^{\overline{u}}[\int_t^\tau L(s, x_{\overline{u}}(s), \overline{u}(s, x_{\overline{u}}(s)))ds + W(\tau, x_{\overline{u}}(\tau))] \quad (3.54)$$

for any stopping time $\tau \in [t,T]$ (dynamic programming principle). This is the case if [after completion with additional state variables $x_{L+1}(t),..,x_{N-1}(t)$] the $N \times N$-matrix b(t) satisfies

$$\xi^T b(t,x,u)\xi \geq \varepsilon \|\xi\|^2, \; \varepsilon > 0 \text{ and } \xi \in R^N, \quad (3.55)$$

a property that implies uniform parabolicity of the associated HJB boundary value problem.

Viscosity Solutions. In the degenerate parabolic case we retain the above standard differentiability and boundedness conditions on the coefficients a(t,x,u) and b(t,x,u) determining the state dynamics and the cost functions L(t,x,u) and $\psi(x)$. Then the value function

$$\begin{aligned} V_{PM}(t,x) &= \inf_{(\Omega,\Phi,\pi,F,W)} V_{(\Omega,\Phi,\pi,F,W)}(t,x) \\ &= \inf_{(\Omega,\Phi,\pi,F,W)} \inf_{u \in A_{tx}} E_{tx}[\int_t^T L(s,x(s),u(s))ds + \psi(x(T))] \end{aligned} \quad (3.56)$$

associated with partial replication of a general contingent claim is continuous in time and state and semiconcave in the state variable x. Furthermore, we have

$$V_{PM}(t,x) \leq E_{tx}[\int_t^\tau L(s,x(s),u(s))ds + V_{PM}(\tau, x(\tau))] \quad (3.57)$$

for every reference probability system (Ω,Φ,π,F,W), every feasible control $u \in \overline{A}_{tx}$ and any stopping time $\tau \in [t,T]$. Also, if $\varepsilon > 0$ is given, then there exists a reference probability system (Ω,Φ,π,F,W) and a feasible control process $u \in \overline{A}_{tx}$ such that

$$V_{PM}(t,x) + \varepsilon \geq E_{tx}[\int_t^\tau L(s,x(s),u(s))ds + V_{PM}(\tau, x(\tau))] \quad (3.58)$$

for any stopping time $\tau \in [t,T]$ (dynamic programming principle). Moreover, the equality

$$V_{PM}(t,x) = V_{(\Omega,\Phi,\pi,F,W)}(t,x) \quad (3.59)$$

holds for every reference probability system $(\Omega, \Phi, \pi, F, W)$ and if in addition $W(t,x)$ is a classical solution of the above HJB boundary value problem, then we have

$$V_{PM}(t,x) = V_{(\Omega,\Phi,\pi,F,W)}(t,x) = W(t,x). \quad (3.60)$$

The dynamic programming principle can therefore also be written in the (generic) form

$$V(t,x) = \inf_{u \in \bar{A}_{tx}} E_{tx}[\int_t^{t+\Delta t} L(s,x(s),u(s))ds + V(t+\Delta t, x(t+\Delta t))]. \quad (3.61)$$

With the two parameter family of non-linear operators

$$[T_{tt+\Delta t}\phi](x) = \inf_{u \in \bar{A}_{tx}} E_{tx}[\int_t^{t+\Delta t} L(s,x(s),u(s))ds + \phi(x(t+\Delta t))] \quad (3.62)$$

on the class of continuous state functions $\phi(x)$ and the family of non-linear, elliptic, second order partial differential operators

$$[G_t\phi](x) = H(t,x,\nabla_x\phi,\nabla_x^2\phi) \quad (3.63)$$

for at least twice continuously differentiable state functions $\phi(x)$ we have then

$$\lim_{\Delta t \downarrow 0} \frac{[T_{tt+\Delta t}\varphi(t+\Delta t,\cdot)](x) - \varphi(t,x)}{\Delta t} = \frac{\partial \varphi}{\partial t}(t,x) - [G_t\varphi(t,\cdot)](x) \quad (3.64)$$

for every $C^{1,2}$ test function $\varphi(t,x)$ [i.e., $\{G_t\}$ is the infinitesimal generator of the operator semigroup $\{T_{tt+\Delta t}\}$ on $C(R^{L+1})$] as well as

$$V(t,x) = [T_{tT}\psi](x) = [T_{tt+\Delta t}V(t+\Delta t,\cdot)](x) \quad (3.65)$$

(abstract dynamic programming principle) and consequently $V(t,x)$ is a uniformly continuous viscosity solution of the (abstract) HJB dynamic programming equation

$$-\frac{\partial V}{\partial t}(t,x) + [G_t V(t,\cdot)](x) = 0 \quad (3.66)$$

which satisfies the boundary condition $V(T,x) = \psi(x)$. If on the other hand $V_1(t,x)$ is a corresponding continuous and bounded viscosity supersolution and $V_2(t,x)$ a continuous and bounded viscosity subsolution, then

$$\sup_{t,x}[V_2(t,x) - V_1(t,x)] = \sup_x[V_2(T,x) - V_1(T,x)] \quad (3.67)$$

holds and therefore $V(t,x)$ is uniquely determined by the Cauchy data

$$V(T,x) = \psi(x). \quad (3.68)$$

<u>Finite Difference Approximation</u>. A discrete approximation $V_h(t,x)$ of the value function $V(t,x)$ associated with the partial replication problem for general contingent claims and a corresponding optimal Markov control policy $\bar{u}_h(t,x)$ can be determined numerically by considering a time discretization

$$t = ih, \ 0 \le i \le m \qquad T = mh \quad (3.69)$$

and a lattice structure

$$x = \delta \begin{bmatrix} j_0 \\ M \\ j_L \end{bmatrix} \quad (3.70)$$

in state space where $j_0, .., j_L$ are integers and the two relevant discretization parameters h and δ satisfy

$$c(t,x,u) = b(t,x,u)b(t,x,u)^T$$

$$c_{ll}(t,x,u) - \sum_{\substack{k=0 \\ k \neq l}}^{L} |c_{kl}(t,x,u)| \geq 0 \qquad (3.71)$$

$$h\sum_{l=0}^{L}\left[c_{ll}(t,x,u) - \frac{1}{2}\sum_{\substack{k=0 \\ k \neq l}}^{L}|c_{kl}(t,x,u)| + \delta|a_l(t,x,u)|\right] \leq \delta^2.$$

We first approximate the controlled continuous-time diffusion process $x(t)$ by a controlled discrete-time Markov chain $x_h(t)$ that evolves on this lattice with one step transition probabilities

$$\pi_t^u(x, x \pm \delta e_l) = \frac{h}{2\delta^2}\left[c_{ll}(t,x,u) - \sum_{\substack{k=0 \\ k \neq l}}^{L}|c_{kl}(t,x,u)| + 2\delta a_l^{\pm}(t,x,u)\right] \qquad (3.72a)$$

$$\pi_t^u(x, x + \delta e_k \pm \delta e_l) = \frac{h}{2\delta^2}c_{kl}^{\pm}(t,x,u)$$
$$\pi_t^u(x, x - \delta e_k \pm \delta e_l) = \frac{h}{2\delta^2}c_{kl}^{\mp}(t,x,u) \qquad [k \neq l] \qquad (3.72b)$$

$$\pi_t^u(x,x) = 1 - \frac{h}{\delta^2}\sum_{l=0}^{L}\left[c_{ll}(t,x,u) - \frac{1}{2}\sum_{\substack{k=0 \\ k \neq l}}^{L}|c_{kl}(t,x,u)| + \delta|a_l(t,x,u)|\right] \qquad (3.72c)$$

[$e_0, .., e_L$ is the standard basis in R^{L+1} and $\pi_t^u(x,y) = 0$ for all other grid points y on the above lattice $X_h \subseteq R^{L+1}$]. The corresponding dynamic programming equation is

$$V_h(t,x) = \min_{u \in \bar{A}_{tx}}\left[\sum_{y \in X_h}\pi_t^{u(t)}(x,y)V_h(t+h,y) + hL(t,x,u(t))\right] \qquad (3.73)$$

with boundary condition $V_h(T,x) = \psi(x)$ and an associated optimal Markov control policy $\bar{u}_h(t,x)$ minimizes the expression

$$\sum_{y \in X_h}\pi_t^{u(t)}(x,y)V_h(t+h,y) + hL(t,x,u(t)) \qquad (3.74)$$

in $\bar{A}_{tx}$ [backwards in time from $T-h$ to 0]. With the finite differences

$$\Delta_t^- V = \frac{V(t,x) - V(t-h,x)}{h} \qquad \Delta_{x_l}^{\pm} V = \pm\frac{V(t,x \pm \delta e_l) - V(t,x)}{\delta}$$
$$\Delta_{x_l}^2 V = \frac{V(t,x+\delta e_l) + V(t,x-\delta e_l) - 2V(t,x)}{\delta^2} \qquad (3.75a)$$

$$\Delta^{+}_{x_k x_l} V = \frac{2V(t,x) + V(t,x+\delta e_k + \delta e_l) + V(t,x-\delta e_k - \delta e_l)}{2\delta^2} - \frac{V(t,x+\delta e_k) + V(t,x-\delta e_k) + V(t,x+\delta e_l) + V(t,x-\delta e_l)}{2\delta^2} \quad (3.75b)$$

$$\Delta^{-}_{x_k x_l} V = \frac{V(t,x+\delta e_k) + V(t,x-\delta e_k) + V(t,x+\delta e_l) + V(t,x-\delta e_l)}{2\delta^2} - \frac{2V(t,x) + V(t,x+\delta e_k - \delta e_l) + V(t,x-\delta e_k + \delta e_l)}{2\delta^2} \quad (3.75c)$$

we then also discretize the continuous-time HJB equation

$$-\frac{\partial V}{\partial t} + H(t,x,\nabla_x V, \nabla_x^2 V) = 0 \quad (3.76)$$

and with

$$H_h(t,x,p_l^{\pm},A_{ll},A_{kl}^{\pm}) = \max_{u\in A_{tx}} \left[\sum_{l=0}^{L} \left(\begin{array}{c} a_l^-(t,x,u(t))p_l^- \\ -a_l^+(t,x,u(t))p_l^+ \\ -\frac{c_{ll}(t,x,u(t))}{2} A_{ll} \\ +\sum_{\substack{k=0 \\ k\neq l}}^{L} \left[\begin{array}{c} \frac{c_{kl}^-(t,x,u(t))}{2} A_{kl}^- \\ -\frac{c_{kl}^+(t,x,u(t))}{2} A_{kl}^+ \end{array} \right] \end{array} \right) - L(t,x,u(t)) \right] \quad (3.77)$$

find that

$$V_h(t-h,x) = V_h(t,x) - hH_h(t,x,\Delta_{x_l}^{\pm} V_h, \Delta_{x_l}^2 V_h, \Delta_{x_k x_l}^{\pm} V_h) \quad (3.78)$$

holds for the value function of the discrete-time Markov chain control problem. This form of the associated dynamic programming equation can now be rewritten as

$$V_h(t,x) = [T_{tt+h}^h V_h(t+h,\cdot)](x) \quad (3.79)$$

with the family of discrete-time operators

$$[T_{tt+h}^h \phi](x) = \phi(x) - hH_h(t+h,x,\Delta_{x_l}^{\pm}\phi, \Delta_{x_l}^2\phi, \Delta_{x_k x_l}^{\pm}\phi) \quad (3.80)$$

for bounded state functions $\phi(x)$ on the lattice $X_h \subseteq R^{L+1}$ which satisfies

$$\lim_{\substack{(\tau,\xi)\to(t,x) \\ h\downarrow 0}} \frac{[T_{tt+h}^h \varphi(\tau+h,\cdot)](\xi) - \varphi(\tau,\xi)}{h} = \frac{\partial \varphi}{\partial t}(t,x) - H(t,x,\nabla_x\varphi, \nabla_x^2\varphi) \quad (3.81)$$

for every $C^{1,2}$ test function $\varphi(t,x)$ (consistency) and consequently we have uniform convergence

$$\lim_{\substack{(\tau,\xi)\to(t,x) \\ h\downarrow 0}} V_h(\tau,\xi) = V(t,x) \quad (3.82)$$

of the discrete-time Markov chain control problem to the continuous-time diffusion process control problem. The same is true on compact sets if instead of the full infinite lattice $X_h \subseteq R^{L+1}$ only a bounded sublattice (with arbitrary definition of the transition probabilities at the boundary) is considered in actual numerical calculations.

Appendix: References

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