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Pricing Dynamic Solvency Insurance and Investment Fund Protection

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Abstract

In the first part of the paper the surplus of a company is modelled by a Wiener process. We consider a dynamic solvency insurance contract. Under such a contract the necessary payments are made instantaneously so that the modified surplus never falls below zero. This means mathematically that the modified surplus process is obtained from the original surplus process by introduction of a reflecting barrier at zero. Theorem 1 gives an explicit expression for the net single premium of such a contract.

In the second part we consider an investment fund whose unit value is modelled by a geometric Brownian motion. Different forms of dynamic investment fund protection are examined. The basic form is a guarantee which provides instantaneously the necessary payments so that the upgraded fund unit value does not fall below a protected level. Theorem 2 gives an explicit expression for the price of such a guarantee. This result can also be applied to price a guarantee where the protected level is an exponential function of time. Moreover it is shown explicitly how the guarantee can be generated by construction of the replicating portfolio. The dynamic investment fund guarantee is compared to the corresponding put option and it is observed that for short time intervals the ratio of the prices is about 2. Finally the price of a more exotic protection is discussed, under which the guaranteed unit value at any time is a fixed fraction of the maximal upgraded unit value that has been observed until then. Several numerical and graphical illustrations show how the theoretical results can be implemented in practice.

1 Introduction

We consider a company whose surplus is a stochastic process which can take on negative values. We propose a contract (labelled *dynamic solvency insurance*) that, whenever the surplus is negative, provides a payment in the amount of the deficit, so that the surplus is immediately reset to zero. The first goal is to determine the net single premium of such a contract, that is the expectation of the sum of the discounted payments. Results of this type have been presented by Pafumi in his discussion of Gerber and Shiu (1998b), as well by the authors in their reply. Their results are for perpetual coverage.

In sections 3 and 4 we examine a model where the unmodified surplus process is a Wiener process. This model does not appear to be as realistic as the compound Poisson model; but it has the advantage that certain calculations can be done very explicitly. Theorem 1 provides an explicit expression for the net single premium for *finite time* coverage. The methods and results of sections 3 and 4 are also useful in the subsequent sections.

Starting with section 5 we look at an investment fund and examine certain forms of *dynamic investment fund protection.* We make the classical assumption, that is that the value of a fund unit follows a geometric Brownian motion. In sections 5 and 6, we analyze a guarantee that provides at any time the necessary payments to prevent the (modified) unit value to fall below a protected level. Theorem 2 gives an explicit formula for the no arbitrage price of the *finite time* guarantee of such a contract. In section 7 we examine the stronger guarantee, where the protected level is an exponential function of time. It is shown that the pricing of such a guarantee can be reduced to the pricing of a guarantee with constant level.

Gerber and Shiu (1998a,c) give results for the price of guarantees with constant and exponential protected levels. Again, their results are for perpetual coverage.

In section 8 it is discussed how the dynamic investment fund protection can be obtained by construction of the replicating portfolio. It is shown explicitly how at any time the total assets should be allocated to the risky and riskless assets.

In section 9 we consider a more exotic guarantee, where the protected level is a fixed percentage of the maximal upgraded unit value that has been observed in the past. The discussion is limited to perpetual coverage. To obtain finite prices we assume that the fund pays cash dividends at a constant rate. The analysis is intimately connected to the analysis of the Russian option in section 10.11 of Boyle et al. (1998).

In section 10 the dynamic investment fund guarantee is compared to the corresponding European put option, which provides a static solution. It is observed that the ratio of the prices tends to 2 for $T \to 0$.

2 A useful identity

The goal of this section is to present an identity that will facilitate the calculations in section 4. Let

$$n(x; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right), \qquad -\infty < x < \infty,$$

denote the probability density function of the normal distribution with mean μ and variance σ^2, and let $\Phi(x)$ denote the standard normal distribution function. Then the following two formulas are easily verified:

$$\int_0^\infty n(x; \mu, \sigma^2) dx = \Phi\left(\frac{\mu}{\sigma}\right), \tag{1}$$

and

$$e^{\kappa x} n(x; \mu, \sigma^2) = e^{\mu\kappa + \frac{1}{2}\sigma^2\kappa^2} n(x; \mu + \kappa\sigma^2, \sigma^2) \tag{2}$$

for any number κ. Combining (1) and (2) we obtain the identity

$$\int_0^\infty e^{\kappa x} n(x; \mu, \sigma^2) dx = e^{\mu\kappa + \frac{1}{2}\sigma^2\kappa^2} \Phi\left(\frac{\mu + \kappa\sigma^2}{\sigma}\right). \tag{3}$$

3 Solvency insurance - perpetual coverage

We consider a company with initial surplus u and surplus $U(t)$ at time t. We suppose that the net income process is a Wiener process with constant parameters μ and σ. Thus

$$U(t) = u + \mu t + \sigma W(t), \qquad t \geq 0,$$

where $\{W(t)\}$ is a standard Wiener process. We consider a contract that provides essentially the following dynamic solvency insurance: whenever the surplus falls below 0, the insurer makes the necessary payment to reset the surplus to 0. Mathematically, this means that the original surplus process is modified by a reflecting barrier at 0. The modified surplus is denoted as $\tilde{U}(t)$. This is illustrated in Figure 1.

Let $P(t)$ denote the cumulative payment of the insurer by time t. Thus $\tilde{U}(t) = U(t) + P(t)$. There is a formula for $P(t)$:

$$P(t) = \max\left\{-\min_{0 \le \tau \le t} U(\tau), 0\right\}.$$

This formula will not be used explicitly in the following.

For a given force of interest $\delta > 0$, let $A(u)$ denote the net single premium of the perpetual solvency protection, that is

$$A(u) = E\left[\int_0^\infty e^{-\delta t} dP(t) \middle| U(0) = u\right]$$

is the expected value of all discounted payments that are made by the insurer.

The function $A(u)$ can be obtained by heuristic reasoning. Consider the "small" time interval from 0 to dt. Then, for $u > 0$, we have

$$\begin{aligned} A(u) &= e^{-\delta dt} E\left[A(U(dt))\right] \\ &= e^{-\delta dt} E\left[A(u + \mu dt + \sigma W(dt))\right] \\ &= (1 - \delta dt)\left[A(u) + \mu A'(u)dt + \frac{1}{2}\sigma^2 A''(u)dt\right], \end{aligned}$$

which yields the differential equation

$$\frac{1}{2}\sigma^2 A''(u) + \mu A'(u) - \delta A(u) = 0.$$

Hence

$$A(u) = C_1 e^{\xi_1 u} + C_2 e^{\xi_2 u}, \qquad u \ge 0,$$

where ξ_1, ξ_2 are the solutions of the equation

$$\frac{1}{2}\sigma^2 \xi^2 + \mu\xi - \delta = 0.$$

Note that $\xi_1 \xi_2 = -2\delta/\sigma^2$. Thus, for example $\xi_1 < 0$, $\xi_2 > 0$. Since $A(u) \to 0$ for $u \to \infty$, it follows that $C_2 = 0$. Hence

$$A(u) = Ce^{-Ru}, \qquad u \ge 0, \tag{4}$$

where $R = |\xi_1|$ is the positive solution of the equation

$$\frac{1}{2}\sigma^2 R^2 - \mu R - \delta = 0, \tag{5}$$

that is

$$R = \frac{\mu + \sqrt{\mu^2 + 2\delta\sigma^2}}{\sigma^2}.$$

To determine the value of the constant C, we examine the function $A(u)$ in a neighborhood of 0. For this purpose, compare two situations, (a) initial surplus 0, (b) initial surplus ε (positive, but "small"). In situation (b) the surplus will instantly hit the level 0, and the modified surplus process will be the same from thereon. Thus the insurer's payments are the same in both situations, except that he has to pay ε more in the very beginning in situation (a). It follows that

$$A(0) \approx A(\varepsilon) + \varepsilon$$

or

$$A'(0) = \lim_{\varepsilon \to 0} \frac{A(\varepsilon) - A(0)}{\varepsilon} = -1. \tag{6}$$

Substituting this in (4), we see that $C = 1/R$. Thus

$$A(u) = \frac{1}{R} e^{-Ru}, \qquad u \geq 0. \tag{7}$$

Remark 1 *An alternative derivation of (4) starts with the observation that the process* $\left\{e^{-\delta t - RU(t)}\right\}$ *is a bounded martingale and uses the optional sampling theorem.*

Remark 2 *A more formal justification of (6) would use Itô calculus.*

Remark 3 *Formula (7) can be found as (R.8) in Gerber and Shiu (1998b).*

4 Solvency insurance - finite time

We now consider a temporary dynamic solvency insurance, where the coverage is provided only up to time T. Let $A(u,T)$ denote the net single premium. We can view the temporary coverage as a difference of a perpetual coverage and a deferred perpetual coverage that starts at time T. It follows that

$$A(u,T) = A(u) - e^{-\delta T} \int_0^\infty A(x) p(x; u, T) dx, \tag{8}$$

where $p(x;u,T)$, $x>0$, denotes the probability density function of the modified surplus at time T. Luckily an explicit expression for this density is available:

$$\begin{aligned} p(x;u,T) &= n(x;u+\mu T,\sigma^2 T) \\ &+e^{-2\mu u/\sigma^2} n(x;-u+\mu T,\sigma^2 T) \\ &-\frac{2\mu}{\sigma^2} e^{2\mu x/\sigma^2}\left[1-\Phi\left(\frac{x+u+\mu T}{\sigma\sqrt{T}}\right)\right], \end{aligned} \tag{9}$$

see formula (91), section 5.7 in Cox and Miller (1995). Now we substitute (7) and(9) in (8). Hence we have to calculate three integrals. The first two can be evaluated directly by using (3), with $\kappa=-R$, and by the fact that R satisfies (5). This way we obtain

$$\begin{aligned} A(u,T) &= \frac{1}{R}e^{-Ru} \\ &-\frac{1}{R}e^{-Ru}\Phi\left(\frac{u+(\mu-R\sigma^2)T}{\sigma\sqrt{T}}\right) \\ &-\frac{1}{R}e^{\left(R-\frac{2\mu}{\sigma^2}\right)u}\Phi\left(\frac{-u+(\mu-R\sigma^2)T}{\sigma\sqrt{T}}\right) \\ &+I \end{aligned} \tag{10}$$

where

$$I=\frac{1}{R}\frac{2\mu}{\sigma^2}e^{-\delta T}\int_0^\infty e^{\left(\frac{2\mu}{\sigma^2}-R\right)x}\left[1-\Phi\left(\frac{x+u+\mu T}{\sigma\sqrt{T}}\right)\right]dx. \tag{11}$$

Observe that

$$\begin{aligned} &\frac{1}{R}\frac{2\mu}{\sigma^2}e^{\left(\frac{2\mu}{\sigma^2}-R\right)x} \\ =\ &\frac{1}{R}\frac{2\mu}{2\mu-\sigma^2 R}\frac{d}{dx}e^{\left(\frac{2\mu}{\sigma^2}-R\right)x} \\ =\ &-\frac{\mu}{\delta}\frac{d}{dx}e^{\left(\frac{2\mu}{\sigma^2}-R\right)x} \end{aligned}$$

by (5). Then integrating (11) by part we obtain

$$\begin{aligned} I &= \frac{\mu}{\delta}e^{-\delta T}\left[1-\Phi\left(\frac{u+\mu T}{\sigma\sqrt{T}}\right)\right] \\ &-\frac{\mu}{\delta}e^{-\delta T}\int_0^\infty e^{\left(\frac{2\mu}{\sigma^2}-R\right)x}n(x;-u-\mu T,\sigma^2 T)dx. \end{aligned}$$

$T \setminus u$	0	1	2	3	4	5
1	1.1456	0.4452	0.1474	0.0408	0.0093	0.0017
2	1.4051	0.6676	0.2953	0.1205	0.0450	0.0153
3	1.5438	0.7923	0.3897	0.1826	0.0811	0.0340
4	1.6291	0.8707	0.4523	0.2274	0.1103	0.0514
5	1.6854	0.9229	0.4953	0.2598	0.1327	0.0658
10	1.7965	1.0275	0.5847	0.3308	0.1858	0.1034
15	1.8219	1.0517	0.6062	0.3487	0.2001	0.1145
20	1.8290	1.0584	0.6122	0.3539	0.2044	0.1179
25	1.8311	1.0605	0.6141	0.3555	0.2058	0.1190
∞	1.8322	1.0615	0.6150	0.3563	0.2064	0.1196

Table 1: Net single premium for dynamic solvency insurance with $\mu = 1$, $\sigma = 2$ and $\delta = 0.05$

Finally we use (3), with $\kappa = \frac{2\mu}{\sigma^2} - R$, and obtain after simplification

$$\begin{aligned} I &= \frac{\mu}{\delta} e^{-\delta T} \left[1 - \Phi \left(\frac{u + \mu T}{\sigma \sqrt{T}} \right) \right] \\ &\quad - \frac{\mu}{\delta} e^{\left(R - \frac{2\mu}{\sigma^2}\right) u} \Phi \left(\frac{-u + (\mu - R\sigma^2)T}{\sigma \sqrt{T}} \right) \end{aligned}$$

Substituting this expression in (10) we obtain the following result:

Theorem 1

$$\begin{aligned} A(u,T) = \frac{1}{R} e^{-Ru} \Phi \left(\frac{-u - (\mu - R\sigma^2)T}{\sigma\sqrt{T}} \right) \\ + \frac{\mu}{\delta} e^{-\delta T} \Phi \left(\frac{-u - \mu T}{\sigma \sqrt{T}} \right) \\ - \left(\frac{1}{R} + \frac{\mu}{\delta} \right) e^{\left(R - \frac{2\mu}{\sigma^2}\right) u} \Phi \left(\frac{-u + (\mu - R\sigma^2)T}{\sigma\sqrt{T}} \right) \end{aligned}$$

For a numerical illustration, suppose $\mu = 1$, $\sigma = 2$ and $\delta = 0.05$. Table 1 gives the net single premium $A(u, T)$ for various values of the initial surplus u and the length of the coverage T. Figure 2 shows, for the same parameters, the net single premium as a function of T for various values of u.

5 Investment fund protection - infinite time

We consider an investment fund. Let $F(t)$ denote the value of a fund unit at time t. We assume the classical model of geometric Brownian motion, that is

$$F(t) = F(0)e^{\mu t+\sigma W(t)}, \qquad t \geq 0,$$

where $\{W(t)\}$ is a standard Wiener process. We also assume a constant risk-less force of interest $r > 0$. The process $\{e^{-rt}F(t)\}$ is a martingale for

$$\mu = r - \frac{1}{2}\sigma^2. \tag{12}$$

We suppose that all dividends are reinvested in the fund. Consequently, the price of a derivative security is calculated as the expected discounted sum of the corresponding payments, with μ given by (12).

Now we consider a contract that upgrades the unit value $F(t)$ to a modified unit value $\tilde{F}(t)$ in the following sense. Let $\tilde{F}(0) = F(0)$ and let K denote the protected unit value $(0 < K \leq F(0))$. If $\tilde{F}(t) > K$ in some time interval, the instantaneous rate of return of $\left\{\tilde{F}(t)\right\}$ is identical to that of $\{F(t)\}$. Whenever $\tilde{F}(t)$ drops to K, just enough money will be added so that $\tilde{F}(t)$ does not drop below K.

We note that this construction can be related to the framework of section 3. It suffices to set

$$\begin{aligned} u &= \ln(f/K), \qquad \text{with } f = F(0), \\ U(t) &= \ln(F(t)/K), \\ \tilde{U}(t) &= \ln(\tilde{F}(t)/K). \end{aligned}$$

Let $V(f)$ denote the price of the contract, and let $\mathcal{V}(u)$, $u \geq 0$, be the function defined by the relation

$$\mathcal{V}(u) = V(f).$$

For $u > 0$, the function $\mathcal{V}(u)$ behaves like the function $A(u)$ in section 3. Thus

$$\mathcal{V}(u) = Ce^{-Ru}, \qquad u \geq 0,$$

for some constant C. If we replace δ by r and μ according to (12) in (5), we see that

$$R = \frac{2r}{\sigma^2}. \tag{13}$$

Then

$$V(f) = \mathcal{V}(\ln(f/K)) = C\,(K/f)^R, \qquad f \geq K.$$

It remains to determine the value of the constant C. For the same reasons that led to (6), we have

$$V'(K) = -1. \tag{14}$$

It follows that $C = K/R$ and

$$V(f) = \frac{K}{R}\left(\frac{K}{f}\right)^R, \qquad f \geq K,$$

or

$$\mathcal{V}(u) = \frac{K}{R}e^{-Ru}, \qquad u \geq 0 \tag{15}$$

6 Investment fund protection - finite time

Suppose now that the dynamic fund protection is only temporary and ends at time T. The price is denoted by $V(f,T)$, or $\mathcal{V}(u,T)$ in terms of the variable $u = \ln(f/K)$.

Observe that

$$\mathcal{V}(u,T) = \mathcal{V}(u) - e^{-rT}\int_0^\infty \mathcal{V}(x)p(x;u,T)dx, \tag{16}$$

with $\mathcal{V}(u)$ as in (15) and $p(x;u,T)$ as in (9). The resulting calculations are essentially the same as those that led to Theorem 1. Thus we can use Theorem 1 with the appropriate substitutions. In this way we find that

$$\begin{aligned}
\mathcal{V}(u,T) &= \frac{K}{R}e^{-Ru}\Phi\left(\frac{-u+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
&\quad +K\left(1-\frac{\sigma^2}{2r}\right)e^{-rT}\Phi\left(\frac{-u-(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
&\quad -Ke^u\Phi\left(\frac{-u-(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right),
\end{aligned}$$

with R given by (13). Finally, we express the price in terms of the initial value f of a fund unit:

$T \setminus K$	80	85	90	95	100
1/12	0.0001	0.0065	0.1304	1.0797	4.5189
2/12	0.0109	0.1093	0.6338	2.3761	6.3359
3/12	0.0624	0.3340	1.2463	3.4770	7.7069
4/12	0.1626	0.6313	1.8676	4.4370	8.8463
5/12	0.3035	0.9659	2.4706	5.2943	9.8376
6/12	0.4746	1.3180	3.0481	6.0732	10.7233
1	1.7709	3.4239	6.0120	9.7476	14.7931
2	4.4061	6.9231	10.3118	14.6840	20.1295
5	10.1373	13.7031	18.0257	23.1640	29.1716
10	15.6391	19.8688	24.7909	30.4504	36.8905
20	20.8713	25.5995	30.9834	37.0626	43.8762
∞	25.6000	30.7063	36.4500	42.8688	50.0000

Table 2: Price of the guarantee with $f = 100$, $\sigma = 0.2$ and $r = 0.04$

Theorem 2

$$\begin{aligned} V(f,T) = {} & \frac{K}{R}\left(\frac{K}{f}\right)^{R} \Phi\left(\frac{\ln(K/f) + \frac{1}{2}\sigma^2 T(R+1)}{\sigma\sqrt{T}}\right) \\ & + K\left(1 - \frac{1}{R}\right) e^{-rT} \Phi\left(\frac{\ln(K/f) - \frac{1}{2}\sigma^2 T(R-1)}{\sigma\sqrt{T}}\right) \\ & - f\Phi\left(\frac{\ln(K/f) - \frac{1}{2}\sigma^2 T(R+1)}{\sigma\sqrt{T}}\right) \end{aligned}$$

For a numerical illustration, suppose $f = 100$ and $\sigma = 0.2$. Tables 2 to 5 give the price of the guarantee for $r = 0.01, 0.02, 0.03, 0.04$. Figure 3 exhibits, for $r = 0.04$, the price of the guarantee as a function of T for various values of K. Since $f = 100$, the protected value K and the prices can be interpreted as percentages of the initial value of the fund unit.

7 Protection with a guaranteed force of return

We now consider a stronger guarantee, where the guaranteed value of a fund unit at time t is $Ke^{\gamma t}$ $(0 < \gamma < r)$. Hence, if $\tilde{F}(t) > Ke^{\gamma t}$, the instantaneous rates of return of $F(t)$ and $\tilde{F}(t)$ are the same, and whenever $\tilde{F}(t)$ reaches the boundary $Ke^{\gamma t}$, just enough money is provided so that the modified unit value does not fall

$T \setminus K$	80	85	90	95	100
1/12	0.0001	0.0068	0.1347	1.1017	4.5613
2/12	0.0116	0.1148	0.6572	2.4346	6.4212
3/12	0.0666	0.3520	1.2965	3.5742	7.8352
4/12	0.1743	0.6671	1.9483	4.5738	9.0180
5/12	0.3261	1.0233	2.5840	5.4712	10.0526
6/12	0.5111	1.3998	3.1955	6.2905	10.9818
1	1.9299	3.6794	6.3760	10.2100	15.3135
2	4.8907	7.5736	11.1267	15.6385	21.1718
5	11.7043	15.5702	20.1716	25.5457	31.7223
10	18.9143	23.6008	28.9444	34.9691	41.6962
20	26.8961	32.2992	38.3106	44.9490	52.2324
∞	38.1622	44.4075	51.2289	58.6432	66.6667

Table 3: Price of the guarantee with $f = 100$, $\sigma = 0.2$ and $r = 0.03$

$T \setminus K$	80	85	90	95	100
1/12	0.0001	0.0071	0.1391	1.1241	4.6040
2/12	0.0124	0.1206	0.6813	2.4942	6.5070
3/12	0.0712	0.3709	1.3484	3.6734	7.9660
4/12	0.1867	0.7047	2.0319	4.7138	9.1930
5/12	0.3501	1.0836	2.7015	5.6526	10.2720
6/12	0.5499	1.4856	3.3485	6.5137	11.2460
1	2.1006	3.9500	6.7573	10.6903	15.8519
2	5.4183	8.2733	11.9943	16.6475	22.2703
5	13.4764	17.6586	22.5512	28.1713	34.5279
10	22.8233	28.0101	33.8137	40.2397	47.2911
20	34.7601	40.9605	47.7147	55.0245	62.8907
∞	64.0000	72.2500	81.0000	90.2500	100.0000

Table 4: Price of the guarantee with $f = 100$, $\sigma = 0.2$ and $r = 0.02$

$T \setminus K$	80	85	90	95	100
1/12	0.0001	0.0074	0.1437	1.1469	4.6471
2/12	0.0132	0.1266	0.7062	2.5548	6.5948
3/12	0.0760	0.3905	1.4019	3.7747	8.0979
4/12	0.1998	0.7440	2.1182	4.8570	9.3706
5/12	0.3756	1.1468	2.8231	5.8385	10.4960
6/12	0.5913	1.5758	3.5071	6.7430	11.5167
1	2.2837	4.2363	7.1562	11.1887	16.4088
2	5.9916	9.0243	12.9165	17.7125	23.4267
5	15.4719	19.9862	25.1821	31.0586	37.6072
10	27.4617	33.1947	39.5000	46.3676	53.7858
20	44.9721	52.1171	59.7563	67.8779	76.4700
∞	143.1084	156.7323	170.7630	185.1891	200.0000

Table 5: Price of the guarantee with $f = 100$, $\sigma = 0.2$ and $r = 0.01$

below the boundary. The processes $\{F(t)\}$ and $\{\tilde{F}(t)\}$ can also be expressed in the language of sections 3 and 4: this time we set

$$\begin{aligned} U(t) &= \ln\left(F(t)/\left(Ke^{\gamma t}\right)\right) \\ &= u + (\mu - \gamma)t + \sigma W(t), \qquad t \geq 0, \end{aligned}$$

and

$$\tilde{U}(t) = \ln\left(\tilde{F}(t)/\left(Ke^{\gamma t}\right)\right), \qquad t \geq 0.$$

Since

$$\left\{e^{-rt+\mu t+\sigma W(t)}\right\}$$

is a martingale for $\mu = r - \frac{1}{2}\sigma^2$, it follows that

$$\left\{e^{-(\mu-\gamma)t+U(t)}\right\}$$

is a martingale. Hence we can use the formulas for $V(f)$ and $V(f,T)$ of the preceding sections: it suffices to replace r by the modified rate $r - \gamma$.

Example *Suppose that* $\sigma = 0.2$, $r = 0.04$, *and* $f = 100$. *Then the price for guarantee with* $\gamma = 0.03$ *can be obtained directly from Table 5 where* $r = 0.04 - 0.03 = 0.01$. *For example, the price for a 2-year guarantee, with* $K = 95$ *and* $\gamma = 0.03$, *is* 17.7125.

8 Synthetic investment fund protection

One way to obtain the dynamic investment fund protection is to use a *replicating portfolio* as an investment strategy. Consider an investor with an initial capital of $a = f + V(f,T)$. Instead of buying the protection from an external agent, he invests initially the amount a, and adopts a strategy so that total assets at any time t are exactly the sum of the upgraded unit value and the price for the remaining guarantee:

$$A(t) = \tilde{F}(t) + V(\tilde{F}(t), T-t), \qquad 0 \le t \le T. \tag{17}$$

The strategy consists of allocating the amount

$$\tilde{F}(t)\left(1 + V_f(\tilde{F}(t), T-t)\right)$$

at time t to the risky investment, and the complement, the amount

$$\begin{aligned} & A(t) - \tilde{F}(t)\left(1 + V_f(\tilde{F}(t), T-t)\right) \\ = \; & V(\tilde{F}(t), T-t) - \tilde{F}(t) V_f(\tilde{F}(t), T-t) \end{aligned}$$

to the riskless investment. This result (where the index f denotes partial derivative with respect to f) follows from a well known formula that can be found, for example, as formula (10.6.6) in Boyle et al. (1998), on page 95 of Baxter and Rennie (1996), or in section 9.3 of Dothan (1990).

For typographical convenience (and without loss of generality) we set $t = 0$, $\tilde{F}(0) = f$, $A(0) = a$ in the following. From Theorem 2 we obtain after simplification the formula

$$\begin{aligned} V_f(f,T) \;=\; & -\left(\frac{K}{f}\right)^{R+1} \Phi\left(\frac{\ln(K/f) + \frac{1}{2}\sigma^2 T(R+1)}{\sigma\sqrt{T}}\right) \\ & -\Phi\left(\frac{\ln(K/f) - \frac{1}{2}\sigma^2 T(R+1)}{\sigma\sqrt{T}}\right) \end{aligned}$$

Hence, the strategy is to invest the amount

$$\begin{aligned} & f\left\{1 - \left(\frac{K}{f}\right)^{R+1} \Phi\left(\frac{\ln(K/f) + \frac{1}{2}\sigma^2 T(R+1)}{\sigma\sqrt{T}}\right)\right. \\ & \left. -\Phi\left(\frac{\ln(K/f) - \frac{1}{2}\sigma^2 T(R+1)}{\sigma\sqrt{T}}\right)\right\} \end{aligned} \tag{18}$$

in the risky asset, and the amount

$$\left(1+\frac{1}{R}\right)K\left(\frac{K}{f}\right)^{R}\Phi\left(\frac{\ln(K/f)+\frac{1}{2}\sigma^2T(R+1)}{\sigma\sqrt{T}}\right)$$
$$+\left(1-\frac{1}{R}\right)Ke^{-rT}\Phi\left(\frac{\ln(K/f)-\frac{1}{2}\sigma^2T(R-1)}{\sigma\sqrt{T}}\right) \tag{19}$$

in the riskless asset. Note that this decomposition of total assets in risky asset and riskless asset is different from the decomposition in upgraded unit value and price of the remaining guarantee as in (17).

For infinite time coverage expressions (18) and (19) can be greatly simplified. Taking the limit $T \to \infty$ in (18) and (19) we see that the two components are

$$f\left\{1-\left(\frac{K}{f}\right)^{R+1}\right\} \tag{20}$$

and

$$f\left(1+\frac{1}{R}\right)\left(\frac{K}{f}\right)^{R+1} \tag{21}$$

For a numerical illustration, suppose $\sigma = 0.2$ and $r = 0.04$ as in Table 2. Tables 6 (for $K = 100$) and 8 (for $K = 95$) show the fund unit value as a function of total assets and remaining time of guarantee, that is f, the solution of $a = f + V(f, T-t)$. For given value of a the fund unit value is of course a decreasing function of both the remaining time of guarantee $T-t$ and the protected value K. This is also highlighted by the charts in Figures 4 and 5. Tables 7 (for $K = 100$) and 9 (for $K = 95$) show how the replicating portfolio has to be constructed. They display the amount invested in the risky asset, see (18) and (20), as a percentage of total assets.

9 Exotic protections

Let us finally consider some exotic schemes, where the guaranteed value of a fund unit is path-dependent. For example, the guaranteed value at time t could be a fixed fraction of the maximal unit value observed up to time t. We shall analyze a slightly different protection where the guaranteed unit value at time t is a fixed fraction of the maximal upgraded unit value that has been observed up to time t.

$a \setminus T-t$	∞	20	10	5	4	3	2	1	6/12	3/12
110	-	-	-	-	-	-	-	-	-	107.38
115	-	-	-	-	-	-	-	102.75	111.91	114.22
120	-	-	-	-	-	-	-	115.07	118.63	119.75
125	-	-	-	-	-	107.79	116.22	122.21	124.35	124.92
130	-	-	-	107.03	114.13	119.52	124.27	128.33	129.69	129.98
135	-	-	-	120.16	123.91	127.55	131.02	133.97	134.85	134.99
140	-	-	115.09	128.79	131.61	134.44	137.15	139.36	139.93	140
145	-	109.03	125.84	136.15	138.44	140.75	142.9	144.60	144.97	155
150	100	122.77	134.27	142.84	144.77	146.70	148.48	149.75	149.99	150
155	120.65	132.18	141.71	149.11	150.78	152.41	153.88	154.85	154.99	155
160	130.75	140.23	148.58	155.11	156.56	157.95	159.16	159.91	160	160
165	139.19	147.56	155.07	160.91	162.17	163.37	164.38	164.94	165	165
170	146.80	154.42	161.29	166.55	167.66	168.70	169.53	169.96	170	170
175	153.89	160.97	167.30	172.08	173.06	173.96	174.65	174.98	175	175
180	160.62	167.27	173.16	177.51	178.39	179.16	179.74	179.99	180	180
185	167.09	173.39	178.90	182.88	183.65	184.32	184.80	184.99	185	185
190	173.36	179.36	184.54	188.18	188.87	189.45	189.85	190	190	190

Table 6: Fund unit value as a function of total assets and remaining time of guarantee, with $K = 100$, $\sigma = 0.2$ and $r = 0.04$

$a \setminus T-t$	∞	20	10	5	4	3	2	1	6/12	3/12
110	-	-	-	-	-	-	-	-	-	56.02
115	-	-	-	-	-	-	-	13.18	62.63	84.49
120	-	-	-	-	-	-	-	59.06	81.71	94.39
125	-	-	-	-	-	23.69	50.12	75.45	90.77	98.09
130	-	-	-	18.55	36.54	51.07	66.26	84.67	95.38	99.40
135	-	-	-	45.65	54.68	64.54	76.07	90.31	97.73	99.83
140	-	-	30.96	58.72	65.50	73.31	82.66	93.85	98.91	99.95
145	-	17.70	47.17	67.43	73.00	79.51	87.28	99.11	99.49	99.99
150	0	38.74	57.13	73.76	78.52	84.08	90.60	97.55	99.77	100
155	33.52	49.76	64.29	78.59	82.72	87.52	93.02	98.46	99.90	100
160	45.17	57.45	69.77	82.35	85.99	90.16	94.80	99.04	99.96	100
165	53.08	63.29	74.12	85.35	88.57	92.20	96.12	99.40	99.98	100
170	59.05	67.94	77.65	87.77	90.63	93.80	97.10	99.63	99.99	100
175	63.80	71.74	80.57	89.74	92.28	95.05	97.83	99.77	100	100
180	67.70	74.91	83.01	91.36	93.62	96.04	98.38	99.86	100	100
185	70.96	77.59	85.08	92.69	94.72	96.83	98.78	99.92	100	100
190	73.73	79.88	86.84	93.81	95.61	97.45	99.09	99.95	100	100

Table 7: Risky investment as a percentage of total assets in the replicating portfolio with $K = 100$, $\sigma = 0.2$ and $r = 0.04$

$a \setminus T-t$	∞	20	10	5	4	3	2	1	6/12	3/12
110	-	-	-	-	-	-	-	100.88	107.44	109.38
115	-	-	-	-	-	-	101.31	110.88	113.89	114.81
120	-	-	-	-	-	106.17	112.62	117.69	119.49	119.94
125	-	-	-	106.76	111.74	116.13	120.17	123.64	124.77	124.99
130	-	-	-	117.39	120.54	123.67	126.67	129.18	129.90	130
135	-	-	114.09	125.40	127.84	130.29	132.64	134.50	134.95	135
140	-	110.80	123.50	132.41	134.42	136.42	138.30	139.70	139.98	140
145	108.76	121.56	131.36	138.87	140.56	142.24	143.77	144.82	144.99	145
150	120.45	130.11	138.45	144.98	146.44	147.86	149.10	149.89	150	150
155	129.40	137.68	145.08	150.85	152.11	153.32	154.34	154.93	155	155
160	137.24	144.68	151.38	156.54	157.64	158.67	159.52	159.96	160	160
165	144.46	151.31	157.45	162.10	163.07	163.95	164.64	164.98	165	165
170	151.26	157.66	163.34	167.56	168.41	169.17	169.74	169.99	170	170
175	157.78	163.81	169.10	172.93	173.68	174.34	174.80	174.99	175	175
180	164.08	169.80	174.75	178.25	178.91	179.47	179.86	180	180	180
185	170.20	175.66	180.31	183.51	184.09	184.58	184.89	185	185	185
190	176.19	181.42	185.80	188.72	189.24	189.66	189.92	190	190	190

Table 8: Fund unit value as a function of total assets and remaining time of guarantee, with $K = 95$, $\sigma = 0.2$ and $r = 0.04$

$a \setminus T-t$	∞	20	10	5	4	3	2	1	6/12	3/12
110	-	-	-	-	-	-	-	28.26	66.82	86.76
115	-	-	-	-	-	-	23.43	63.47	84.17	95.50
120	-	-	-	-	-	34.07	55.34	78.36	92.29	98.58
125	-	-	-	30.75	43.62	56.07	69.83	86.75	96.30	99.59
130				51.21	59.10	68.16	78.78	91.81	98.26	99.89
135	-	-	39.05	62.76	68.96	76.17	84.80	94.93	99.21	99.97
140	-	30.15	52.36	70.66	75.81	81.84	88.99	96.87	99.65	99.99
145	25.01	45.12	61.16	76.46	80.86	86.01	91.97	98.08	99.85	100
150	40.91	54.49	67.62	80.87	84.70	89.13	94.12	98.83	99.94	100
155	50.45	61.30	72.62	84.32	87.68	91.51	95.68	99.29	99.97	100
160	57.32	66.57	76.61	87.06	90.02	93.33	96.82	99.57	99.99	100
165	62.65	70.80	79.85	89.26	91.87	94.75	97.66	99.74	100	100
170	66.94	74.28	82.53	91.04	93.36	95.85	98.28	99.85	100	100
175	70.48	77.19	84.77	92.50	94.55	96.71	98.73	99.91	100	100
180	73.46	79.66	86.68	93.70	95.52	97.39	99.06	99.95	100	100
185	76.00	81.77	88.28	94.69	96.30	97.92	99.31	99.97	100	100
190	78.20	83.59	89.66	95.51	96.94	98.35	99.49	99.98	100	100

Table 9: Risky investment as a percentage of total assets in the replicating portfolio with $K = 95$, $\sigma = 0.2$ and $r = 0.04$

Let $0 < \varphi < 1$ be the guaranteed fraction of the maximum $\tilde{M}(t)$. Thus if $\tilde{F}(t) > \varphi\tilde{M}(t)$, the instantaneous rates of return of $\tilde{F}(t)$ and $F(t)$ are the same, and whenever $\tilde{F}(t)$ reaches the barrier $\varphi\tilde{M}(t)$, just enough money is provided so that the modified unit value does not fall below this barrier. The maximum process $\left\{\tilde{M}(t)\right\}$ is defined as

$$\tilde{M}(t) = \max\left\{m, \max_{0\le\tau\le t} \tilde{F}(\tau)\right\}$$

Here m is an initial value such that $\varphi m \le f \le m$.

We limit ourselves to the infinite time case. As we shall see later, the price of this contract is unfortunately infinite, if it is calculated as in section 5. Therefore we make the perhaps opportunistic assumption that the price of a security is the expected discounted sum of the corresponding payments, where the expectation is now calculated according to

$$\mu = r - d - \frac{1}{2}\sigma^2, \qquad \text{with } d > 0. \tag{22}$$

An explanation is that the fund pays cash dividends at a constant proportional rate d, so that the process

$$\left\{e^{-(r-d)t}F(t)\right\}$$

is a martingale.

Let $V(f, m; \varphi)$ denote the price of the guarantee. First we observe that this is a homogeneous function of degree 1 of the variables f and m. Thus

$$V(f, m; \varphi) = mV(f/m, 1; \varphi). \tag{23}$$

Let $\varphi m < f < m$. By distinguishing whether the process $F(t)$ first falls to the level φm or rises to the level m, we see that

$$V(f/m, 1; \varphi) = V(\varphi, 1; \varphi)A(f/m; \varphi, 1) + V(1, 1; \varphi)B(f/m; \varphi, 1). \tag{24}$$

The functions A and B are defined as in section 10.10 of Boyle et al. (1998). They are both linear combinations of $(f/m)^{\theta_1}$ and $(f/m)^{\theta_2}$, where θ_1 and θ_2 are solutions of the quadratic equation

$$\frac{1}{2}\sigma^2\theta^2 + \mu\theta - r = 0,$$

with μ given by (22). If θ_1 denotes the smaller solution, then $\theta_1 < 0$ and $\theta_2 > 1$. From (24) it follows that $V(f/m, 1; \varphi)$ is also a linear combination of $(f/m)^{\theta_1}$ and $(f/m)^{\theta_2}$. Hence

$$V(f/m, 1; \varphi) = C_1(f/m)^{\theta_1} + C_2(f/m)^{\theta_2},$$

where the coefficients C_1 and C_2 depend on φ only. Then by (23)

$$V(f, m; \varphi) = C_1 f^{\theta_1} m^{1-\theta_1} + C_2 f^{\theta_2} m^{1-\theta_2}, \qquad \varphi m \leq f \leq m. \tag{25}$$

To determine the coefficients, we examine this function at the boundaries. For the same reason that led to (6) and (14), we must have

$$\left.\frac{\partial V}{\partial f}\right|_{f=\varphi m} = -1. \tag{26}$$

This yields the condition

$$C_1 \theta_1 \varphi^{\theta_1 - 1} + C_2 \theta_2 \varphi^{\theta_2 - 1} = -1. \tag{27}$$

For the same reason that leads to (10.11.7) in Boyle et al. (1998), we have

$$\left.\frac{\partial V}{\partial m}\right|_{f=m} = 0,$$

which yields the condition

$$C_1(1 - \theta_1) + C_2(1 - \theta_2) = 0. \tag{28}$$

Solving equations (27) and (28) we get

$$C_1 = \frac{\frac{1}{1-\theta_1}}{-\frac{\theta_1}{1-\theta_1}\varphi^{\theta_1 - 1} - \frac{\theta_2}{\theta_2 - 1}\varphi^{\theta_2 - 1}},$$

$$C_2 = \frac{\frac{1}{\theta_2 - 1}}{-\frac{\theta_1}{1-\theta_1}\varphi^{\theta_1 - 1} - \frac{\theta_2}{\theta_2 - 1}\varphi^{\theta_2 - 1}}.$$

Substituting these values in (25), we obtain

$$V(f, m; \varphi) = m \frac{\frac{1}{1-\theta_1}(f/m)^{\theta_1} + \frac{1}{\theta_2 - 1}(f/m)^{\theta_2}}{-\frac{\theta_1}{1-\theta_1}\varphi^{\theta_1 - 1} - \frac{\theta_2}{\theta_2 - 1}\varphi^{\theta_2 - 1}}. \tag{29}$$

Note that the numerator is positive, but that the denominator is only positive, if

$$\varphi < \left(\frac{-\theta_1}{1-\theta_1} \frac{\theta_2 - 1}{\theta_2} \right)^{\frac{1}{\theta_2 - \theta_1}} \tag{30}$$

Thus the price of the guarantee is given by formula (29), provided that (30) holds. If (30) does not hold, the "price" of the guarantee would be infinite. In the limit $d \to 0$ ($\theta_1 \to 0$, $\theta_2 \to 1$) the expression on the right hand side of (30) is 0. Thus if $d = 0$, the "price" would be infinite for any $\varphi > 0$.

The expression on the right of (30) appears in formula (10.11.11) of Boyle et al. (1998) as $\tilde{\varphi}$, the optimal value in the context of a Russian option. Hence (30) states that φ must be smaller than $\tilde{\varphi}$.

In particular, for $\varphi = \tilde{\varphi}$ the "price" would be infinite. This can also be explained as follows. Consider a Russian option and its value function $R(f, m; \varphi)$ (which is denoted by the symbol $V(f, m; \varphi)$ in section 10.11 of Boyle et al. (1998)). Then for $\varphi m < f < m$ the price of the guarantee can be written as

$$V(f, m; \varphi) = R(f, m; \varphi) V(\varphi, 1; \varphi).$$

From this relation it follows that

$$\left.\frac{\partial V}{\partial f}\right|_{f=\varphi m} = \left.\frac{\partial R}{\partial f}\right|_{f=\varphi m} V(\varphi, 1; \varphi).$$

But, for $\varphi = \tilde{\varphi}$ we have the smooth pasting condition

$$\left.\frac{\partial R}{\partial f}\right|_{f=\tilde{\varphi} m} = 0,$$

see formula (9.12) of Gerber and Shiu (1996). This shows that for $\varphi = \tilde{\varphi}$ condition (26) cannot be satisfied. Hence for $\varphi = \tilde{\varphi}$ there is no finite price for the guarantee.

10 Concluding remarks

In this paper we analyze alternative solutions of two classical problems. The first problem consists of insuring the solvency of an insurance company. If the aim is that the company is solvent at time T, the classical minimal solution is provided by a stop-loss contract that covers the claims experience of the interval from 0 to T

and where the deductible is the initial capital. With this solution the intermediate surplus can be negative and there may be little hope for a positive surplus at time T. This is different for dynamic solvency insurance. Here the necessary payments are made instantaneously to avoid a negative surplus. As a consequence at any time there is hope for a substantially positive surplus at time T.

The second problem concerns the protection of an investment fund. If the goal is that the initial investment of f is worth at least K at time T, this can be accomplished by a European put option with strike price K and time to maturity T. This static solution has the following unattractive feature: if the investment fund does not develop favorably, so that the corresponding put option is deeply in-the-money, there may be little hope that the investment is worth more than K at time T. With dynamic fund protection, this situation can be avoided: at any time the investor is assured that his investment will be worth more than K at time T.

Evidently a better protection has a higher price. For an illustration consider an investment horizon of $T = 1$ with $f = 100$, $\sigma = 0.2$ and $r = 0.04$. Table 10 compares the prices of the static protection with those of the dynamic protection. The prices of the latter are from Table 2 and are more than twice as high as the prices of the corresponding put options. A more extensive comparison is provided by Table 11. We note that the ratio between the price of the dynamic fund protection and the price of the corresponding European put option increases progressively with T. We also observe that the ratio approaches the value 2 for small values of T. In fact, 2 is the limit for $T \to 0$ in general. To see this, suppose first that $f = K$. By Theorem 2 we have

$$\begin{aligned} V(K,T) &= \frac{K}{R}\Phi\left(\frac{1}{2}\sigma\sqrt{T}(R+1)\right) \\ &+K\left(1-\frac{1}{R}\right)e^{-rT}\Phi\left(-\frac{1}{2}\sigma\sqrt{T}(R-1)\right) \\ &-K\Phi\left(-\frac{1}{2}\sigma\sqrt{T}(R+1)\right). \end{aligned}$$

Using the expansion

$$\Phi(\varepsilon) = \frac{1}{2} + \frac{\varepsilon}{\sqrt{2\pi}} + \ldots \tag{31}$$

we see that

$$V(K,T) \sim K\frac{2\sigma\sqrt{T}}{\sqrt{2\pi}} - \frac{1}{2}K\left(r - \frac{1}{2}\sigma^2\right)T + \ldots \qquad \text{for } T \to 0.$$

K	80	85	90	95	100
European put option	0.7693	1.4654	2.5315	4.0325	6.0040
Dynamic fund protection	1.7709	3.4239	6.0120	9.7476	14.7931
Ratio	2.30	2.34	2.37	2.42	2.46

Table 10: Comparison of the prices with $f = 100$, $T = 1$, $\sigma = 0.2$ and $r = 0.04$

The price of the corresponding put option is

$$\begin{aligned} P(K,T) &= Ke^{-rT}\Phi\left(-\frac{r}{\sigma}\sqrt{T}+\frac{1}{2}\sigma\sqrt{T}\right) \\ &\quad -K\Phi\left(-\frac{r}{\sigma}\sqrt{T}-\frac{1}{2}\sigma\sqrt{T}\right). \end{aligned}$$

Using (31) we gather that

$$P(K,T) \sim K\frac{\sigma\sqrt{T}}{\sqrt{2\pi}} - \frac{1}{2}KrT + \ldots \qquad \text{for } T \to 0.$$

Thus indeed

$$\lim_{T\to 0}\frac{V(K,T)}{P(K,T)} = 2.$$

Now suppose that $f > K$. Let $g(t)$, $t > 0$, denote the probability density function of the first time when $F(t) = K$. By conditioning on the first passage time we see that

$$\begin{aligned} V(f,T) &= \int_0^T e^{-rt}V(K,T-t)g(t)dt, \\ P(f,T) &= \int_0^T e^{-rt}P(K,T-t)g(t)dt. \end{aligned}$$

Hence

$$\begin{aligned} V(f,T) &\sim K\frac{2\sigma}{\sqrt{2\pi}}\int_0^T \sqrt{T-t}g(t)dt \qquad \text{for } T \to 0, \\ P(f,T) &\sim K\frac{\sigma}{\sqrt{2\pi}}\int_0^T \sqrt{T-t}g(t)dt \qquad \text{for } T \to 0, \end{aligned}$$

which explains why $V(f,T)/P(f,T) \to 2$ for $T \to 0$.

$T \setminus K$	80	85	90	95	100
1/12	2.04	2.05	2.06	2.08	2.12
2/12	2.07	2.09	2.11	2.13	2.17
3/12	2.10	2.12	2.14	2.17	2.21
4/12	2.13	2.15	2.18	2.21	2.24
5/12	2.15	2.18	2.20	2.24	2.28
6/12	2.18	2.20	2.23	2.27	2.31
1	2.30	2.34	2.37	2.42	2.46
2	2.52	2.56	2.61	2.66	2.72
5	3.09	3.16	3.24	3.32	3.40
10	4.07	4.19	4.32	4.45	4.58
20	6.61	6.87	7.13	7.40	7.67

Table 11: Ratios of the prices with $f = 100$, $\sigma = 0.2$ and $r = 0.04$

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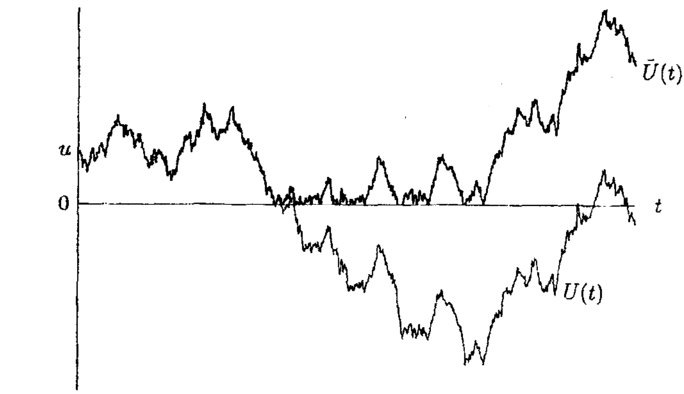

Figure 1: The surplus with and without dynamic solvency insurance

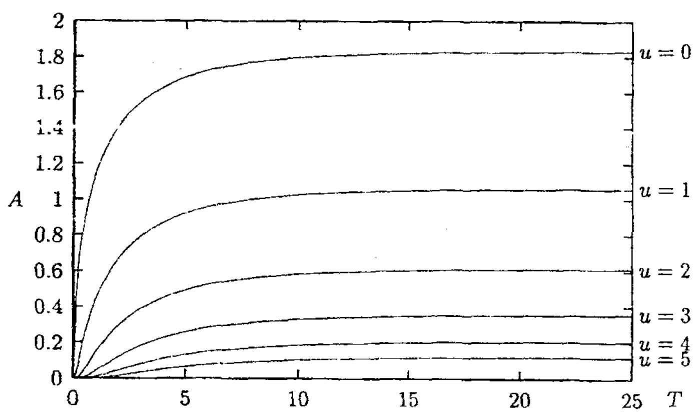

Figure 2: Net single premium for dynamic solvency insurance as a function of T with $\mu = 1$, $\sigma = 2$ and $\delta = 0.05$

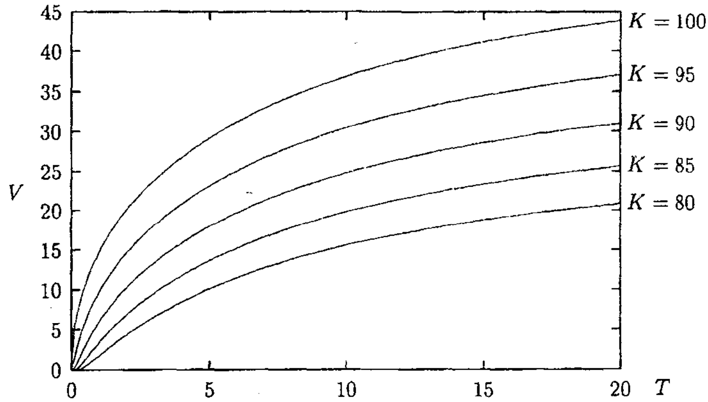

Figure 3: Price of the guarantee as a function of T with $f = 100$, $\sigma = 0.2$ and $r = 0.04$

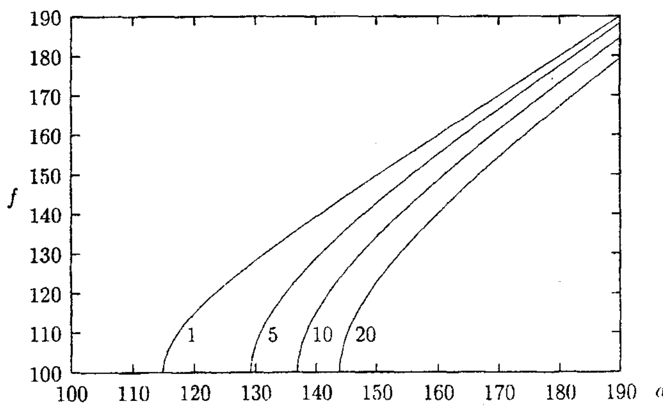

Figure 4: Fund unit value as a function of total assets with $K = 100$, $\sigma = 0.2$ and $r = 0.04$, for $T - t = 1, 5, 10, 20$

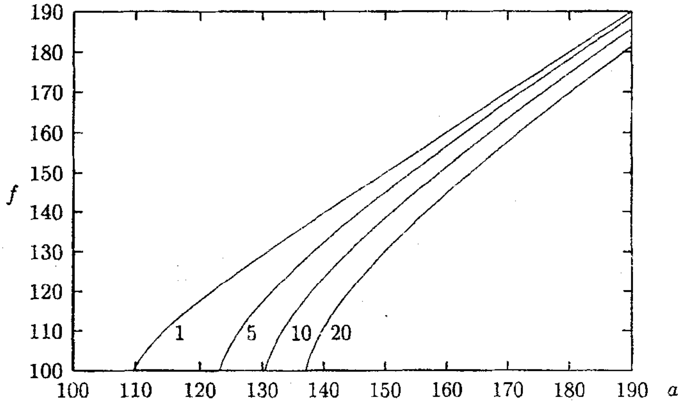

Figure 5: Fund unit value as a function of total assets with $K = 95$, $\sigma = 0.2$ and $r = 0.04$, for $T - t = 1, 5, 10, 20$