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STOCHASTIC AMORTIZATION OF A DEBT

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Summary

The purpose of the paper is to present a stochastic approach to the classical actuarial paradigm of amortization of a debt. In a first section, we consider the situation of a floating contract where the borrower accepts each year a complete variation of his payments in function of the financial market.

In a second section, we try to develop a model where the financial risk generated by the brownian motion is shared between the two parties.

Keywords : stochastic interest rates ; Wiener process ; amortization.

1. INTRODUCTION :

Each actuary (I hope so!) knows the classical formulas of amortization under constant interest rate . What does it happen when the financial environment becomes uncertain ? More precisely , we suppose that the constant interest rate is replaced by a stochastic Wiener process . This kind of modelization is classic and has been used to analyse the phenomenon of actualization and to price capitalization contract under stochastic interest rates (see DEVOLDER 86 and 91). The purpose is now to develop a stochastic amortization schedule of a loan given explicit formulas for the annual payments and the outstanding liabilities.
It should be also interesting in this stochastic environment to compare loans with fixed rates and loans with floating rates .

In a first section , we consider the situation of a floating contract where the borrower accepts each year a complete variation of his payments in function of the financial market. The sequence of payments can be computed and their mathematical expectation can be interpretated in term of financial risk.The results are compared with the classical deterministic theory.

In a second section , we try to develop a model where the financial risk generated by the brownian motion is shared between the two parties in function of a parameter k ($0 \leq k \leq 1$) fixed at the beginning of the contract and reflecting the level of risk accepted by the borrower. The case k=0 gives back the deterministic formula with an explicit value of the average rate to apply.

2. CLASSICAL AMORTIZATION OF A DEBT :

A debt will be said to be amortized when all liabilities are discharged by a sequence of equal payments made at equal intervals of time. If we consider a loan of a unitary capital made at t=0 for a duration N , repaid by constant annual payments under a constant interest rate i , we know that the characteristics of the loan are given by the following formulas :

-annual payment : $P=1/\, a_N$ (2.1)

where $a_N = (1-(1+i)^{-N})/i$

- outstanding liability at time t ($0 \leq t \leq N$) :

$R(t) = P \,.\, a_{N-t}$ (prospective approach) (2.2)

$=(1+i)^t - P \,.\, s_t$ (retrospective approach) (2.3)

where $s_N = ((1+i)^N-1)/i$.

3. STOCHASTIC FINANCING PLAN :

Consider a loan of a unitary capital at t=0 for a duration N ,repaid now by variable annual payments , $\{ P_1 ,P_2 ,...,P_N \}$ made successively at t = 1, 2,..., N.
In a stochastic environment, these payments will become random variables , governed by the evolution of the rates on the market.
To modelize the financial risk on this product , we have to introduce a stochastic model of evolution of the rates.We will use for this the brownian approach developed for instance in DEVOLDER(1986). In this context , the interest rates are generated by a process called "flow of interest" :

$$I(t) = \int_0^t \delta(s)\, ds + \int_0^t \sigma(s)\, dw(s) \tag{3.1}$$

where : -w is a standard brownian motion ;
-δ is the instantaneous average rate ;
-σ is the volatility on the rates.

An initial amount C(0) capitalized with this flux has an evolution following a stochastic differential equation :

$$C(t) = C(0) + \int_0^t C(s)\, dI(s) \tag{3.2}$$

It can be shown that the solution of this stochastic differential equation is given by:

$$C(t) = C(0).\ \exp\left(\int_0^t (\delta(s) - \sigma^2(s)/2)\, ds\right).\ \exp\left(\int_0^t \sigma(s)\, dw(s)\right) \tag{3.3}$$

The mathematical expectation of this process is given by :

$$EC(t) = C(0)\ .\ \exp\left(\int_0^t \delta(s)\, ds\right) \tag{3.4}$$

We have also introduced in the same model an actualisation process which reflects in this stochastic world the present value of a future amount :

$$\varphi(t) = 1 / C(t) \tag{3.5}$$

This actualisation process is also solution of a stochastic differential equation and has the following properties :

$$d\varphi(t) = -(\delta(t) - \sigma^2(t))\, \varphi(t)\, dt - \sigma(t)\, \varphi(t)\, dw(t) \tag{3.6}$$

and $$E\varphi(t) = \varphi(0) . \exp(\int_0^t -(\delta(s) - \sigma^2(s))\, ds \tag{3.7}$$

The corrected rates $(\delta - \sigma^2)$ can be called no-risk rates (see DEVOLDER 1986).

Let us come back now to our problem of financing plan of a debt.
In this new context , the outstanding liability R of the loan becomes a stochastic process governed by the annual payments. The evolution of this process must take into account two elements : the capitalization under market conditions driven by the process " flow of interest " I and the annual payments.
These payments $\{ P_1, P_2, .., P_N \}$ are now supposed to be random variables measurable with respect to the filtration generated by the brownian motion .
The stochastic differential equation of the outstanding liability is given by :

$$R(t) = 1 - \sum_{i=1}^{N} P_i\, 1_{t \le i} + \int_0^t R(s)\, dI(s) \tag{3.8}$$

A *balanced financing plan* consists of a series of measurable random variables $\{ P_1, P_2, ..., P_N \}$ such that the loan can be repaid on maturity :

$$P_N = R(N^-)$$

The solution of equation (3.8) , taking into account formula (3.3) , is then :

$$R(t) = \exp \int_0^t (\delta(s) - 1/2\, \sigma^2(s))\, ds. \exp \int_0^t \sigma(s)\, dw(s)$$

$$\sum_{i=1}^{[t]} P_i . \exp \int_i^t (\delta(s) - 1/2\, \sigma^2(s))\, ds. \exp \int_i^t \sigma(s)\, dw(s) \tag{3.9}$$

This outstanding liability is defined retrospectively (capitalized value of the borrowed amount - sum of the capitalized partial redemptions).
On the other hand , a prospective approach can be considered , based on the present value of future redemptions , noted R_p :

$$R_p(t) = \sum_{i=1+[t]}^{N} P_i \,.\, \exp \int_t^i -(\delta(s) - 1/2\,\sigma^2(s))\,ds.\ \exp \int_t^i -\sigma(s)\,dw(s) \qquad (3.10)$$

Unlike process R, this process R_p is not adapted to the natural filtration of the brownian motion.
On the due dates, (i.e. for t=j; j=0,1,2,...N) and supposing as usual that the future redemptions will remain equal, we have the following retrospective and prospective expressions for the outstanding liability :

$$R(j) = \exp \int_0^j (\delta(s) - 1/2\,\sigma^2(s))\,ds.\ \exp \int_0^j \sigma(s)\,dw(s)$$

$$- \sum_{i=1}^{j} P_i \,.\, \exp \int_i^j (\delta(s) - 1/2\,\sigma^2(s))\,ds.\ \exp \int_i^j \sigma(s)\,dw(s)$$

$$R_p(j) = P_j \sum_{i=j+1}^{N} \exp \int_j^i -(\delta(s) - 1/2\,\sigma^2(s))\,ds.\ \exp \int_j^i -\sigma(s)\,dw(s) \qquad (3.11)$$

In the deterministic theory, the prospective and retrospective loan balances are equal. In this stochastic environment, this equality is not confirmed but a balanced financing plan can be determined by the following rule : each payment is defined such as the retrospective balance is equal to the conditionnal expectation of the prospective balance under the natural filtration. This means that the payment P_j^* is such that :

$$R(j) = E\,(\,R_p(j)\ /\,F_j\,)$$

These two elements are given by :

$$E\,(\,R_p(j)\ /\,F_j\,) = P_j^* \,.\, \sum_{i=j+1}^{N} \exp \int_j^i (\sigma^2(s) - \delta(s))\,ds$$

and $$R(j) = \exp \int_0^j (\delta(s) - 1/2\,\sigma^2(s))\,ds.\ \exp \int_0^j \sigma(s)\,dw(s)$$

$$- \sum_{i=1}^{j-1} P_i^* \,.\, \exp \int_i^j (\delta(s) - 1/2\,\sigma^2(s))\,ds.\ \exp \int_i^j \sigma(s)\,dw(s)\ - P_j^*$$

By equalling the two expressions, it comes :

$$P_j^* = \{ \exp \int_0^j (\delta(s) - 1/2\,\sigma^2(s))\,ds . \exp \int_0^j \sigma(s)\,dw(s)$$

$$- \sum_{i=1}^{j-1} P_i^* . \exp \int_i^j (\delta(s) - 1/2\,\sigma^2(s))\,ds . \exp \int_i^j \sigma(s)\,dw(s) \;\} / \ddot{a}_{N-j+1} \tag{3.12}$$

where $\ddot{a}_{N-j+1} = \sum_{i=j}^{N} \exp \int_j^i (\sigma^2(s) - \delta(s))\,ds$ is a generalised annuity computed with the average rates δ corrected for the financial risk by the volatility σ^2

This formula allows to calculate the annual redemptions by recurrence.
The first annual redemption, at $t = 1$,is given by :

$$P_1^* = \{ \exp \int_0^1 (\delta(s) - 1/2\,\sigma^2(s))\,ds . \exp \int_0^1 \sigma(s)\,dw(s) \} / \ddot{a}_N$$

$$= \{ \exp \int_0^1 \sigma(s)\,dw(s) . \exp \int_0^1 1/2\,\sigma^2(s)\,ds \} / a_N \tag{3.13}$$

where $a_N = \sum_{i=1}^{N} \exp \int_0^i (\sigma^2(s) - \delta(s))\,ds$

The formula (3.13) can be compared with the deterministic model (cf. (2.1)).

The mathematical expectation of this first stochastic redemption is :

$$E\,P_1^* = E\,\{ \exp \int_0^1 \sigma(s)\,dw(s) . \exp \int_0^1 1/2\,\sigma^2(s)\,ds \} / a_N$$

$$= (\exp \int_0^1 \sigma^2(s)\,ds) / a_N \tag{3.14}$$

Let us analyse the behaviour of this plan near the expiry of the loan.
As the last adjustment by a control variable is made at $t = N$, we have :

$$E\,(R_p(N) / F_N) = 0 = R\,(N)$$

Thus :

$$R(N) = \exp \int_0^N (\delta(s) - 1/2\, \sigma^2(s))\, ds.\ \exp \int_0^N \sigma(s)\, dw(s)$$

$$- \sum_{i=1}^{N-1} P_i^* .\ \exp \int_i^N (\delta(s) - 1/2\, \sigma^2(s))\, ds.\ \exp \int_i^N \sigma(s)\, dw(s)\ - P_N^*$$

$$= R(N) - P_N^* = 0$$

So the last payment is :

$$P_N^* = \exp \int_0^N (\delta(s) - 1/2\, \sigma^2(s))\, ds.\ \exp \int_0^N \sigma(s)\, dw(s)$$

$$- \sum_{i=1}^{N-1} P_i^* .\ \exp \int_i^N (\delta(s) - 1/2\, \sigma^2(s))\, ds.\ \exp \int_i^N \sigma(s)\, dw(s)$$

The average evolution of the redemptions can also be studied ,starting from (3.12) :

$$\mathbf{E}(P_j^*) = \{\exp \int_0^j \delta(s)\ ds - \sum_{i=1}^{j-1} \mathbf{E}(P_i^*) .\ \exp \int_i^j \delta(s)\ ds\ \} / \ddot{a}_{N-j+1}$$

and therefore also :

$$\mathbf{E}(P_{j+1}^*) = \{\exp \int_0^{j+1} \delta(s)\ ds - \sum_{i=1}^{j} \mathbf{E}(P_i^*) .\ \exp \int_i^{j+1} \delta(s)\ ds\ \} / \ddot{a}_{N-j}$$

By multiplying in this last expression the upper part and the lower part by

$M=\exp(- \int_j^{j+1} \delta(s)\ ds)$,we have :

$$\mathbf{E}(P_{j+1}^*) = \{\exp \int_0^j \delta(s)\ ds - \sum_{i=1}^{j-1} \mathbf{E}(P_i^*) .\ \exp \int_i^j \delta(s)\ ds - \mathbf{E}(P_j^*)\ \} / \{\ \ddot{a}_{N-j}.M\}$$

$$= \{\ \mathbf{E}(P_j^*).(\ \ddot{a}_{N-j+1} - 1)\} / \{\ \ddot{a}_{N-j}.M\}$$

But :

$$(\ddot{a}_{N-j+1} - 1) / \{ \ddot{a}_{N-j}.M\} = \sum_{i=j+1}^{N} \exp \int_j^i (\sigma^2(s) - \delta(s))\, ds \,/\{K \sum_{i=j+1}^{N} \exp \int_j^i (\sigma^2(s) - \delta(s))\, ds \}$$

where $K = \exp \int_j^{j+1} -\sigma^2(s)\; ds$.

Finally :

$$E\,(P_{j+1}^*) = E\,(P_j^*).\exp \int_j^{j+1} \sigma^2\,(s)\, ds \tag{3.15}$$

This annual correction , $\exp \int_j^{j+1} \sigma^2\,(s)\, ds$, reflects the variance of the underlying financial risk between $t = j$ and $t = j+1$ .To illustrate this point , we can start from the capitalization formula (3.3) from j to j+1 :

$$\begin{aligned} C\,(j+1) &= C\,(j).\ \exp(\int_j^{j+1} (\delta\,(s) - \sigma^2\,(s)/2)\, ds\,).\ \exp(\int_j^{j-1} \sigma\,(s)\, dw(s)) \\ &= C\,(j).\ \exp(\int_j^{j+1} \delta\,(s)\, ds)\,.\ \alpha\,(j\,, j+1) \end{aligned}$$

where $\alpha\,(j\,, j+1)$ is the stochastic perturbation of the model.
The variance of this coefficient is given by :

$$\mathrm{var}\,(\alpha\,(j\,, j+1)\,/\,F_j\,) = (\exp \int_j^{j-1} \sigma^2\,(s)\, ds\; - 1).$$

So the recurrence formula (3.15) can be rewritten :

$$E\,(P_{j+1}^*) = E\,(P_j^*).(1 + \mathrm{var}\,(\alpha\,(j\,, j+1)\,/\,F_j\,)) \tag{3.16}$$

CONCLUSION : the average rating is rising; each year, the variance of the financial risk is added to the expectation of the previous payment.

The combination of (3.14) and (3.16) gives the explicit formula for each redemption:

$$E\,(P_j^*) = (\,\exp \int_0^j \sigma^2\,(s)\, ds\,)\,/\,a_N \tag{3.17}$$

This last formula can be interpreted in the following way :
1) $1 / a_N$ is the annual payment calculated in the deterministic way at a no-risk rate equal to $\delta - \sigma^2$; the use of this lower rate that the average rate δ reflects the fact that the borrower agrees to take on the risk on the rates .

2) $\exp \int_0^j \sigma^2 (s)\, ds$ gives the capitalised value of a unitary capital for a rate equalling the differential between the risk and no-risk rates .

Everything happens as if the loan were granted with the average rate $\delta - \sigma^2$, but the basic capital were indexed with the financial uncertainty reflected by σ^2 .

4. SHARING OF THE LOAN RATE VARIABILITY RISK :

In the previous paragraph , we considered the case of a borrower who agreed to undergo the market rates and to have the charges of his loan vary sometimes heavily to the hazards of the market .
Some borrowers could prefer a better stability of their charges in time , even if on average this means a higher rate. It is natural to introduce a concept of risk sharing : the borrower who agrees to take on a part of the risk should be able to benefit from a discount on his average interest rate.

If $dI(s) = \delta(s)\, ds + \sigma(s)\, dw(s)$ represents the market process ,applied normally when the borrower agrees the totality of the risk , we have to modelize a new process noted I_k , used when the borrower wants to take on a part k ($0 \leq k \leq 1$) of the rate variability risk. The case k=1 (totality of the risk) has been developped in the previous paragraph . The other limit case k=0 concerns a loan with fixed rates.

This new flux can be written :

$$dI_k(s) = \alpha_k(s)\, ds + k\,\sigma(s)\, dw(s) \; . \qquad (4.1)$$

The coherence of the rates requires that :

$$\alpha_k(s) < \alpha_j(s) \qquad \text{for} \quad k<j$$

If we use as tool to price this risk ,the concept of no-risk rate (cf.(3.7)) , we must take into account the fact that the interests are not received but paid by the client. It is then natural to talk of negative debit interest and to use the negative flows .

Let us then consider the no-risk rates for these flows:

-no-risk flow related to -I : $-\delta(s)-(-\sigma(s))^2$
-no-risk flow related to $-I_k$: $-\alpha_k(s)-(-k\sigma(s))^2$

We suppose now that all borrowers , regardless of their attitude towards risk (coefficient k) , use the same no-risk rates.

For the average rates , we then obtain :

$$\alpha_k(s) = \delta(s) + (1-k^2)\,\sigma^2(s) \tag{4.2}$$

For k=1 , the borrower takes on the whole variation risk (cf. previous paragraph) and the average rates are : $\alpha_1(s) = \delta(s)$.
For k=0 , the borrower does not want to take any hazard for his repayment ; so the deterministic rates which should be applied to him are : $\alpha_0(s) = \delta(s) + \sigma^2(s)$
The flow I_k can generally be noted in the following way :

$$I_k(s) = \mathbf{E}\, I(s) + k.\int_0^s \sigma(r)\, dw(r) + \mathrm{Var}\, I(s) - \mathrm{Var}(k.\int_0^s \sigma(r)\, dw(r)) \tag{4.3}$$

The first term is the average rate ;the second term is the part of the risk accepted by the borrower; the third term is a penalty on the average rates , related to the existence of a financial hazard ; the fourth term is a discount for the borrower's acceptance of a part of the financial hazard.

Suppose a borrower uses this scale of rates { α_k }. The flow structure then becomes :

$$dI_k(s) = (\delta(s) + (1-k^2)\,\sigma^2(s))\, ds + k\,\sigma(s)\, dw(s) \tag{4.4}$$

The loan is supposed to be repaid by annual payments { $P_1, P_2, \ldots, P_N$ }.
The formulas of paragraph 3 apply provided δ is now replaced by ($\delta + (1-k^2)\,\sigma^2$) and σ by $k\sigma$.
In particular , for the balanced financing plan , the formula (3.12) becomes :

$$P_j^*(k) = \{ \exp\int_0^j (\delta(s)+(1-3/2\,k^2)\sigma^2(s))\, ds .\ \exp\int_0^j k\sigma(s)\, dw(s)$$

$$- \sum_{i=1}^{j-1} P_i^*(k) \exp\int_i^j (\delta(s))+(1-3/2\,k^2)\sigma^2(s))ds .\ \exp\int_i^j k\sigma(s)\, dw(s) \ \}/\ \ddot{a}_{N-j+1}^{\,k} \tag{4.5}$$

where $\ddot{a}_{N-j+1}{}^{k} = \sum_{i=j}^{N} \exp \int_{j}^{i} -(\delta(s)+(1-2k^2)\sigma^2(s))\, ds$ is a generalised annuity computed with the average rates ($\delta + (1-k^2)\sigma^2$) corrected for the financial risk by the volatility $k^2\sigma^2$.

The average evolution of these payments is given by (see (3.17)) :

$$E(P_j^*(k)) = (\exp k^2 \int_0^j \sigma^2(s)\, ds) / a_N^k \tag{4.6}$$

where $a_N^k = \sum_{i=1}^{N} \exp \int_0^i -(\delta(s)+(1-2k^2)\sigma^2(s))\, ds$

In particular :

1) for <u>k=1 :</u> this is again the plan of paragraph 3 (all the risk accepted by the borrower).

2) for <u>k=0 :</u> no risk accepted by the borrower ; we obtain :

$$P_j^*(0) = \{\exp \int_0^j (\delta(s)+\sigma^2(s))\, ds - \sum_{i=1}^{j-1} P_i^*(0) \exp \int_i^j (\delta(s))+\sigma^2(s))ds\} / \ddot{a}_{N-j+1}{}^{0}$$

So the loan is granted at the rate $\delta + \sigma^2$ and we have :

$$P_j^*(0) = P_{j+1}^*(0) = P^*(0) = 1 / a_N^0 \tag{4.7}$$

where $a_N^0 = \sum_{i=1}^{N} \exp \int_0^i -(\delta(s)+\sigma^2(s))\, ds$

We thus find a classical financing plan with level payments , calculated at the deterministic rate $\delta + \sigma^2$ (higher than the average rate δ). The price of absolute safety is reflected by the correction σ^2 .

In general , the formula (4.6) shows that the higher k (i.e. the more risk the borrower takes on) :

- the smaller the corrected average rate ($\delta + (1-2k^2)\sigma^2$) in the annuity a_N^k ;

- the higher the yearly increase factor of the averages charges .

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